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NON-INTEGRAL TOROIDAL SURGERY ON HYPERBOLIC KNOTS IN S^3

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Abstract. It is shown that a hyperbolic knot in S^3 admits at most one non-integral Dehn surgery producing a toroidal manifold.

Let K be a knot in the 3-sphere S^3 and $M = M_K$ the complement of an open regular neighborhood of K in S^3. As usual, the set of slopes on the torus ∂M (i.e. the set of isotopy classes of essential simple loops on ∂M) is parameterized by

$$\{m/n : \ m, n \in \mathbf{Z}, n > 0, (m, n) = 1\} \cup \{1/0\},$$

so that $1/0$ is the meridian slope and $0/1$ is the longitude slope. A slope m/n is called *non-integral* if $n \geq 2$. The manifold obtained by Dehn surgery on S^3 along the knot K (equivalently, Dehn filling on M along the torus ∂M) with slope m/n, is denoted by $M(m/n)$. Now suppose that $K \subset S^3$ is a hyperbolic knot, i.e. the interior of M has a complete hyperbolic metric of finite volume. A basic question in Dehn surgery theory is: when can a surgery on K produce a non-hyperbolic 3-manifold? A special case of this question is: when can a surgery on K produce a toroidal 3-manifold, i.e. a 3-manifold which contains an (embedded) incompressible torus? In [GL1], Gordon and Luecke showed that if m/n is a non-integral slope and $M(m/n)$ contains an incompressible torus, then $n = 2$. In this paper, we show that there is at most one such surgery slope.

Theorem 1. *For a hyperbolic knot in S^3, there is at most one surgery with non-integral slope producing a manifold containing an incompressible torus.*

There are examples of hyperbolic knots in S^3 which admit toroidal surgeries with non-integral slopes [EM]. The best known example is the $(-2, 3, 7)$-pretzel knot, on which the surgery with slope $37/2$ gives a toroidal 3-manifold. It is a conjecture ([Go1], [K, Problem 1.77]) that for a hyperbolic knot in S^3, there is at most one surgery with non-integral slope producing a non-hyperbolic 3-manifold, and further, if there is such a surgery, it must be a toroidal surgery and the knot must belong to the collection of examples given in [EM]. Theorem 1 provides some supporting evidence for this conjecture.

We now go on to the proof of Theorem 1. Our argument is based on applications of results and combinatorial techniques developed in [CGLS], [GL1], [Go2].

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Recall that the *distance* between two slopes m_1/n_1 and m_2/n_2 is defined as the number $\Delta = \Delta(m_1/n_1, m_2/n_2) = |m_1n_2 - n_1m_2|$, which is equal to the minimal geometric intersection number between simple loops representing the two slopes on ∂M. Suppose that there are two slopes $m_1/2$ and $m_2/2$ such that both $M(m_1/2)$ and $M(m_2/2)$ contain incompressible tori. It follows from [Go2, Theorem 1.1] that there are exactly four hyperbolic manifolds that admit two toroidal Dehn fillings at distance more than 5, but for homological reasons, only one of these is the complement of a knot in S^3, namely the figure 8 knot complement, and by [Th] every non-integral surgery on this manifold is hyperbolic. Hence we have $\Delta(m_1/2, m_2/2) = |2m_1 - 2m_2| = 2|m_1 - m_2| \leq 5$. Note that both m_1 and m_2 are odd integers. So $|m_1 - m_2|$ is even and thus must be equal to 2. Hence the distance between the two slopes is exactly 4. Our task here is to show that this is impossible. Note, however, that 4 can be realized as the distance between integral toroidal surgery slopes for a hyperbolic knot in S^3. For instance the slopes 16 and 20 are both toroidal surgery slopes for the $(-2, 3, 7)$-pretzel knot. The reason that distance 4 is impossible in our situation is mainly due, as we will see, to the fact that the first homology of $M(m_i/2)$ with $\mathbf{Z}_2$ coefficients is trivial for $i = 1, 2$.

By [GL1] and [GL2] (see [GL1, Theorem 1.2]), for $i = 1, 2$, there is an incompressible torus $\widehat{T}_i$ in $M(m_i/2)$ such that $M \cap \widehat{T}_i = T_i$ is an incompressible, ∂-incompressible, twice punctured torus properly embedded in M with each component of ∂T_i having slope $m_i/2$ in ∂M. Note that T_i separates M since $M(m_i/2)$ has finite first homology. By an isotopy of T_i, we may assume that T_1 and T_2 intersect transversely, and $T_1 \cap T_2$ has the minimal number of components. So $T_1 \cap T_2$ is a set of finitely many circle components and arc components properly embedded in T_i, $i = 1, 2$. Furthermore, no circle component of $T_1 \cap T_2$ bounds a disk in T_i and no arc component of $T_1 \cap T_2$ is boundary parallel in T_i, $i = 1, 2$, since T_i is incompressible and ∂-incompressible.

We shall use the indices i and j to denote 1 or 2, with the convention that, when they are used together, $\{i, j\} = \{1, 2\}$ as a set.

Let V_i denote the solid torus that is attached to M in forming $M(m_i/2)$. The torus $\widehat{T}_i$ intersects V_i in two disks $B_{i(1)}$ and $B_{i(2)}$, which cut V_i into two 2-handles, which we denote by $H_{i(1)}$ and $H_{i(2)}$. Correspondingly, $\partial T_i = \partial B_{i(1)} \cup \partial B_{i(2)}$ cuts ∂M into two annuli $A_{i(1)}$ and $A_{i(2)}$, where $A_{i(k)} \subset \partial H_{i(k)}$. Each $\partial T_j \cap A_{i(k)}$ consists of exactly 8 essential arcs on $A_{i(k)}$.

The torus $\widehat{T}_i$ separates $M(m_i/2)$ into two submanifolds which we denote by $\widehat{X}_{i(1)}$ and $\widehat{X}_{i(2)}$, with $H_{i(k)} \subset \widehat{X}_{i(k)}$. Correspondingly T_i separates M into two pieces, denoted by $X_{i(1)}$ and $X_{i(2)}$. Thus $X_{i(k)} = M \cap \widehat{X}_{i(k)}$. Note that $\widehat{X}_{i(k)}$ is obtained by attaching the 2-handle $H_{i(k)}$ to $X_{i(k)}$ along the annulus $A_{i(k)}$, $k = 1, 2$. Also note that $F_{i(k)} = \partial X_{i(k)} = T_i \cup A_{i(k)}$ is a closed surface of genus two, $k = 1, 2$.

Lemma 2. *For $i = 1, 2$, $k = 1, 2$, we have $H_1(\widehat{X}_{i(k)}, \widehat{T}_i; \mathbf{Z}_2) = 0$.*

Proof. Since $H_1(M(m_i/2); \mathbf{Z}_2) = 0$, and $\widehat{T}_i$ is connected, we have

$$0 = H_1(M(m_i/2), \widehat{T}_i; \mathbf{Z}_2) = H_1(\widehat{X}_{i(1)}, \widehat{T}_i; \mathbf{Z}_2) \oplus H_1(\widehat{X}_{i(2)}, \widehat{T}_i; \mathbf{Z}_2);$$

hence each $H_1(\widehat{X}_{i(k)}, \widehat{T}_i; \mathbf{Z}_2) = 0$. □

Now, as in [CGLS, 2.5], and [Go2], we construct two graphs Γ_1 and Γ_2 in $\widehat{T}_1$ and $\widehat{T}_2$ respectively by taking the arc components of $T_1 \cap T_2$ as edges and $\widehat{T}_i - int(T_i) =$

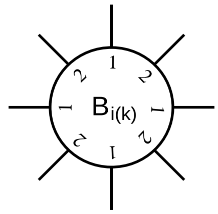

FIGURE 1.

$B_{i(1)} \cup B_{i(2)}$ as (fat) vertices. So Γ_i is a graph on a torus with two vertices. The exterior of the graph Γ_i in $\widehat{T}_i$ is the set of faces of Γ_i. Each face of the graph Γ_i is a surface properly embedded in $X_{j(k)}$ for some $k = 1$ or 2. Since each component of ∂T_1 intersects each component of ∂T_2 in exactly 4 points, each of the two vertices of Γ_i has valency (i.e. the number of edge endpoints at the vertex) 8 and there are exactly 8 edges in Γ_i. If e is an edge with an endpoint at a vertex $B_{i(k)}$ of Γ_i ($k = 1$ or 2), then that endpoint is in $\partial B_{i(k)} \cap \partial B_{j(l)}$ for some vertex $B_{j(l)}$ ($l = 1$ or 2) of Γ_j, and the endpoint of e at the vertex $B_{i(k)}$ is given the label l. So the labels of edge endpoints around each of the vertices $B_{i(k)}$, $i = 1, 2$, $k = 1, 2$, are as shown in Figure 1.

A *cycle* in Γ_i is a subgraph homeomorphic to a circle (where the fat vertices of Γ_i are considered as points). The number of edges in a cycle is its *length*. A length one cycle is also called a *loop*. Since no arc component of $T_1 \cap T_2$ is boundary parallel on T_1 or T_2, no loop of Γ_i bounds a disk face. We shall consider two parallel loops as forming a cycle of length two. (Two edges of Γ_i are said to be *parallel* in Γ_i if they, together with two arcs in ∂T_i, bound a disk in T_i.) A cycle in Γ_i consisting of two parallel adjacent loops is called an *S-cycle*. This definition is a specialization to our current situation of the usual definition of an S-cycle (i.e. a length 2 Scharlemann cycle) given in [W], [GL1]. The disk face of Γ_i bounded by an S-cycle in Γ_i is called the *S-disk* of the S-cycle.

Let $\overline{\Gamma}_i$ be the reduced graph of Γ_i, i.e. the graph obtained from Γ_i by amalgamating each complete set of mutually parallel edges of Γ_i to a single edge. Then up to homeomorphism of $\widehat{T}_i$, $\overline{\Gamma}_i$ is a subgraph of the graph illustrated in Figure 2 (for a proof of this, see [Go2, Lemma 5.2]). It follows in particular that in Γ_i the number of loops at the vertex $B_{i(1)}$ is equal to the number of loops at the vertex $B_{i(2)}$, and thus the number of loops of Γ_i is even. From now on we will take Figure 2 as a fixed model graph of which our $\overline{\Gamma}_i$ is a subgraph.

If Γ_i contains some loops, then they cut T_i into disks together with two annuli, which will be called the *loop complement annuli.*

Lemma 3. (1) *No pair of edges can be parallel in both* Γ_1 *and* Γ_2.

(2) *Any closed curve* α *on* T_i *intersects* Γ_i *in an even number of points. In particular, any loop complement annulus contains an even number of (non-loop) edges.*

(3) (The parity rule) *A component* e *of* $T_1 \cap T_2$ *is a loop on* Γ_1 *if and only if it is a non-loop on* Γ_2. *An edge of* Γ_i *has the same label at its two endpoints if and only if it is not a loop.*

(4) *Each non-loop edge in* $\overline{\Gamma}_i$ *represents at most two edges of* Γ_i.

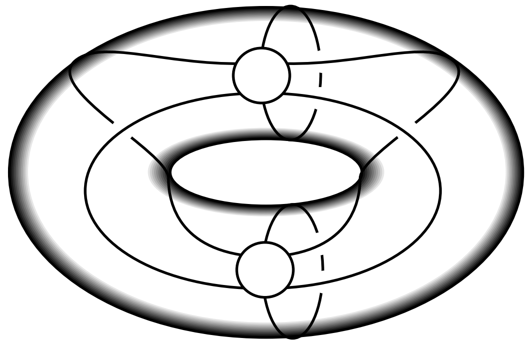

FIGURE 2.

Proof. (1) Otherwise, the manifold M would be cabled by [Go2, Lemma 2.1], contradicting the assumption that M is hyperbolic.

(2) We have seen above that $\widehat{T}_j$ is a separating torus; hence α intersects $\widehat{T}_j$ in an even number of points. Since $\alpha \cap \Gamma_i = \alpha \cap \widehat{T}_j$, the result follows. For the statement about a loop complement annulus A, just take α to be the central curve of A.

(3) We fix an orientation on T_i and let each component of ∂T_i have the induced orientation. Then the two components of ∂T_i are not homologous in ∂M since T_i separates M. (The corresponding vertices of Γ_i are *anti-parallel* in the terminology of [CGLS, p. 278]). Since M, T_1 and T_2 are all orientable, the parity rule given in [CGLS, Page 279] holds and has the above special form in our present situation.

(4) By the parity rule, a non-loop edge of Γ_i is a loop edge of Γ_j. Since there are at most two families of parallel loops on Γ_j, a set of three or more parallel non-loop edges would contain a pair of edges parallel in both Γ_i and Γ_j, contradicting (1). $\square$

Lemma 4. *Γ_i cannot have two disk faces D_1, D_2, each with an even number of edges, such that ∂D_1 and ∂D_2 are nonparallel, nonseparating curves on $F_{j(k)}$ for some $k = 1, 2$.*

Proof. Otherwise, since $F_{j(k)}$ has genus 2, the disks D_i cut $X_{j(k)}$ into a manifold with sphere boundary, which must be a 3-ball B because $X_{j(k)}$, as a submanifold of S^3 with a single boundary component, is irreducible. Thus we have

$$\widehat{X}_{j(k)} = \widehat{T}_j \cup H_{j(k)} \cup D_1 \cup D_2 \cup B.$$

From now on we shall regard $H_{j(k)}$ as a 1-handle. Now $H_1(\widehat{T}_j \cup H_{j(k)}, \widehat{T}_j; \mathbf{Z}_2) = \mathbf{Z}_2$ is generated by the core of $H_{j(k)}$. Since ∂D_i has an even number of edges, it runs over $H_{j(k)}$ an even number of times, so it represents zero in $H_1(\widehat{T}_j \cup H_{j(k)}, \widehat{T}_j; \mathbf{Z}_2)$. Therefore,

$$H_1(\widehat{X}_{j(k)}, \widehat{T}_{j(k)}; \mathbf{Z}_2) = H_1(\widehat{T}_j \cup H_{j(k)}, \widehat{T}_{j(k)}; \mathbf{Z}_2)/\left\langle \partial D_1, \partial D_2 \right\rangle = \mathbf{Z}_2/\left\langle 0, 0 \right\rangle = \mathbf{Z}_2.$$

This contradicts Lemma 2, completing the proof. $\square$

Lemma 5. *Let ∂_i be a boundary component of T_i, $i = 1, 2$. If the four points of $\partial_1 \cap \partial_2$ appear in the order x_1, x_2, x_3, x_4 on ∂_1, then they also appear in the same* *order on ∂_2, in some direction. In particular, if two of the points x_p, x_q are adjacent on ∂_1 among the four points, then they are also adjacent on ∂_2.*

Proof. We may choose coordinates on ∂M so that ∂_1 is the $1/0$ curve and ∂_2 is the $1/4$ curve. The lemma is obvious by drawing such curves on a torus. □

We will often apply this lemma to the endpoints of a pair of edges e_1, e_2. If each e_i has an endpoint x_i at the vertex $B_{1(l)}$, with label k, then at $B_{2(k)}$ e_1 and e_2 both have label l. Lemma 5 says that x_1, x_2 are adjacent on $\partial B_{1(l)}$ among all edge endpoints labeled k if and only if they are adjacent on $\partial B_{2(k)}$ among all edge endpoints labeled l.

By the parity rule, one of Γ_1 and Γ_2, say Γ_1, contains at least 4 loops. So Γ_1 has either 8, 6 or 4 loops.

Lemma 6. *Γ_i cannot have 8 loops.*

Proof. Suppose Γ_1, say, has 8 loops. Then the four loops based at the vertex $B_{1(1)}$ in Γ_1 form three S-cycles, bounding three S-disks D_1, D_2 and D_3. The disks D_1 and D_3 are on the same side of T_2, say $X_{2(1)}$. The four corresponding edges of Γ_2 connect the two vertices $B_{2(1)}$ and $B_{2(2)}$, and by Lemma 3(1) they are mutually nonparallel in Γ_2. Thus ∂D_1 and ∂D_3 are nonparallel curves on the genus 2 surface $F_{2(1)} = \partial X_{2(1)}$. Since each ∂D_i is an S-cycle, it runs over $H_{2(1)}$ twice in the same direction, so it is a nonseparating curve on $F_{2(1)}$. This contradicts Lemma 4. □

Lemma 7. *Γ_i cannot have 6 loops.*

Proof. Suppose Γ_1, say, has 6 loops. By Lemma 3(2) the number of non-loop edges in each loop complement annulus A_k is even. Thus the two non-loop edges must be on the same A_k. Up to isomorphism, the graph Γ_1 is one of the two graphs shown in Figure 3(a)–(b). However, Figure 3(a) is not possible because the longitude of the torus intersects the graph in an odd number of points, which contradicts Lemma 3(2).

Now consider Γ_2. For the same reason as above, the number of non-loop edges on each loop complement annulus A_k is even, and by Lemma 3(4) each non-loop edge of $\overline{\Gamma}_2$ represents at most two parallel edges, so we have two possibilities, shown in Figure 3(c)–(d). Again, Figure 3(c) can be ruled out by looking at the intersection number between Γ_2 and a longitude of the torus.

Consider the two edges e_1, e_2 in Figure 3(d). The two endpoints x_1, x_2 of e_1, e_2 on $\partial B_{2(2)}$ are adjacent among all points of $\partial B_{1(1)} \cap \partial B_{2(2)}$, so by Lemma 5 they are also adjacent on $\partial B_{1(1)}$. Examining the labeling of the edge endpoints at $\partial B_{1(1)}$ we see that the rightmost loop based at $B_{1(1)}$ is one of the e_1, e_2; hence the other two endpoints y_1, y_2 of e_1, e_2 are also adjacent on $\partial B_{1(1)}$ among all points of $\partial B_{1(1)} \cap \partial B_{2(1)}$. However, on Figure 3(d) the two endpoints of e_1, e_2 on $\partial B_{2(1)}$ are not adjacent among all edge endpoints labeled 1, which contradicts Lemma 5. □

We may now assume that Γ_1 has exactly four loops. By the parity rule Γ_2 also has exactly four loops. By Lemma 3(2) each loop complement annulus A_k contains an even number of edges, so there are four possible configurations for the graph Γ_i, according to whether each A_k contains exactly two edges, and if there are two, whether they are parallel. See Figure 4(a)–(d). However, Figure 4(a) is impossible for each of Γ_1 and Γ_2 because the longitude of the torus intersects the graph in an odd number of points.

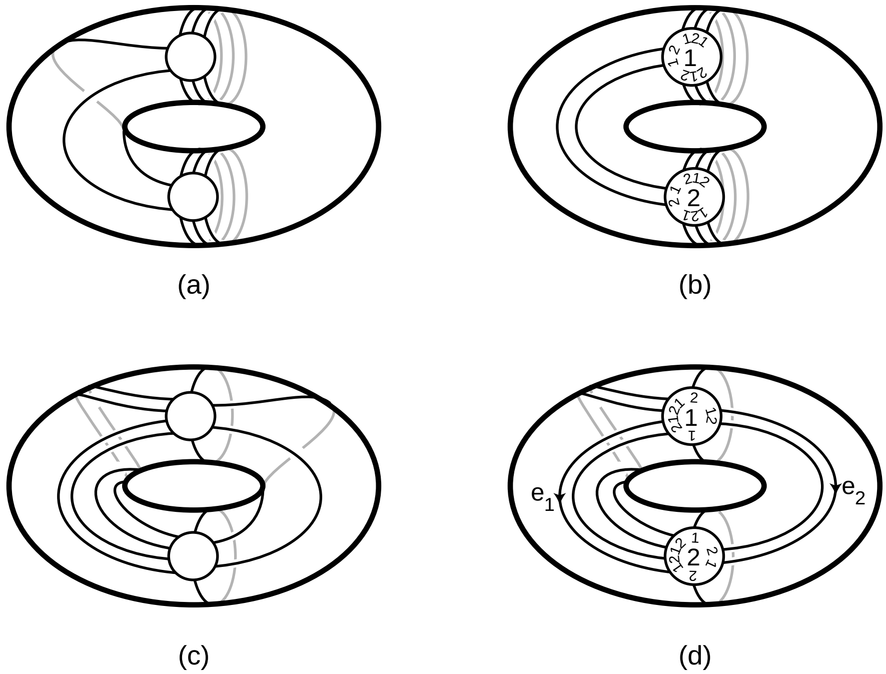

Figure 3.

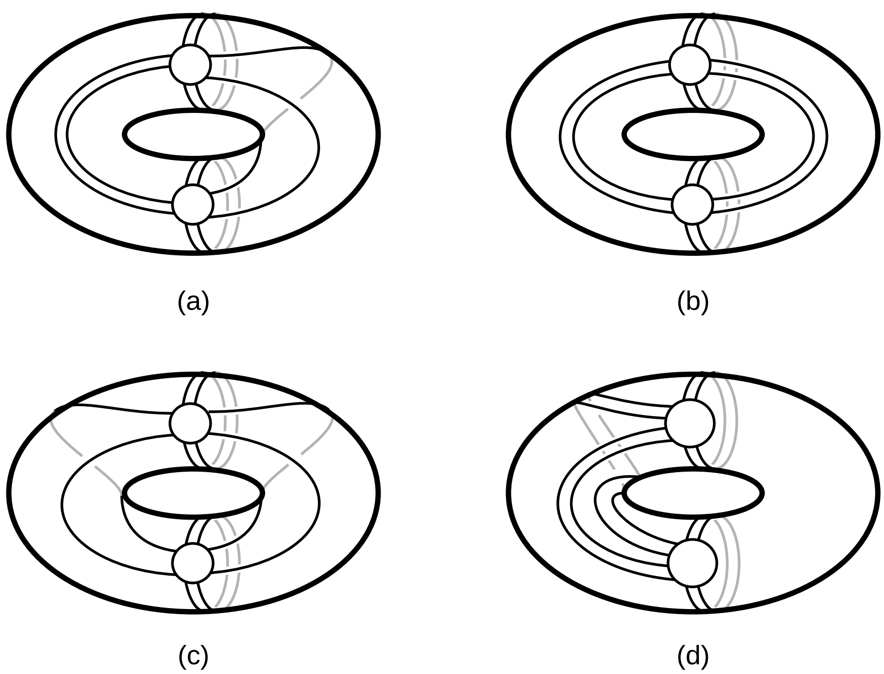

Figure 4.

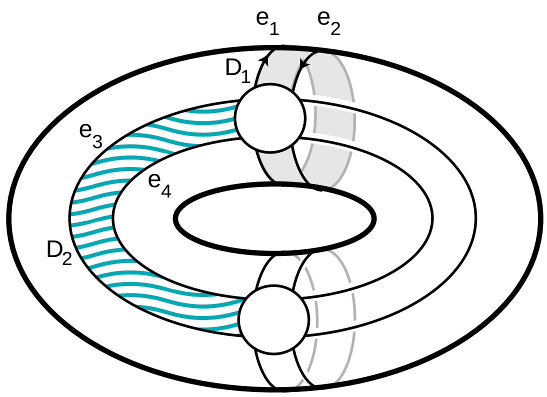

FIGURE 5.

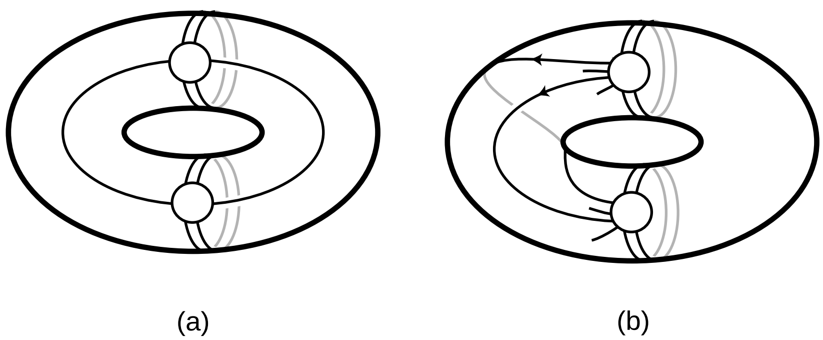

FIGURE 6.

Lemma 8. *Figure 4(b) is impossible for each of* Γ_1 *and* Γ_2.

Proof. Suppose that one of Γ_1 and Γ_2, say Γ_1, is as shown in Figure 4(b). Let D_1 be the S-disk bounded by the S-cycle $\{e_1, e_2\}$, and D_2 the disk bounded by the cycle $\{e_3, e_4\}$, as shown in Figure 5.

Then D_1 and D_2 are contained in the same side of T_2, say in $X_{2(1)}$. In Γ_2, the two edges $\{e_1, e_2\}$ also form a cycle which we denote by σ_1. There are two possibilities for $\sigma_1 = \{e_1, e_2\}$, depending on whether or not e_1 and e_2 lie on the same loop complement annulus. See Figure 6.

It is clear that ∂D_1 and ∂D_2 are nonparallel curves on $F_{2(1)}$. Also, as the boundary of an S-disk, ∂D_1 is always nonseparating. Now ∂D_2 consists of one loop at each vertex of Γ_2, together with two arcs on the boundary of the handle $H_{2(1)}$. In the case of Figure 6(a), there is an arc with its endpoints on ∂D_1, intersecting ∂D_2 at a single point, and hence ∂D_2 is also nonseparating. This contradicts Lemma 4.

In the case of Figure 6(b), the two endpoints x_1, x_2 of e_1 and e_2 on $\partial B_{2(1)}$ are labeled 1, say, and are adjacent among all points of $\partial B_{1(1)} \cap \partial B_{2(1)}$ on $\partial B_{2(1)}$. Orient e_1, e_2 as in Figure 6. Then on Γ_1 the two endpoints x_1, x_2 are labeled 1, so they cannot be adjacent edge endpoints on $\partial B_{1(1)}$. Thus from Figure 5 we can see that the two parallel edges e_1, e_2 on Γ_1 have opposite orientations. But then

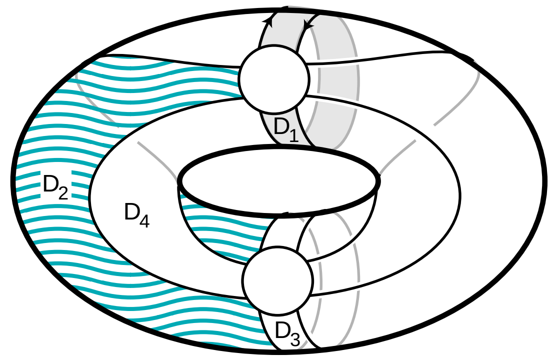

FIGURE 7.

x_1, x_2 are not adjacent among all edge endpoints labeled 1 on $\partial B_{1(1)}$, contradicting Lemma 5. □

Lemma 9. Γ_i *cannot be as in Figure 4(c).*

Proof. Suppose Γ_1, say, is as shown in Figure 4(c). Consider the disks D_1, D_2 shown in Figure 7. They are on the same side of T_2, say in $X_{2(1)}$. The curve ∂D_1 is an S-cycle on Γ_1, so it is automatically nonseparating on $F_{2(1)}$. The curve ∂D_2 is also nonseparating, because it runs over $H_{2(1)}$ three times. Clearly they are nonparallel, so by the proof of Lemma 4 we see that

$$\widehat{X}_{2(1)} = \widehat{T}_2 \cup H_{2(1)} \cup D_1 \cup D_2 \cup B,$$

where B is a 3-ball. Similarly

$$\widehat{X}_{2(2)} = \widehat{T}_2 \cup H_{2(2)} \cup D_3 \cup D_4 \cup B.$$

Hence

$$\begin{aligned} H_1(M(m_2/2); \mathbf{Z}_2) &= H_1(\widehat{T}_2 \cup V_2; \mathbf{Z}_2) / \langle \partial D_1, \partial D_2, \partial D_3, \partial D_4 \rangle \\ &= (\mathbf{Z}_2)^4 / \langle \partial D_1, \dots \partial D_4 \rangle . \end{aligned}$$

Each of ∂D_1 and ∂D_3 runs over $H_{2(k)}$ an even number of times, so if we denote by σ_1, σ_3 the corresponding cycles on Γ_2 with each fat vertex shrunk to a point, then homologically we have $\partial D_i = \sigma_i$ in $H_1(\widehat{T}_2 \cup V_2; \mathbf{Z}_2)$, for $i = 1, 3$.

Consider Γ_2. We have shown so far that Γ_2 must be the graph in either Figure 4(c) or 4(d). In either case we have $\sigma_1 + \sigma_3 = 0$ in $H_1(\widehat{T}_2; \mathbf{Z}_2)$, so homologically we have $\partial D_1 = \partial D_3$. Therefore,

$$0 = H_1(M(m_2/2); \mathbf{Z}_2) = (\mathbf{Z}_2)^4 / \langle \partial D_1, \partial D_2, \partial D_4 \rangle \neq 0.$$

This contradiction completes the proof of the lemma. □

Lemma 10. Γ_1 *and* Γ_2 *cannot both be as in Figure 4(d).*

Proof. In this case the graph Γ_i is contained in an annulus Q_i in $\widehat{T}_i$, so we can redraw the graph on Q_i, $i = 1, 2$, as in Figure 8. Orient the non-loop edges in both Γ_i so that they go from $B_{i(1)}$ to $B_{i(2)}$. This determines the orientations of the loop edges: they must go from the endpoints labeled 1 to those labeled 2. Use e_i^-, e_i^+ to denote the tail and head of e_i, respectively.

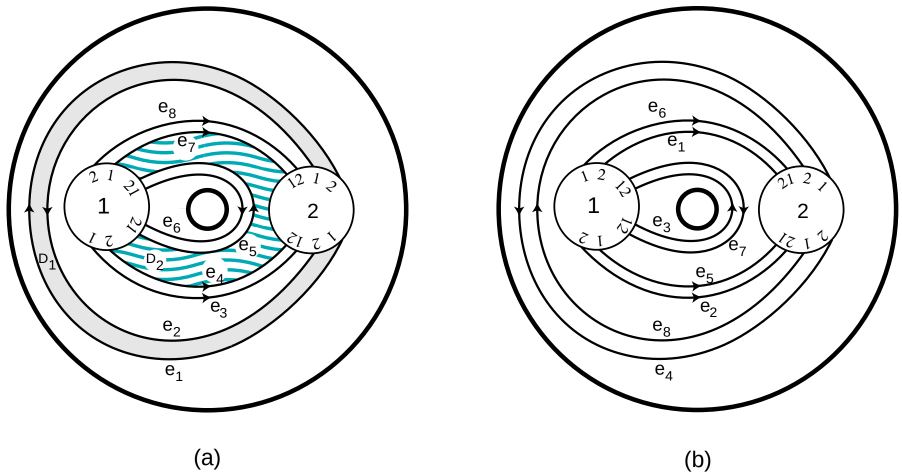

Figure 8.

Label the edges of Γ_1 as in Figure 8(a). There is a correspondence between the edges of Γ_1 and Γ_2. Up to isomorphism we may assume that e_1 on Γ_2 is as shown in Figure 8(b). There are only two non-loop edges on Γ_2 labeled 2. They correspond to loops in Γ_1 based at $B_{1(2)}$, so the other one must be e_2. Now consider the four points of $\partial B_{1(2)} \cap \partial B_{2(1)}$. On $\partial B_{1(2)}$ they appear in the order $e_1^-, e_2^-, e_7^+, e_3^+$, so by Lemma 5 they appear in the same order on $\partial B_{2(1)}$. This determines the edges e_7 and e_3. Similarly one can determine the correspondence between the other edges. The labeling of the edges are shown in Figure 8.

Let D_1, D_2 be the two disk faces of Γ_1 shown in Figure 8(a). Note that they are on different sides of T_2. Since D_1 is an S-disk, and D_2 has an odd number of edges, both ∂D_1 and ∂D_2 are nonseparating curves on $F_{2(1)}$ and $F_{2(2)}$, respectively. Let P be the twice punctured annulus obtained from Q_2 by removing the interiors of the fat vertices. Consider the complex $Y = \partial M \cup P \cup D_1 \cup D_2$. Denote by $N(S)$ the regular neighborhood of a subcomplex S in M. Notice that $N(\partial M \cup P)$ has boundary the torus ∂M and a surface F of genus three. We have shown that $\partial D_1, \partial D_2$ are nonseparating. Since they lie on different sides of P, they are also nonparallel. Thus after adding the two 2-handles $N(D_1)$ and $N(D_2)$ to $N(\partial M \cup P)$, the manifold $N(Y)$ has boundary consisting of two tori.

We would like to calculate the homology of $N(Y)$. Consider the complex $P \cup \partial M$ shown in Figure 9(a). We have determined the graph Γ_2 on P. Up to homeomorphism there is a unique way of connecting the endpoints of e_1, e_2 by arcs on $A_{2(1)}$ to form the loop ∂D_1 and similarly for ∂D_2. See Figure 9(a)–(b). This also determines the boundary slope $m_1/2$ on ∂M, as shown in Figure 9(c). Now pick a basis $\{u, v, x_1, x_2\}$ for $H_1(\partial M \cup P) = \mathbf{Z}^4$ as in Figure 9(d), where u is the loop $\partial B_{2(1)}$, v is the inner boundary circle of Q_2, and x_j runs over the annulus $A_{2(j)}$.

Consider the curves $\partial D_1, \partial D_2$ shown in Figure 9(a)–(b). Calculating their homology classes in $H_1(\partial M \cup P)$, we have

$$\partial D_1 = x_1 + v + x_1 - u = 2x_1 + v - u,$$
$$\partial D_2 = -x_2 + v + x_2 + u + v - x_2 - u = 2v - x_2.$$

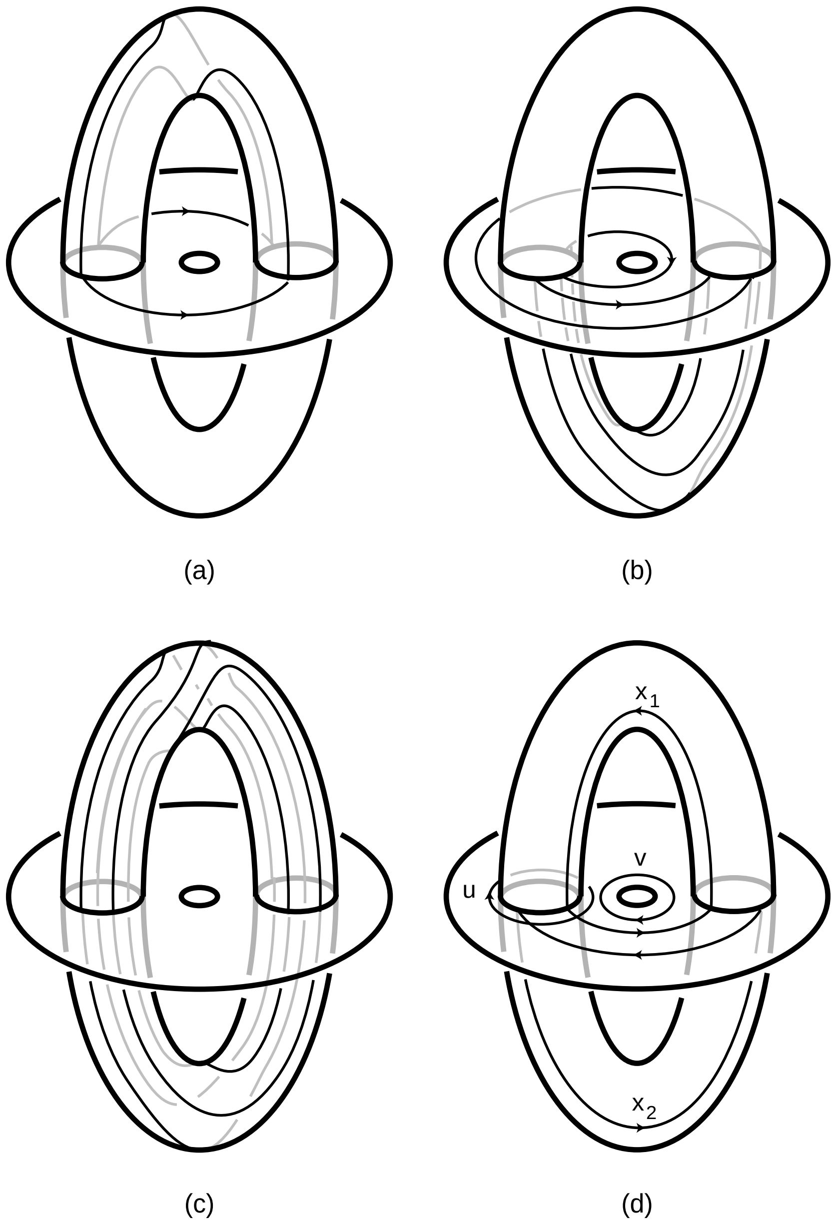

FIGURE 9.

The boundary slope $m_2/2$ is represented by u. From Figure 9(c) we can see that the slope $m_1/2$ is a $(1,-4)$ curve, represented by $u-4(x_1+x_2)$. The meridian slope $r_0 = 1/0$ of ∂M is characterized by the property that it has geometric intersection number 2 with each of $m_1/2$ and $m_2/2$, so it must be a $(1,-2)$ curve, represented by $u-2(x_1+x_2)$. Thus in homology we have

$$r_0 = u - 2(x_1 + x_2).$$

Denote by W the manifold obtained from $N(Y)$ by Dehn filling on ∂M along the meridian slope. That is, $W = N(Y)(1/0) = N(Y)(r_0)$. Then W is obtained from $\partial M \cup P$ by adding two 2-handles along $\partial D_1, \partial D_2$, then adding a solid torus along the slope r_0. So $H_1(W)$ has the presentation

$$\begin{aligned}&\langle u, v, x_1, x_2 \,:\, 2x_1 - u + v = 2v - x_2 = u - 2(x_1 + x_2) = 0\rangle \\ &= \langle x_1, x_2 \,:\, 3x_2 = 0\rangle = \mathbf{Z} \oplus \mathbf{Z}_3.\end{aligned}$$

We have seen that $\partial N(Y)$ consists of two tori; hence $W = N(Y)(1/0)$ has boundary a single torus. On the other hand, W is a submanifold of $M(r_0) = S^3$ with boundary a torus; hence $H_1(W) = \mathbf{Z}$. This contradiction completes the proof of the lemma, hence the proof of Theorem 1. □

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