FQE Problems for 9/15/03

1. Let \( A \) be an \( n \times n \) matrix with complex entries. If \( A^m = 0 \) for some positive integer \( m \), prove that \( A^n = 0 \).

2. Let \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) be the linear transformation represented by the matrix
\[
\begin{bmatrix}
0 & 1 & 2 & -2 \\
1 & 0 & 2 & -3 \\
1 & 1 & 4 & -5
\end{bmatrix}
\]
(a) Find a basis for the kernel (null space) of \( T \)
(b) Find a basis for the range (image) of \( T \).

3. Let \( A \) be a square matrix with distinct eigenvalues \( \lambda_1, \ldots, \lambda_m \) and corresponding eigenvectors \( v_1, \ldots, v_m \). Prove that \( v_1, \ldots, v_m \) are linearly independent.

4. Let \( V \) be an \( (n-1) \)-dimensional subspace of \( \mathbb{R}^n \). Prove that there exists \( x_0 \) in \( \mathbb{R}^n \) such that
\[
V = \{ x \in \mathbb{R}^n \mid x \cdot x_0 = 0 \}
\]

5. Let \( K \) be the vector space of \( n \times n \) skew-symmetric matrices \( (A^t = -A) \) with real coefficients, and let \( S \) be the vector space of \( n \times n \) symmetric matrices \( (A^t = A) \) with real coefficients. Define a linear transformation \( T : K \rightarrow S \) as follows. If the \( (i,j) \)-th entry of \( A \) is \( a_{ij} \), then the \( (i,j) \)-th entry of \( T(A) \) is given by
\[
T(A)_{ij} = \begin{cases} 
  a_{ij} & \text{if } i < j \\
  0 & \text{if } i = j \\
  -a_{ij} & \text{if } i > j 
\end{cases}
\]

i. Is \( T \) one to one? Explain why or why not.

ii. Is \( T \) onto? Explain why or why not.

6. Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation. Show that if \( T \) is onto (surjective), then \( n \geq m \).
7. Prove that a $2 \times 2$ matrix $A$ with positive entries is diagonalizable over $\mathbb{R}$.

8. (08/23/01) Let $\mathcal{P}_{\xi}$ denote the vector space consisting of all real polynomials of the form $p(x) = a_0 + a_1 x + a_2 x^2$. For any two polynomials $p$ and $q$ in $\mathcal{P}_{\xi}$, define a symmetric, bilinear form $\langle p, q \rangle$ by $\langle p, q \rangle = p(0)q(0) + p(\frac{1}{2})q(\frac{1}{2}) + p(1)q(1)$.

a. Show that $\langle p, q \rangle$ is an inner product on $\mathcal{P}_{\xi}$. (Note: it is routine to check that $\langle p, q \rangle$ is symmetric and bilinear. You need not do this part of the argument.)

b. Find an orthonormal basis for the subspace of $\mathcal{P}_{\xi}$ that is spanned by $p(x) = x$ and $q(x) = x^2$.

9. Denote by $P_n$ the vector space of all real polynomials of degree $\leq n$. It is well known that $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n$.

(a) Must every basis for $P_n$ contain a polynomial of degree $n$? Verify your assertion.

(b) Find a basis for $P_n$ which consists entirely of polynomials of degree $n$.

10. Suppose $A$ and $B$ are linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^5$, both of rank 3. Show that there are non-zero vectors $x$ and $y$ such that $Ax = By$.

11. Let $V$ be the vector space of real polynomials of degree $\leq 2$. Define a linear transformation $L : V \to V$ by

$$L(P(x)) = (-3x + x^2)P''(x) + 3P'(x) + P(x) + 3xP(0),$$

where $P(x) = ax^2 + bx + c$.

a) Find the matrix representations of $L$ and $L^{-1}$ with respect to the basis $x^2, x, 1$ in $V$.

b) Find a basis for $V$ consisting of eigenvectors for $L$.

12. Let $V$ be a vector space over a field $F$ and let $T : V \to V$ be a linear transformation which is nilpotent ($T^k = 0$ for some $k > 0$). Prove that $I + T$ is invertible.