FQE Problems for 9/29/03

1. Let $Q$ be the additive group of rational numbers. The additive group $Z$ is a subgroup. Show that $Q/Z$ has infinite order, but each element of $Q/Z$ has finite order.

2. Let $G = \{(g_1, g_2) \in \mathbb{R}^2 \mid g_1 \neq 0\}$ and let $e = (1, 0)$. Then $G$ is a group under the multiplication $(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, g_1 h_2 + g_2)$, and $e$ is the identity of $G$ (you may assume all of this).

a. Let $H$ be the subgroup of $G$ defined by $H = \{(1, h) \mid h \in \mathbb{R}\}$. Show that $H$ is normal in $G$.

b. Show that $G/H$ is isomorphic to the multiplicative group $\mathbb{R}^*$ of non-zero real numbers. Hint: find a homomorphism of $G$ onto $\mathbb{R}^*$ with kernel $H$.

3. Let $n$ be a positive integer, let $G_n = \{(g_1, g_2) \in \mathbb{R}^2 \mid g_1 \neq 0\}$, and let $e = (1, 0)$. Then $G_n$ is a group under the multiplication $(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, g_1 h_2 + g_2 h_1^n)$, and $e$ is the identity of $G$ (you may assume all of this). Prove that for $n > 1$, the center $Z(G_n)$ of $G_n$ is

$$Z(G_n) = \begin{cases} \{(1,0)\} & \text{for } n \text{ even} \\ \{(1,0), (-1,0)\} & \text{for } n \text{ odd}. \end{cases}$$

4. Let $G$ be a group and let $H$ be the subgroup generated by the squares of elements in $G$ (so $h \in H$ if and only if $h$ is of the form $h = g_1^2 g_2^2 \cdots g_k^2$ where $k$ is some positive integer and $g_1, \ldots, g_k$ are in $G$).

(a) Show that $H$ is normal in $G$.

(b) Show that the quotient group $G/H$ is Abelian.

5. How many homomorphisms are there from the group $\mathbb{Z}/\langle 20 \rangle$ to the group $\mathbb{Z}/\langle 8 \rangle$? How many of these homomorphisms are onto?

6. Show that there is no non-trivial automorphism of the field $\mathbb{Q}$ of rational numbers.

7. Let $F$ be a field and let $P(x) = a_0 + \cdots + a_n x^n$ be an irreducible polynomial of degree $n \geq 2$ in $F[x]$. Let $\alpha = x + \langle P(x) \rangle$, where $\langle P(x) \rangle$
denotes the ideal generated by $P(x)$. Express $\alpha^{-1}$ in $F[x]/\langle P(x) \rangle$ in terms of $\alpha^0, \ldots, \alpha^{n-1}$.