FQE Problems for 10/27/03

1. Let

\[ f(x, y) = \begin{cases} 
  x^2 y & \text{for } (x, y) \neq (0, 0) \\
  x^6 + 3y^2 & \text{for } (x, y) = (0, 0)
\end{cases} \]

(a) Use the definition of directional derivative to show that \( f \) has a directional derivative in every direction at \((0, 0)\).

(b) Show that \( f \) is not continuous at \((0, 0)\).

2. Prove or disprove the following:

(a) If \( \sum_{n=1}^{\infty} a_n \) converges absolutely, then so does \( \sum_{n=1}^{\infty} a_n^2 \).

(b) If \( \sum_{n=1}^{\infty} a_n \) converges and \( \lim_{n \to \infty} \frac{b_n}{a_n} = 1 \), then \( \sum_{n=1}^{\infty} b_n \) converges.

3. Does \( \lim_{n \to \infty} x^n(1 - x^n) = 0 \) uniformly on the closed interval \( 0 \leq x \leq 1 \)? Justify your answer.

4. Let \( f(x) = \sqrt{x} \) for \( x \in A = [0, \infty) \).

(a) Prove that \( f \) is uniformly continuous on \( A \).

(b) Does there exist a real constant \( K \) such that

\[ |f(x) - f(y)| \leq K|x - y| \]

for all \( x \) and \( y \) in \( A \)?

5. For each continuously differentiable simple closed curve \( C \) (oriented counter-clockwise) in \( \mathbb{R}^2 \), define \( \tau(C) \) by

\[ \tau(C) = \int_C 2y^3 \, dx + (3x - 2x^3) \, dy. \]

Use Green’s Theorem to find the continuously differentiable simple closed curve \( C \) for which \( \tau(C) \) is maximal.

6. Let \( f: [a, b] \to \mathbb{R} \) be continuous. If \( \int_a^b f(x)g(x) \, dx = 0 \) for all continuous functions \( g \) on \( [a, b] \) such that \( g(a) = g(b) = 0 \), prove that \( f \) is identically 0.
7. Determine the convergence or divergence of the sequence
\[ a_n = \frac{1}{n + 1} + \cdots + \frac{1}{2n} \].

8. Let
\[ F(u, v, x, y) := u^2 + v^3 + x^2 - 3y \]
\[ G(u, v, x, y) := u^2 + v^4 + 3x + y^4. \]
a. Justify the existence of a unique solution \( x = h(u, v), y = k(u, v) \) to the simultaneous equations \( F(u, v, x, y) = 0, G(u, v, x, y) = 0 \) on a neighborhood of the point \((u, v, x, y) := (1, 1, -1, 1)\), where \( h \) and \( k \) are continuously differentiable.
b. Let \( x = h(u, v), y = k(u, v) \) be the solution from part a. Calculate \( \frac{\partial h}{\partial u}(1, 1) \).

9. Let \( f(x) \) be a real-valued function, defined and continuous for each \( x \) in \( \mathbb{R} \). Let \( a \) be a fixed real number. Prove there is a point on the graph of \( f(x) \) whose distance to \((a, 0)\) is a minimum.

10. Let \( f(x) \) be continuous on \([0, 1]\) and let \( n \geq 2 \) be a fixed integer. If \( f(0) = f(1) \), prove that there exists a number \( c \) in \([0, 1]\) such that \( f(c) = f(c + \frac{1}{n}) \). Hint: consider the function \( g(x) := f(x + \frac{1}{n}) - f(x) \).

11. Let \( f: [a, b] \rightarrow \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\). Prove that if \( \lim_{x \to a^+} f'(x) = A \), then the (one-sided) derivative \( f'(a) \) exists and \( f'(a) = A \).

12. Assume \( f \) is a real-valued continuous function on the rectangle \( \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\} \). Let
\[ F(x) = \int_c^d f(x, y)dy. \]
Prove that \( F \) is a continuous function on \([a, b] \).