$\mathbf{K_1}$ of twisted rings of polynomials

Bernard Badzioch

University of Notre Dame

Department of Mathematics

Notre Dame, IN 46556, USA

Warsaw University

Department of Mathematics

ul. Banacha 2

Warsaw, Poland

Abstract. We prove that for an arbitrary endomorphism α of a ring R the group $K_1(R_\alpha[t])$ splits into the direct sum of $K_1(R)$ and $\widetilde{Nil}(R;\alpha)$. Moreover, for any such R and α $\widetilde{Nil}(R;\alpha)$ is isomorphic to $\widetilde{Nil}(R';\alpha')$ for some ring R' with $\alpha'\colon R'\to R'$ an isomorphism.

Introduction. Let R be a ring with a unit $1\in R$ and $\alpha\colon R\to R$ an endomorphism preserving the unit. We define $R_\alpha[t]$, the α - twisted ring of polynomials with coefficients in R, so that additively $R_\alpha[t]=R[t]$ and the multiplication is given by the formula: $(rt^i)(st^j)=r\alpha^i(s)t^{i+j}$ for $r,s\in R$. Investigating the group $K_1(R_\alpha[t])$ H. Bass (in the case $\alpha=id_R$) and F.T. Farrell and W.C. Hsiang (in the case α – an automorphism of R) have shown that it splits into the direct sum of $K_1(R)$ and $\widetilde{Nil}(R;\alpha)$. The definition of the last group is recalled below. The aim of this paper is to generalize those results. We will prove:

Theorem. *For any endomorphism* $\alpha\colon R\to R$, $K_1(R_\alpha[t])\simeq K_1(R)\oplus\widetilde{Nil}(R;\alpha)$.

Moreover, from the arguments used in the proof we will get the following

Corollary. *If α is an arbitrary endomorphism of R then there exists a ring R', an automorphism $\alpha'\colon R' \to R'$ and a ring homomorphism $\iota\colon R \to R'$ such that $\iota_*\colon \widetilde{Nil}(R;\alpha) \to \widetilde{Nil}(R';\alpha')$ is an isomorphism.*

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All the rings considered in this paper are assumed to be rings with a unit, homomorphisms of rings preserve the unit. By a R-module we will mean a right R-module.

Let M, N be R - modules and $\alpha\colon R \to R$ an endomorphism. We will call a map $\varphi\colon M \to N$ α - linear if φ is additive and $\varphi(mr) = \varphi(m)\alpha(r)$ for all $m \in M$, $r \in R$. In the case when M and N are free, finitely generated modules with fixed bases, the map φ can be represented by a matrix with entries in R (see [F-H]).

We define the group $\widetilde{Nil}(R;\alpha)$ as follows. Let $\widetilde{\mathcal{N}il}(R;\alpha)$ be a category in which objects are pairs (F,φ), where F is a free, finitely generated R - module and $\varphi\colon F \to F$ is an α - linear, nilpotent endomorphism. A morphism $f\colon (F,\varphi) \to (F',\varphi')$ is a R - linear homomorphism $f\colon F \to F'$ satisfying $\varphi' f = f\varphi$.

Now, $\widetilde{Nil}(R;\alpha)$ is an abelian group generated by the isomorphism classes of objects of $\widetilde{\mathcal{N}il}(R;\alpha)$ and by relations:

(A) $[F,\varphi] = [F',\varphi'] + [F'',\varphi'']$ for

$$0 \to (F',\varphi') \longrightarrow (F,\varphi) \longrightarrow (F'',\varphi'') \to 0$$

a short exact sequence in $\widetilde{\mathcal{N}il}(R;\alpha)$;

(O) $[F,0] = 0 \in \widetilde{Nil}(R;\alpha)$.

We can consider $\widetilde{\mathrm{Nil}}$ as a functor from the category of pairs (R,α), where R and α are as above, and ring homomorphisms $f\colon R \to R'$ satisfying $f\alpha = \alpha' f$ to the category of abelian groups.

Let $\varepsilon\colon R_\alpha[t] \longrightarrow R$ be the evaluation homomorphism, $\varepsilon(w(t)) = w(0)$ and let $i\colon R \hookrightarrow R_\alpha[t]$ be the inclusion. We have the induced homomorphisms ε_*, i_* of groups K_1, and since $\varepsilon i = \mathrm{id}_R$ we get $\varepsilon_* i_* = \mathrm{id}_{K_1(R)}$. Let $[R^n, \varphi] \in \widetilde{Nil}(R;\alpha)$ and let M_φ represent φ in the coordinates of the standard basis of R^n. As in [B] we define a homomorphism $k\colon \widetilde{Nil}(R;\alpha) \longrightarrow K_1(R_\alpha[t])$ by $k[R^n, \varphi] = [I - M_\varphi t]$, where I is the identity matrix. Furthermore, using the same arguments as in [B] for $\alpha = \mathrm{id}_R$ one can check that the sequence

$$\widetilde{Nil}(R;\alpha) \xrightarrow{k} K_1(R_\alpha[t]) \xrightarrow{\varepsilon_*} K_1(R) \to 0$$

is exact. Therefore showing that k is a monomorphism is enough to prove that $K_1(R_\alpha[t])$ splits. Our objective will be to reduce the general situation when α is an arbitrary endomorphism, to the case when it is an automorphism and when (by [F-H]) k is known to be injective.

Lemma 1. *Let $\iota\colon (R,\alpha) \longrightarrow (R', \alpha')$ satisfy*

(i) $\ker\,\iota \subseteq \bigcup_i \ker\,\alpha^i$

(ii) $\forall_{r' \in R'} \exists_{j \geq 0}\ (\alpha')^j(r') \in \mathrm{im}\ \iota$.

Then $\iota_*\colon \widetilde{Nil}(R;\alpha) \longrightarrow \widetilde{Nil}(R';\alpha')$ is an isomorphism.

Let $\alpha\colon R \to R$ be an endomorphism and let $R' = \varinjlim\,(R \xrightarrow{\alpha} R \xrightarrow{\alpha} R \xrightarrow{\alpha} \cdots)$ with $\alpha'\colon R' \longrightarrow R'$ induced by α. It is easy to check that the homomorphism $\iota\colon (R,\alpha) \longrightarrow (R',\alpha')$ satisfies the conditions of lemma 1. Since α' is an isomorphism we get the corollary and the theorem (modulo the proof of lemma 1) follows from the commutativity of the diagram:

$$\begin{array}{ccc}
\widetilde{Nil}(R;\alpha) & \xrightarrow{k} & K_1(R_\alpha[t]) \\
\simeq \big\downarrow \iota_* & & \big\downarrow \bar{\iota}_* \\
\widetilde{Nil}(R';\alpha') & \xrightarrow[1-1]{k} & K_1(R'_{\alpha'}[t])
\end{array}$$

where $\bar{\iota}\colon R_\alpha[t] \longrightarrow R'_{\alpha'}[t]$ is a prolongation of ι defined by $\bar{\iota}(t) = t$.

To prove lemma 1 we will need the following fact, the proof of which will be postponed until the end of this paper.

Lemma 2. *The homomorphism $\alpha_*\colon \widetilde{Nil}(R;\alpha) \longrightarrow \widetilde{Nil}(R;\alpha)$ is the identity of $\widetilde{Nil}(R;\alpha)$ for any* *endomorphism* $\alpha\colon R \to R$.

(In the case when α is an automorphism of R this fact has been proven in [F-H]).

Proof of lemma 1. Let us notice that ι_* can be described as follows: if $[R^n, \varphi] \in \widetilde{Nil}(R;\alpha)$ and $M_\varphi = (r_{ij})_{ij}$ is the matrix of φ with respect to the standard basis of R^n, then $\iota_*([R^n, \varphi]) = [(R')^n, \iota(\varphi)]$ where in the coordinates of the standard basis of $(R')^n$ $\iota(\varphi)$ is represented by the matrix $\iota(M_\varphi) = (\iota(r_{ij}))_{ij}$.

We will define a map:

$$q\colon \{\textit{isomorphism classes of } \mathrm{Ob}\widetilde{\mathcal{N}il}(R;\alpha)\} \longrightarrow \widetilde{Nil}(R;\alpha)$$

Let $((R')^n, \varphi') \in \mathrm{Ob}\widetilde{\mathcal{N}il}(R';\alpha')$ and $M_{\varphi'} = ({r'}_{ij})_{ij}$ be the matrix representing φ' with respect to the standard basis of $(R')^n$. Let $k = \min\{l \geq 0 | \forall_{i,j}\ {\alpha'}^l({r'}_{ij}) \in \mathrm{im}\ \iota\}$ (such k exists by (ii)). Let us choose $s_{ij} \in \iota^{-1}({\alpha'}^k({r_{ij}}'))$. We define $q((R')^n, \varphi') = [R^n, q(\varphi')]$, where $q(\varphi')$ is the α - linear homomorphism represented (with respect to the standard basis of R^n) by the matrix $M_{q(\varphi')} = (s_{ij})_{ij}$. It is easy to check that $q((R')^n, \varphi') \in \widetilde{Nil}(R;\alpha)$, e.i. that $q(\varphi')$ is nilpotent. Furthermore, $q((R')^n, \varphi')$ does not depend on the choice of s_{ij} from $\iota^{-1}({\alpha'}^k({r'}_{ij}))$. Indeed, let ${s'}_{ij} \in \iota^{-1}({\alpha'}^k({r'}_{ij}))$, and let $\varphi\colon R^n \longrightarrow R^n$ be the α - linear homomorphism represented by the matrix $M = ({s'}_{ij})_{ij}$. We have ${s'}_{ij} = s_{ij} + c_{ij}$ for some $c_{ij} \in \ker\ \iota \subseteq \bigcup_i \ker\ \alpha^i$. Therefore, for some $r \geq 0$, $\alpha^r(M_{q(\varphi')}) = \alpha^r(M)$. It follows that ${\alpha_*}^r[R^n, \varphi] = {\alpha_*}^r[R^n, q(\varphi')]$, and applying lemma 2 we get $[R^n, \varphi] = [R^n, q(\varphi')]$. By a similar application of lemma 2 one can show that $q((R')^n, \varphi')$ does not depend on the choice of the element from the class of isomorphism of $\mathrm{Ob}\widetilde{\mathcal{N}il}(R';\alpha')$. Thus q is a well defined map. Moreover, since $q((R')^n, 0) = [R^n, 0]$, and

$$q((R')^n, \begin{pmatrix} \varphi' & \star \\ 0 & \varphi'' \end{pmatrix}) = [R^n, \begin{pmatrix} \alpha^r q(\varphi') & \star \\ 0 & \alpha^s q(\varphi'') \end{pmatrix}]$$

for some $r, s \geq 0$, q factorizes to a homomorphism $q_*\colon \widetilde{Nil}(R';\alpha') \longrightarrow \widetilde{Nil}(R;\alpha)$. It is easy to check that $q_*\iota_* = \mathrm{id}_{\widetilde{Nil}(R;\alpha)}$, $\iota_* q_* = \mathrm{id}_{\widetilde{Nil}(R';\alpha')}$, so ι_* is an isomorphism. This completes the proof of lemma 1.

Proof of lemma 2. Let $[R^n, \varphi] \in \widetilde{Nil}(R;\alpha)$, and let $M_\varphi = (a_{ij})_{ij}$ be the matrix of φ with respect to the standard coordinates in R^n. We want to show that $[R^n, \varphi] = [R^n, \alpha(\varphi)]$, where $M_{\alpha(\varphi)} =$

$(\alpha(a_{ij}))_{ij}$. Let $b_{ij} = \alpha(a_{ij})$. We have a short exact sequence in $\widetilde{Nil}(R;\alpha)$:

$$0 \longrightarrow (R^{n^3}, 0) \xrightarrow{f} (R^{n^3+n}, \varphi_1') \xrightarrow{g} (R^n, \varphi) \longrightarrow 0$$

where

$$f((r_{ijk})_{1\le i,j,k\le n}) = ((r_{ijk})_{1\le i,j,k\le n}, -\sum_{ij} a_{ij} r_{1ji}, \sum_{ij} a_{ij} r_{2ji},\ \ldots,\ -\sum a_{ij} r_{nji})$$

$$g((r_{ijk})_{1\le i,j,k\le n},\ r_1,\ r_2,\ \ldots,\ r_n) = (r_1 + \sum_{ij} a_{ij} r_{1ji},\ r_2 + \sum_{ij} a_{ij} r_{2ji},\ \ldots,\ r_n + \sum a_{ij} r_{nji})$$

$(r_{ijk})_{1\le i,j,k\le n} = (r_{111},\ r_{112},\ \ldots,\ r_{11n},\ r_{121},\ \ldots,\ r_{1nn},\ r_{211},\ \ldots,\ r_{nnn})$, and φ_1' is the α - linear map represented in the standard coordinates ofR^{n^3+n} by the matrix:

$$M_{\varphi'_1} = \begin{pmatrix} B_1 & 0 & 0 & 0 & \ldots & 0 & e_{11} \\ 0 & B_1 & 0 & 0 & \ldots & 0 & e_{12} \\ 0 & 0 & B_1 & 0 & \ldots & 0 & e_{13} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & B_1 & e_{1n} \\ B_2 & 0 & 0 & 0 & \ldots & 0 & e_{21} \\ 0 & B_2 & 0 & 0 & \ldots & 0 & e_{22} \\ \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \\ 0 & 0 & 0 & 0 & \ldots & B_n & e_{nn} \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

Here $B_i \in M_{n,n^2}(R)$ has the i-th row of the form $(b_{11},\ b_{21},\ \ldots,\ b_{n1},\ b_{12},\ \ldots,\ b_{nn})$ and all other entries equal 0, and $e_{ij} \in M_{n,n}(R)$ has only one non-zero ij-th entry equal 1. Thus in $\widetilde{Nil}(R;\alpha)$ we get

$$[R^n, \varphi] = [R^{n^3+n}, {\varphi_1}'] - [R^{n^3}, 0] = [R^{n^3+n}, {\varphi_1}'] = [R^{n^3}, \varphi_1]$$

where the matrix M_{φ_1} representing φ_1 is obtained by deleting the last n rows and columns of $M_{{\varphi_1}'}$.

Let us notice that M_{φ_1} has (n − 1)n^2 rows with zero entries only. Permuting the elements of the standard basis e_1, e_2,..., e_{n^3} of R^{n^3} so that the elements whose indices correspond to the indices of the zero rows come last, and the relative order of other elements is unchanged, we get a basis with respect to which φ_1 is represented by the matrix:

$$
N_{\varphi_1} = \begin{pmatrix}
C_1 & 0 & \dots & 0 & \star \\
0 & C_2 & \dots & 0 & \star \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & C_n & \star \\
C_1 & 0 & \dots & 0 & \star \\
0 & C_2 & \dots & 0 & \star \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & C_n & \star \\
 & & & & \\
\dots & \dots & \dots & \dots & \dots \\
 & & & & \\
C_1 & 0 & \dots & 0 & \star \\
0 & C_2 & \dots & 0 & \star \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & C_n & \star \\
0 & 0 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & 0
\end{pmatrix}
$$

where $C_k \in M_{1,n}(R) = (b_{k1},\ b_{k2}, \dots,\ b_{kn})$ is the k-th row of $M_{\alpha(\varphi)}$, and each C_k repeats n times in N_{φ_1}. Let $\varphi_2 \colon R^{n^3} \longrightarrow R^{n^3}$ be the α- linear homomorphism such that $M_{\varphi_2} = N_{\varphi_1}$. Since φ_2 is obtained by conjugation of φ_1 with an R- linear automorphism of R^{n^3}, we have $[R^{n^3}, \varphi_1] = [R^{n^3}, \varphi_2]$, and $[R^{n^3}, \varphi_2] = [R^{n^2}, \varphi_3]$, where we get M_{φ_3} by deleting the last $(\mathrm{n}-1)\mathrm{n}^2$ rows and columns of M_{φ_2}. The matrix M_{φ_3} has n different rows, each one repeating n times. Conjugating φ_3 with a suitable R-linear automorphism of R^{n^2} we can get an α - linear homomorphism $\psi \colon R^{n^2} \longrightarrow R^{n^2}$, the matrix M_ψ of which will have only n non-zero rows. In particular, if the conjugating automorphism will assign to the element e_i of the standard basis of R^{n^2} the vector $e'_i = \sum_{k=0}^{(n-1)} e_{nk+i}$ for $i \leq n$ and $e'_i = e_i$ for $i > n$ we will get:

$$
M_\psi = \begin{pmatrix} M_{\alpha(\varphi)} & \star \\ 0 & 0 \end{pmatrix}
$$

So finally: $[R^{n^2}, \varphi_3] = [R^{n^2}, \psi] = [R^n, \alpha(\varphi)]$.

References.

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