

THE CANONICAL DELOOPING MACHINE

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ABSTRACT. We suggest a new delooping machine, which is based on recognizing an n -fold loop space by a collection of operations acting on it, like the traditional delooping machines of James, Stasheff, May, Boardman-Vogt, Segal, and Bousfield. Unlike the traditional delooping machines, which carefully select a nice space of such operations, we consider all natural operations on n -fold loop spaces, resulting in the algebraic theory $\text{Map}_*(\bigvee_{\bullet} S^n, \bigvee_{\bullet} S^n)$. The advantage of this new approach is that the delooping machine is universal in a certain sense, the proof of the recognition principle is more conceptual, works the same way for all values of n , and does not need the test space to be connected.

1. INTRODUCTION

The goal of this paper is to give a proof of the following characterization of n -fold loop spaces. In the category Spaces_* of pointed spaces, consider the full subcategory generated by the wedges $\bigvee_k S^n$ of n -dimensional spheres for $k \geq 0$ (where $\bigvee_0 S^n = *$). Let \mathcal{T}^{S^n} denote the opposite category, see Figure 1. Since $\bigvee_k S^n$ is a k -fold coproduct of S^n 's in Spaces_* , in \mathcal{T}^{S^n} it is a k -fold categorical product of S^n 's.

Theorem 1.1. *A space $Y \in \text{Spaces}_*$ is weakly equivalent to an n -fold loop space, iff there exists a product preserving functor $\tilde{Y}: \mathcal{T}^{S^n} \rightarrow \text{Spaces}_*$ such that $\tilde{Y}(S^n)$ is weakly equivalent to Y .*

The category \mathcal{T}^{S^n} is in fact an *algebraic theory* (see 2.1). From this point of view, one can regard the above theorem as a *recognition principle*: a loop space structure is detected by the structure of an *algebra over the algebraic theory \mathcal{T}^{S^n}* .

We will actually prove a stronger version (see Theorem 4.8) of Theorem 1.1: given a product preserving functor $\tilde{Y}: \mathcal{T}^{S^n} \rightarrow \text{Spaces}_*$, one can construct a space $B_n \tilde{Y}_c$ such that $\Omega^n B_n \tilde{Y}_c \simeq \tilde{Y}(S^n)$, thereby *delooping* the space $\tilde{Y}(S^n)$.

This description of iterated loop spaces is in some sense an extreme delooping machine. By Yoneda's lemma the theory \mathcal{T}^{S^n} encodes all natural maps $(\Omega^n X)^k \rightarrow (\Omega^n X)^l$, and we use all this structure in order to detect loop spaces. This stands in contrast to the approach of James [13], Stasheff [21], May [16], Boardman-Vogt [4], Segal [20], Bousfield [5], or Kriz [14], where only carefully chosen sets of maps between loop spaces are used for the same purpose. Our indiscriminate method

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FIGURE 1. A morphism $\bigvee_3 S^n \rightarrow \bigvee_4 S^n$ in category \mathcal{T}^{S^n} .

however brings some advantages. First of all, as in [5], Theorem 1.1 is true for all, not necessarily connected, loop spaces. Kriz's machine [14] does not require connectivity, either, but deloops only infinite loop spaces. Beck's machine [3] works for all loop spaces, but detects loop spaces by an action of a monad $\Omega^n \Sigma^n$ rather than a space of operations, so that the recognition principle becomes almost tautological and arguably less practical. Also, since we avoid making particular choices of operations on loop spaces, our delooping machine provides a convenient ground for proving uniqueness theorems of the kind of May and Thomason [17], [22]. Namely, given an operad, a PROP, or a semi-theory (i.e., a machine of the type of Segal's Γ -spaces, see [2]), one can replace it by an algebraic theory describing the same structure on spaces. On the other hand, it is relatively easy to compare homotopy theories of objects described by various algebraic theories. This implies Theorem 4.10 – a uniqueness result for "delooping theories".

Most of the arguments and constructions we use are formal and do not depend on any special properties of loop spaces. Indeed, at least one implication of the statement of Theorem 1.1 holds when we replace S^n with an arbitrary pointed space A . If \mathcal{T}^A is an algebraic theory constructed analogously to \mathcal{T}^{S^n} above, then for any mapping space $Y = \text{Map}_*(A, X)$, we can define a product preserving functor $\tilde{Y}: \mathcal{T}^A \rightarrow \text{Spaces}_*$ such that $\tilde{Y}(A) = Y$. It is not true that for an arbitrary A also the opposite statement will hold, i.e. that any such functor will come from some mapping space. A counterexample (following an idea of A. Przeździecki) can be obtained as follows. Assume that for some space A every \mathcal{T}^A -algebra can be identified with a mapping space $\text{Map}_*(A, X)$ for some X . As a consequence of [1, Corollary 1.4] we get that if F is a functor from the category of pointed spaces to itself, such that F preserves weak equivalences and preserves products up to a weak equivalence, then for any mapping space $\text{Map}_*(A, X)$ the space $F(\text{Map}_*(A, X))$ must be weakly equivalent to some mapping space $\text{Map}_*(A, X')$. Take $A = S^2 \vee S^3$, $X = K(\mathbb{Z}, 3)$, the Eilenberg-Mac Lane space, and let F be the functor picking the connected component of the basepoint. We have $\text{Map}_*(A, X) = \mathbb{Z} \times K(\mathbb{Z}, 1)$, and so $F(\text{Map}_*(A, X)) = K(\mathbb{Z}, 1)$. Since $K(\mathbb{Z}, 1)$ does not decompose non-trivially into a product of spaces, it follows that it is not of the form $\text{Map}_*(A, X')$ for any space X' . In other words, we can put a \mathcal{T}^A -algebra structure on $K(\mathbb{Z}, 1)$ which does not come from any mapping space.

It should be true that if for a given space A , the mapping spaces from A can be described as algebras over some operad, PROP, semi-theory, algebraic theory, or using some other formalism employing only finitary operations on a space, then they must be characterized by means of the theory \mathcal{T}^A . Therefore the example described above shows that that for $A = S^2 \vee S^3$ none of these formalisms will work.

Another advantage of the proposed recognition principle is that the argument seems to be more conceptual than in the previously known cases. For example, we get an analogue (Corollary 4.9) of May's approximation theorem [16] as a simple consequence of, rather than a hard step towards, the recognition principle.

This simplicity comes, no wonder, with a price tag attached: the theory \mathcal{T}^{S^n} is more cumbersome than the other devices used in delooping, such as the little n -disks operad. For example, while the homology of the little n -disks operad has a neat description as the operad describing n -algebras, see F. Cohen [8, 9], even

the rational homology of the corresponding PROP $\text{Map}_*(\bigvee_l S^n, \bigvee_k S^n)$ is harder to come by, see the thesis [7] of the second author.

The theory \mathcal{T}^{S^n} bears resemblance to the cacti operad, see [10], which consists of (unpointed) continuous maps from a sphere S^n to a tree-like joint of spheres S^n at finitely many points. This operad was invented as a bookkeeping device for operations on free sphere spaces arising in string topology, see [6].

Also, the operadic part $\mathcal{O}_n := \text{Map}_*(S^n, \bigvee S^n)$ of \mathcal{T}^{S^n} has been described as a "universal operad of n -fold loop spaces" by P. Salvatore in [18]. As was also noted by Salvatore, while the space underlying a connected algebra over this operad is weakly equivalent to an n -fold loop space, in general a loop space will admit several actions of \mathcal{O}_n . Therefore, connected \mathcal{O}_n -algebras can be seen as loop spaces equipped with some extra structure.

Notation 1.2.

- Let Spaces_* denote the category of pointed compactly generated (but not necessarily Hausdorff) topological spaces. From the perspective of homotopy theory, there is no difference between this category and the category of all pointed topological spaces. The category Spaces_* has a model category structure with the usual notions of weak equivalences, fibrations and cofibrations, and it is Quillen equivalent to the category of pointed topological spaces, see [12]. The assumption that all spaces are compactly generated has the advantage that for any space X , the smash product functor $Z \mapsto Z \wedge X$ is left adjoint to the mapping space functor $Z \mapsto \text{Map}_*(X, Z)$. This has some further useful consequences which we will invoke.
- If X is an unpointed space by X_+ we will denote the space X with an adjoined basepoint.
- All functors are assumed to be covariant.
- If \mathcal{C} is a category, then \mathcal{C}^{op} will denote the opposite category of \mathcal{C} .

2. ALGEBRAIC THEORIES AND THEIR ALGEBRAS

Definition 2.1. An *algebraic theory* \mathcal{T} is a small category with objects T_0, T_1, \dots together with, for each n , a choice of morphisms $p_1^n, \dots, p_n^n \in \text{Mor}_{\mathcal{T}}(T_n, T_1)$ such that for any k, n the map

$$\prod_{i=1}^n (p_i^n)_* : \text{Mor}_{\mathcal{T}}(T_k, T_n) \rightarrow \prod_{i=1}^n \text{Mor}_{\mathcal{T}}(T_k, T_1)$$

is an isomorphism. In other words, the object T_n is an n -fold categorical product of T_1 's, and p_i^n 's are the projection maps. In particular T_0 is the terminal object in \mathcal{T} . We will also assume that it is an initial object. A morphism of algebraic theories is a functor $\mathcal{T} \rightarrow \mathcal{T}'$ preserving the projection maps. We will consider algebraic theories enriched over Spaces_* ; in particular, the sets of morphisms will be provided with a pointed topological space structure. We will also regard Spaces_* as a category enriched over itself. Accordingly, all functors between categories enriched over Spaces_* will be assumed continuous and basepoint preserving.

Given an algebraic theory \mathcal{T} , a \mathcal{T} -*algebra* \tilde{Y} is a product preserving functor $\tilde{Y} : \mathcal{T} \rightarrow \text{Spaces}_*$. A morphism of \mathcal{T} -algebras is a natural transformation of functors. A *left \mathcal{T} -module* is any functor $\mathcal{T} \rightarrow \text{Spaces}_*$. A *right \mathcal{T} -module* is a functor $\mathcal{T}^{\text{op}} \rightarrow \text{Spaces}_*$.

We will say that a space Y admits a \mathcal{T} -algebra structure, if there is a \mathcal{T} -algebra \tilde{Y} and a homeomorphism $\tilde{Y}(T_1) \cong Y$.

For an algebraic theory \mathcal{T} , by $\mathcal{Alg}_{\mathcal{T}}$ we will denote the category of all \mathcal{T} -algebras and their morphisms.

Example 2.2. For any pointed space $A \in \text{Spaces}_*$ we can define an algebraic theory \mathcal{T}_A enriched over Spaces_* by setting

$$\text{Mor}_{\mathcal{T}_A}(T_m, T_n) := \text{Map}_*(A^m, A^n).$$

Thus, \mathcal{T}_A is isomorphic to the full subcategory of Spaces_* generated by the spaces A^n for $n \geq 0$.

For any $X \in \text{Spaces}_*$, we can consider a product preserving functor

$$\mathcal{T}_A \rightarrow \text{Spaces}_*, \quad T_n \mapsto \text{Map}_*(X, A^n).$$

This shows that any mapping space $\text{Map}_*(X, A)$ has a canonical structure of a \mathcal{T}_A -algebra.

Example 2.3. Let A be again a pointed space, and let \mathcal{T}^A be a category with objects T_0, T_1, \dots and morphisms

$$\text{Mor}_{\mathcal{T}^A}(T_m, T_n) = \text{Map}_*\left(\bigvee_n A, \bigvee_m A\right).$$

In other words, \mathcal{T}^A is isomorphic to the opposite of the full subcategory of Spaces_* generated by the finite wedges of A . Since $\bigvee_n A$ is an n -fold coproduct of A in Spaces_* , T_n is an n -fold categorical product of T_1 's in \mathcal{T}^A . It follows that \mathcal{T}^A is an algebraic theory. For $X \in \text{Spaces}_*$, we can define a functor

$$\mathcal{T}^A \rightarrow \text{Spaces}_*, \quad T_n \mapsto \text{Map}_*\left(\bigvee_n A, X\right).$$

Therefore the mapping space $\text{Map}_*(A, X)$ has a canonical structure of a \mathcal{T}^A -algebra. In particular, if $A = S^n$ we get that any n -fold loop space canonically defines an algebra over \mathcal{T}^{S^n} .

2.4. A special instance of an algebraic theory \mathcal{T}^A is obtained when we take $A = S^0$. The category \mathcal{T}^{S^0} is equivalent to the opposite of the category of finite pointed sets. One can check that the forgetful functor

$$U_{\mathcal{T}^{S^0}} : \mathcal{Alg}^{\mathcal{T}^{S^0}} \rightarrow \text{Spaces}_*, \quad U_{\mathcal{T}^{S^0}}(\tilde{Y}) = \tilde{Y}(T_1),$$

gives an isomorphism of categories. Also, for any algebraic theory \mathcal{T} there is a unique map of algebraic theories $I_{\mathcal{T}} : \mathcal{T}^{S^0} \rightarrow \mathcal{T}$. If $U_{\mathcal{T}} : \mathcal{Alg}^{\mathcal{T}} \rightarrow \text{Spaces}_*$ is the forgetful functor, $U_{\mathcal{T}}(\tilde{Y}) = \tilde{Y}(T_1)$, then we have $U_{\mathcal{T}} = U_{\mathcal{T}^{S^0}} \circ I_{\mathcal{T}}^*$ where $I_{\mathcal{T}}^* : \mathcal{Alg}^{\mathcal{T}} \rightarrow \mathcal{Alg}^{\mathcal{T}^{S^0}}$ is the functor induced by $I_{\mathcal{T}}$.

3. TENSOR PRODUCT OF FUNCTORS

The following general construction will be used in the case when \mathcal{C} is an algebraic theory, F a right \mathcal{C} -module, and G a \mathcal{C} -algebra.

Definition 3.1. Let \mathcal{C} be a small topological category, i.e., a small category enriched over Spaces_* , and $F \in \text{Spaces}_*^{\mathcal{C}^{\text{op}}}$, $G \in \text{Spaces}_*^{\mathcal{C}}$. The tensor product $F \otimes_{\mathcal{C}} G$ is the coequalizer

$$F \otimes_{\mathcal{C}} G := \text{colim} \bigvee_{(c,d) \in \mathcal{C} \times \mathcal{C}} F(c) \wedge \text{Mor}(d, c) \wedge G(d) \begin{array}{c} \xrightarrow{j_1} \\ \xrightarrow{j_2} \end{array} \bigvee_{c \in \mathcal{C}} F(c) \wedge G(c) .$$

The map j_1 is the wedge of the maps $\text{ev} \wedge \text{id}: (F(c) \wedge \text{Mor}(d, c)) \wedge G(d) \rightarrow F(d) \wedge G(d)$, where ev is the evaluation map, and j_2 is similarly induced by the evaluation maps $\text{ev}: \text{Mor}(d, c) \wedge G(d) \rightarrow G(c)$.

The most important – from our perspective – property of the tensor product is given by the following

Proposition 3.2. *Let \mathcal{C} be a small topological category and $F \in \text{Spaces}_*^{\mathcal{C}^{\text{op}}}$. Consider the functor*

$$\text{Map}_*(F, -): \text{Spaces}_* \rightarrow \text{Spaces}_*^{\mathcal{C}}, \quad X \mapsto \text{Map}_*(F, X).$$

The left adjoint of $\text{Map}_(F, -)$ exists and is given by*

$$F \otimes_{\mathcal{C}} -: \text{Spaces}_*^{\mathcal{C}} \rightarrow \text{Spaces}_*, \quad G \mapsto F \otimes_{\mathcal{C}} G.$$

For a proof see, e.g., [15].

3.3. Assume now that we have two small categories \mathcal{C} and \mathcal{D} enriched over Spaces_* and two functors $F: \mathcal{C}^{\text{op}} \rightarrow \text{Spaces}_*$ and $G: \mathcal{C} \times \mathcal{D}^{\text{op}} \rightarrow \text{Spaces}_*$. For every $d \in \mathcal{D}$, the functor G defines $G(d): \mathcal{C} \rightarrow \text{Spaces}_*$ by $G(d)(c) = G(c, d)$. Applying the tensor product construction, we obtain a new functor $F \otimes_{\mathcal{C}} G: \mathcal{D}^{\text{op}} \rightarrow \text{Spaces}_*$ such that $(F \otimes_{\mathcal{C}} G)(d) = F \otimes_{\mathcal{C}} G(d)$. Since smash product in Spaces_* commutes with colimits, for any $H: \mathcal{D} \rightarrow \text{Spaces}_*$, we have a natural isomorphism

$$(F \otimes_{\mathcal{C}} G) \otimes_{\mathcal{D}} H \cong F \otimes_{\mathcal{C}} (G \otimes_{\mathcal{D}} H) \in \text{Spaces}_* .$$

3.4. Our main interest lies in the following instances of these constructions:

1) For $A \in \text{Spaces}_*$, let \mathcal{T}^A be the algebraic theory defined in Example 2.3. Consider the functor

$$\Omega^A: \text{Spaces}_* \rightarrow \text{Spaces}_*^{\mathcal{T}^A}$$

given by $\Omega^A(X)(T_k) := \text{Map}_*(\bigvee_k A, X)$. By Proposition 3.2, Ω^A has a left adjoint

$$B_A: \text{Spaces}_*^{\mathcal{T}^A} \rightarrow \text{Spaces}_*,$$

given by $B_A(Y) = \bigvee_{\bullet} A \otimes_{\mathcal{T}^A} Y$. Here $\bigvee_{\bullet} A$ denotes the right \mathcal{T}^A -module defined as the functor from $(\mathcal{T}^A)^{\text{op}}$ to Spaces_* such that $\bigvee_{\bullet} A(T_k) = \bigvee_k A$. Note that $\Omega^A(X)$ preserves products, and so Ω^A takes values in the full subcategory $\text{Alg}^{\mathcal{T}^A} \subset \text{Spaces}_*^{\mathcal{T}^A}$. Thus, we get an adjoint pair (B_A, Ω^A) of functors between $\text{Alg}^{\mathcal{T}^A}$ and Spaces_* .

2) For $A \in \text{Spaces}_*$, consider

$$\text{End}_A(T_k, T_l) := \text{Mor}_{\mathcal{T}^A}(T_k, T_l) = \text{Map}_*\left(\bigvee_l A, \bigvee_k A\right)$$

as a functor $(\mathcal{T}^A)^{\text{op}} \times \mathcal{T}^A \rightarrow \text{Spaces}_*$. Using the canonical map $I_{\mathcal{T}^A}: \mathcal{T}^{S^0} \rightarrow \mathcal{T}^A$ (see 2.4), we can view End_A as a functor on the category $(\mathcal{T}^{S^0})^{\text{op}} \times \mathcal{T}^A$. For $X \in \text{Spaces}_*$, define

$$F_{\mathcal{T}^A}(X) := \text{End}_A \otimes_{\mathcal{T}^{S^0}} \Omega^{S^0}(X) \in \text{Spaces}_*^{\mathcal{T}^A} .$$

One can check that $F_{\mathcal{T}^A}(X)$ preserves products, i.e., defines a \mathcal{T}^A -algebra. Thus we get a functor

$$F_{\mathcal{T}^A}: \text{Spaces}_* \rightarrow \text{Alg}^{\mathcal{T}^A}, \quad X \mapsto F_{\mathcal{T}^A}(X),$$

which is left adjoint to the forgetful functor

$$U_{\mathcal{T}^A}: \text{Alg}^{\mathcal{T}^A} \rightarrow \text{Spaces}_*, \quad U_{\mathcal{T}^A}(\tilde{Y}) = \tilde{Y}(T_1).$$

We will call $F_{\mathcal{T}^A}$ the *free \mathcal{T}^A -algebra functor* and $F_{\mathcal{T}^A}(X)$ the *free \mathcal{T}^A -algebra generated by X* .

3) Consider again an algebraic theory \mathcal{T}^A and let Δ^{op} be the simplicial category. Let $\tilde{Y}_\bullet: \Delta^{\text{op}} \times \mathcal{T}^A \rightarrow \text{Spaces}_*$ be a simplicial \mathcal{T}^A -algebra. Let $\Delta[\bullet]_+: \Delta \rightarrow \text{Spaces}_*$ denote the pointed cosimplicial space $[n] \mapsto \Delta[n]_+$. In this case the tensor product $\Delta[\bullet]_+ \otimes_{\Delta^{\text{op}}} \tilde{Y}_\bullet =: |\tilde{Y}_\bullet|$ gives the geometric realization of \tilde{Y}_\bullet . Since realization preserves products in Spaces_* , we see that $|\tilde{Y}_\bullet|$ is a \mathcal{T}^A -algebra.

3.5. Notice that the isomorphism of Section 3.3 shows that for a pointed simplicial space X_\bullet we have $|F_{\mathcal{T}^A}X_\bullet| \cong F_{\mathcal{T}^A}|X_\bullet|$, and that similarly for a simplicial \mathcal{T}^A -algebra \tilde{Y}_\bullet we get $|B_A\tilde{Y}_\bullet| \cong B_A|\tilde{Y}_\bullet|$.

3.6. Finally, consider the functors Ω^A and $U_{\mathcal{T}^A}$ of Section 3.4. The composition $U_{\mathcal{T}^A} \circ \Omega^A: \text{Spaces}_* \rightarrow \text{Spaces}_*$ is given by $U_{\mathcal{T}^A} \circ \Omega^A(X) = \text{Map}_*(A, X)$. As a result its left adjoint $B_A \circ F_{\mathcal{T}^A}$ is the smash product $B_A \circ F_{\mathcal{T}^A}(X) = X \wedge A$. This observation indicates that the algebraic theory \mathcal{T}^A may be suitable for describing mapping spaces from A , at least in some cases.

Lemma 3.7. *For any pointed finite set Z , we have a canonical isomorphism*

$$F_{\mathcal{T}^A}Z \cong \text{Map}_*(A, Z \wedge A)$$

of \mathcal{T}^A -algebras.

Proof. For a finite pointed set Z , the \mathcal{T}^A -algebra $\text{Map}_*(A, Z \wedge A)$, as a functor $\mathcal{T}^A \rightarrow \text{Spaces}_*$, is representable by T_{k-1} , where k is the cardinality of Z . Thus, by Yoneda's lemma, $\text{Mor}_{\text{Alg}^{\mathcal{T}^A}}(\text{Map}_*(A, Z \wedge A), \tilde{Y}) \cong \text{Map}_*(Z, U_{\mathcal{T}^A}(\tilde{Y}))$. The adjointness of $F_{\mathcal{T}^A}$ and $U_{\mathcal{T}^A}$ yields the result. \square

Combining this isomorphism with the equality $B_A(F_{\mathcal{T}^A}(Z)) = Z \wedge A$, we see that B_A acts as a classifying space for $\text{Map}_*(A, Z \wedge A)$. Our goal will be to show that when we take $A = S^n$, this construction works for any \mathcal{T}^{S^n} -algebra.

4. MODEL CATEGORIES AND QUILLEN EQUIVALENCES

Our strategy of approaching Theorem 1.1 will be to reformulate it in the language of model categories and prove it in this form. Below we describe model category structures we will encounter in this process. As it was the case so far, most of our setup will apply to mapping spaces $\text{Map}_*(A, X)$ from an arbitrary space A , and only in the proof of Theorem 4.8, we will specialize to $A = S^n$.

For any algebraic theory \mathcal{T} , the category of \mathcal{T} -algebras $\text{Alg}^{\mathcal{T}}$ has a model category structure with weak equivalences and fibrations defined objectwise, i.e., via the forgetful functor $U_{\mathcal{T}}$, [19]. For a CW-complex $A \in \text{Spaces}_*$, let $R_A \text{Spaces}_*$ denote the category of pointed compactly generated spaces together with the following choices of classes of morphisms:

- a map $f: X \rightarrow Z$ is a weak equivalence in $R_A \text{Spaces}_*$, if $f_*: \text{Map}_*(A, X) \rightarrow \text{Map}_*(A, Z)$ is a weak equivalence of mapping spaces;
- a map f is a fibration if it is a Serre fibration;
- a map f is a cofibration if it has the left lifting property with respect to all fibrations which are weak equivalences in $R_A \text{Spaces}_*$.

Proposition 4.1. *The category $R_A \text{Spaces}_*$ is a model category.*

Proof. The statement follows from a general result on the existence of right localizations of model categories, see [11, 5.1, p. 65]. \square

Note that for $A = S^0$, this defines the standard model category structure on Spaces_* .

In order to avoid confusing $R_A \text{Spaces}_*$ with Spaces_* , we will call weak equivalences (respectively, fibrations and cofibrations) in $R_A \text{Spaces}_*$ *A-local equivalences* (respectively, *fibrations* and *cofibrations*). Notice that a map $f: X \rightarrow Z$ is an S^n -local equivalence iff it induces isomorphisms $f_*: \pi_q(X) \rightarrow \pi_q(Z)$ for $q \geq n$.

4.2. A cofibrant resolution of a \mathcal{T}^A -algebra. Directly from the definition of the model structure on $\text{Alg}^{\mathcal{T}^A}$, it follows that every \mathcal{T}^A -algebra is a fibrant object. The structure of cofibrant algebras is more complicated (see [19]). For an arbitrary algebra $\tilde{Y} \in \text{Alg}^{\mathcal{T}^A}$, one can however describe its cofibrant replacement as follows. Recall the adjoint pair

$$F_{\mathcal{T}^A}: \text{Spaces}_* \rightleftarrows \text{Alg}^{\mathcal{T}^A} : U_{\mathcal{T}^A}$$

of Section 3.4.2.

Proposition 4.3. *For any CW-complex $A \in \text{Spaces}_*$, the functors*

$$F_{\mathcal{T}^A}: \text{Spaces}_* \rightleftarrows \text{Alg}^{\mathcal{T}^A} : U_{\mathcal{T}^A}$$

form a Quillen pair, see, e.g., [11, Definition 8.5.2]. In particular, the two functors induce an adjoint pair of functors between the homotopy categories.

Proof. The functor $U_{\mathcal{T}^A}$ sends weak equivalences and fibrations in $\text{Alg}^{\mathcal{T}^A}$ to weak equivalences and fibrations in Spaces_* , respectively, thus the conclusion follows. \square

Next, consider the adjoint functors

$$|\cdot|: \text{SSets}_* \rightleftarrows \text{Spaces}_* : \text{Sing}_\bullet$$

between the categories of pointed spaces and pointed simplicial sets, where Sing_\bullet is the singularization functor and $|\cdot|$ is geometric realization. We will denote by $F'_{\mathcal{T}^A}: \text{SSets}_* \rightarrow \text{Alg}^{\mathcal{T}^A}$ the composition of $|\cdot|$ and $F_{\mathcal{T}^A}$ and by $U'_{\mathcal{T}^A}: \text{Alg}^{\mathcal{T}^A} \rightarrow \text{SSets}_*$ the functor obtained by composing $U_{\mathcal{T}^A}$ with Sing_\bullet . The functors $F'_{\mathcal{T}^A}, U'_{\mathcal{T}^A}$ form again a Quillen pair. Therefore for any \mathcal{T}^A -algebra \tilde{Y} , they define a simplicial object $F'_{\mathcal{T}^A}U'_{\mathcal{T}^A}_\bullet \tilde{Y}$ in the category $\text{Alg}^{\mathcal{T}^A}$ which has the algebra $(F'_{\mathcal{T}^A}U'_{\mathcal{T}^A})^{(k+1)} \tilde{Y}$ in its k -th simplicial dimension. Its face and degeneracy maps are defined using the counit and the unit of adjunction, respectively (compare [16, Chapter 9]). Let $|F'_{\mathcal{T}^A}U'_{\mathcal{T}^A}_\bullet \tilde{Y}|$ denote the objectwise geometric realization of $F'_{\mathcal{T}^A}U'_{\mathcal{T}^A}_\bullet \tilde{Y}$.

Lemma 4.4. *$|F'_{\mathcal{T}^A}U'_{\mathcal{T}^A}_\bullet \tilde{Y}|$ is a \mathcal{T}^A -algebra.*

Proof. Clearly, $|F'_{\mathcal{T}^A} U'_{\mathcal{T}^A} \bullet \tilde{Y}|$ is a functor from \mathcal{T}^A to Spaces_* . Also, since we are working in the category of compactly generated spaces, realization preserves products, and so $|F'_{\mathcal{T}^A} U'_{\mathcal{T}^A} \bullet \tilde{Y}|$ is a \mathcal{T}^A -algebra. \square

Similarly to [1, 3.5, p. 903], we get

Lemma 4.5. *For any $\tilde{Y} \in \text{Alg}^{\mathcal{T}^A}$ there is a canonical weak equivalence*

$$|F'_{\mathcal{T}^A} U'_{\mathcal{T}^A} \bullet \tilde{Y}| \rightarrow \tilde{Y}.$$

The above lemma remains to be true, if we replace the functors $F'_{\mathcal{T}^A}$ and $U'_{\mathcal{T}^A}$ with $F_{\mathcal{T}^A}$ and $U_{\mathcal{T}^A}$, respectively. What we will use in the sequel (see Step 3 of the proof of Theorem 4.8) though is that the free algebras $(F'_{\mathcal{T}^A} U'_{\mathcal{T}^A})_n \tilde{Y}$ are generated by spaces obtained as realizations of simplicial sets. The algebra $|F'_{\mathcal{T}^A} U'_{\mathcal{T}^A} \bullet \tilde{Y}|$ can be taken as a cofibrant replacement of \tilde{Y} , since we have

Lemma 4.6. *For any $\tilde{Y} \in \text{Alg}^{\mathcal{T}^A}$ the algebra $|F'_{\mathcal{T}^A} U'_{\mathcal{T}^A} \bullet \tilde{Y}|$ is a cofibrant object in $\text{Alg}^{\mathcal{T}^A}$.*

Proof. This is a consequence of [19], which describes the structure of cofibrant objects in the model category $\text{Alg}^{\mathcal{T}}$. \square

Next, let $A \in \text{Spaces}_*$. Recall (Section 3.4.1) that we have an adjoint pair of functors (B_A, Ω^A) . Moreover the following holds:

Proposition 4.7. *For any CW-complex $A \in \text{Spaces}_*$, the functors*

$$B_A: \text{Alg}^{\mathcal{T}^A} \rightleftarrows R_A \text{Spaces}_*: \Omega^A$$

form a Quillen pair.

Proof. The functor Ω^A sends A -local equivalences and A -local fibrations to weak equivalences and fibrations in $\text{Alg}^{\mathcal{T}^A}$, respectively, which yields the statement. \square

Our main result, Theorem 1.1, can now be restated more precisely as follows:

Theorem 4.8. *For $n \geq 0$ the Quillen pair*

$$B_n: \text{Alg}^{\mathcal{T}^{S^n}} \rightleftarrows R_{S^n} \text{Spaces}_*: \Omega^n,$$

where $B_n := B_{S^n}$ and $\Omega^n := \Omega^{S^n}$, is a Quillen equivalence, see, e.g., [11, Definition 8.5.20]. In particular, the two functors induce an equivalence of the homotopy categories.

Corollary 4.9 (Approximation theorem). *For any CW-complex $X \in \text{Spaces}_*$, the following \mathcal{T}^{S^n} -algebras are weakly equivalent:*

$$F_n X \xrightarrow{\sim} \Omega^n \Sigma^n X,$$

where $F_n X$ denotes the free \mathcal{T}^{S^n} -algebra $F_{\mathcal{T}^{S^n}} X$ on X and $\Sigma^n X = S^n \wedge X$ is the reduced suspension. Moreover, these equivalences establish an equivalence of monads $F_n \sim \Omega^n \Sigma^n$ on the category of CW-complexes.

Let us first deduce Theorem 1.1 and Corollary 4.9 from Theorem 4.8.

Proof of Theorem 1.1. Let \tilde{Y} be any \mathcal{T}^{S^n} -algebra, and let $\tilde{Y}_c \xrightarrow{\sim} \tilde{Y}$ be its cofibrant replacement. Like any other object in $R_{S^n} \text{Spaces}_*$, $B_n \tilde{Y}_c$ is fibrant and therefore Theorem 4.8 implies that the adjoint $\tilde{Y}_c \rightarrow \Omega^n B_n \tilde{Y}_c$ of the identity isomorphism $B_n \tilde{Y}_c \xrightarrow{\sim} B_n \tilde{Y}_c$ is a weak equivalence of \mathcal{T}^{S^n} -algebras. Therefore $\tilde{Y}(T_1) \simeq \Omega^n B_n \tilde{Y}_c(T_1)$, and we indeed recover the statement of Theorem 1.1. \square

Proof of Corollary 4.9. By [19] the free F_n -algebra generated by a CW-complex X is cofibrant in $\mathcal{Alg}^{\mathcal{T}^{S^n}}$. The space $B_n F_n X$ is fibrant, as any object of $R_{S^n} \text{Spaces}_*$. Then the isomorphism $B_n F_n X \xrightarrow{\text{id}} B_n F_n X$ implies by Theorem 4.8 that the adjoint $F_n X \rightarrow \Omega^n B_n F_n X$ is a weak equivalence. On the other hand, $B_n F_n X = \Sigma^n X$ by 3.6. Thus, we get a weak equivalence $F_n X \xrightarrow{\sim} \Omega^n \Sigma^n X$. It defines an equivalence of monads, because of the naturality of the construction. \square

Proof of Theorem 4.8. It is enough to show that for every cofibrant \mathcal{T}^{S^n} -algebra \tilde{Y} , the unit $\eta_{\tilde{Y}}: \tilde{Y} \rightarrow \Omega^n B_n \tilde{Y}$ of the adjunction (B_n, Ω^n) is a weak equivalence in $\mathcal{Alg}^{\mathcal{T}^{S^n}}$. Indeed, for $\tilde{Y} \in \mathcal{Alg}^{\mathcal{T}^{S^n}}$, $X \in \text{Spaces}_*$, and $f: \tilde{Y} \rightarrow \Omega^n X$, we have a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\eta_{\tilde{Y}}} & \Omega^n B_n \tilde{Y} \\ & \searrow f & \downarrow \Omega^n f^b \\ & & \Omega^n X, \end{array}$$

where f^b is the adjoint to f . Assume that \tilde{Y} is cofibrant and $\eta_{\tilde{Y}}$ is a weak equivalence in $\mathcal{Alg}^{\mathcal{T}^{S^n}}$. If f is also a weak equivalence, then so is $\Omega^n f^b$. In particular the map

$$\Omega^n f^b(T_1): \Omega^n(B_n \tilde{Y}) = (\Omega^n B_n \tilde{Y})(T_1) \rightarrow (\Omega^n X)(T_1) = \Omega^n X$$

is a weak equivalence of spaces, or, in other words, f^b is an S^n -local weak equivalence.

Conversely, if f^b is an S^n -local equivalence, then $\Omega^n f^b$ is an objectwise weak equivalence, and so is f .

The proof of the fact that, for a cofibrant $\tilde{Y} \in \mathcal{Alg}^{\mathcal{T}^{S^n}}$, the map $\eta_{\tilde{Y}}$ is a weak equivalence follows from a bootstrap argument below.

1) Let $\tilde{Y} = F_n(Z)$, where Z is an arbitrary pointed discrete space. Since F_n is a left adjoint functor, it commutes with colimits. Therefore, since Z is the colimit of the poset of finite subsets X of Z containing the basepoint, we get:

$$F_n(Z) = \text{colim}_{X \subseteq Z} F_n(X) = \text{colim}_{X \subseteq Z} \text{Map}_*(S^n, X \wedge S^n).$$

The second equality follows from 3.7. Furthermore, since S^n is a compact space, we have $\text{colim}_{X \subseteq Z} \text{Map}_*(S^n, X \wedge S^n) = \text{Map}_*(S^n, Z \wedge S^n)$. Therefore, the map $\eta_{\tilde{Y}}$ is an isomorphism of \mathcal{T}^{S^n} -algebras by 3.6.

2) Let Z_\bullet be a pointed simplicial set, and let $\tilde{Y} = F'_n(Z_\bullet)$, where $F'_n = F'_{\mathcal{T}^{S^n}}$. We have by 3.5

$$F'_n(Z_\bullet) = F_n(|Z_\bullet|) \cong |F_n Z_\bullet|,$$

where $F_n Z_\bullet$ denotes the simplicial \mathcal{T}^{S^n} -algebra obtained by applying F_n in each simplicial dimension of Z_\bullet . By Step 1 for every $k \geq 0$, we have an isomorphism

$\eta_k: F_n(Z_k) \rightarrow \Omega^n B_n F_n(Z_k)$, assembling into a simplicial map by naturality. Thus, the map

$$|\eta_\bullet|: \tilde{Y} \rightarrow |\Omega^n B_n F_n(Z_\bullet)|$$

is also an isomorphism. Next, notice that by 3.6, we have $B_n F_n(Z_k) = Z_k \wedge S^n$, so it is an $(n-1)$ -connected space. Therefore (see [16, Theorem 12.3]), we have a natural weak equivalence $|\Omega^n B_n F_n(Z_\bullet)| \simeq \Omega^n |B_n F_n(Z_\bullet)|$. (A technical condition of properness of $B_n F_n(Z_\bullet)$, needed for applying May's theorem, is satisfied here, as Z_\bullet is discrete and B_n and F_n are admissible functors, see [16, Definitions 11.2 and A.7].) Combining this with the isomorphism $|B_n F_n(Z_\bullet)| \cong B_n |F_n(Z_\bullet)|$ we get a weak equivalence

$$|\Omega^n B_n F_n(Z_\bullet)| \simeq \Omega^n B_n |F_n(Z_\bullet)| \cong \Omega^n B_n \tilde{Y}$$

It follows that $\eta_{\tilde{Y}}$ is a weak equivalence.

3) Let \tilde{Y} be any \mathcal{T}^{S^n} -algebra and $F'_n U'_{n\bullet} \tilde{Y}$ its simplicial resolution as in Section 4.2, where $U'_n = U'_{\mathcal{T}^{S^n}}$. Note that, in every simplicial dimension k , the algebra $(F'_n U'_n)_k \tilde{Y}$ is of the form considered in Step 2. It follows that for $k \geq 0$, we have a weak equivalence

$$(1) \quad \eta_k: (F'_n U'_n)_k \tilde{Y} \xrightarrow{\sim} \Omega^n B_n (F'_n U'_n)_k \tilde{Y}.$$

To see that the map

$$|\eta_\bullet|: |F'_n U'_{n\bullet} \tilde{Y}| \rightarrow |\Omega^n B_n F'_n U'_{n\bullet} \tilde{Y}|$$

is also a weak equivalence, we can use a result of May [16, Theorem 11.13]. The assumption of strict properness [16, Definition 11.2] of the simplicial spaces $F'_n U'_{n\bullet} \tilde{Y}$ and $\Omega^n B_n F'_n U'_{n\bullet} \tilde{Y}$, needed for May's theorem, is not hard to verify, since all the functors $F_n, U_n, |\text{Sing}_\bullet(\cdot)|, B_n$, and Ω^n are admissible in the sense of [16, Definition A.7]. May also assumes that the realizations of the simplicial spaces are connected H-spaces, which will not be satisfied in our case, in general. His result however readily generalizes to the case of simplicial spaces whose realizations are H-spaces with π_0 's having a group structure, as it is the case for the simplicial spaces at hand for $n \geq 1$. The H-space structure is not there for $n = 0$, but in this case, the statement of the theorem is trivial, anyway.

Using arguments similar to those employed in Step 2, we get from here that

$$\eta: |F'_n U'_{n\bullet} \tilde{Y}| \rightarrow \Omega^n B_n |F'_n U'_{n\bullet} \tilde{Y}|$$

is a weak equivalence.

4) Let \tilde{Y} be any cofibrant algebra. We have a commutative diagram:

$$\begin{array}{ccc} |F'_n U'_{n\bullet} \tilde{Y}| & \xrightarrow{\eta} & \Omega^n B_n |F'_n U'_{n\bullet} \tilde{Y}| \\ h \downarrow \sim & & \downarrow \Omega^n B_n h \\ \tilde{Y} & \xrightarrow{\eta_{\tilde{Y}}} & \Omega^n B_n \tilde{Y}, \end{array}$$

where h is the weak equivalence of Lemma 4.5. The functor B_n is a left Quillen functor and as such it preserves weak equivalences between cofibrant \mathcal{T}^{S^n} -algebras, while Ω^n preserves all weak equivalences. Therefore $\Omega^n B_n h$ is a weak equivalence, and, as a consequence, so is $\eta_{\tilde{Y}}$. \square

Theorem 4.10. *Suppose \mathcal{T} is an algebraic theory such that it*

- (1) acts on n -fold loop spaces $\Omega^n X$ by natural operations $(\Omega^n X)^k \rightarrow (\Omega^n X)^l$, i.e., admits a morphism $\phi : \mathcal{T} \rightarrow \mathcal{T}^{S^n}$, and
- (2) via this action deloops n -fold loop spaces in the sense of Theorem 4.8, i.e., the loop functor $R_{S^n} \text{Spaces}_* \xrightarrow{\Omega^n} \text{Alg}^{\mathcal{T}^{S^n}} \xrightarrow{\phi^*} \text{Alg}^{\mathcal{T}}$ establishes a Quillen equivalence.

Then $\phi : \mathcal{T} \rightarrow \mathcal{T}^{S^n}$ is a weak equivalence of topological theories.

This theorem is, in fact, an obvious corollary of a uniqueness theorem [2, Theorem 1.6] (theories considered in [2] are enriched over simplicial sets, but the proof of this result holds for topological theories with little changes).

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