

1 | Some Set Theory

Sets. In general sets will be denoted by capital letters: $A, B, C, \dots$. We will also use the following notation for sets that will be of a particularly interest:

$\emptyset$ = the empty set (i.e. the set that contains no elements)

$\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers

$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ the set of positive integers

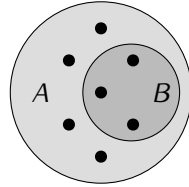
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ the set of integers

$\mathbb{Q}$ = the set of rational numbers

$\mathbb{R}$ = the set of real numbers

We will write $a \in B$ to denote that a is an element of the set B and $a \notin B$ to indicate that a is not an element of B .

1.1 Definition. A set B is a *subset* of a set A if every element of B is in A . In such case we write $B \subseteq A$.



A set B is a *proper subset* of A if $B \subseteq A$ and $B \neq A$.

1.3 Example. Here are some often used subsets of $\mathbb{R}$:

1) an open interval:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$



2) a closed interval:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$



3) a half open interval:

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$



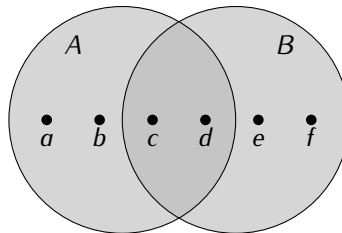
1.4 Definition. The *union* of sets A and B is the set $A \cup B$ that consists of all elements that belong to either A or B :

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The *intersection* of sets A and B is the set $A \cap B$ that consists of all elements that belong to both A and B :

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

1.5 Example. If $A = \{a, b, c, d\}$, $B = \{c, d, e, f\}$



1.7 Definition. If $A \cap B = \emptyset$ then we say that A and B are *disjoint sets*.

Definition 1.4 can be extended to unions and intersections of arbitrary families of sets. If $\{A_i\}_{i \in I}$ is a family of sets then

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$
$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

1.10 Definition. The *difference* of sets A and B is the set $A \setminus B$ consisting of the elements of A that do not belong to B :

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

1.12 Definition. If $A \subseteq B$ then the set $B \setminus A$ is called the *complement* of A in B .

1.13 Properties of the algebra of sets.

Distributivity:

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

De Morgan's Laws:

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

1.14 Definition. The *Cartesian product* of sets A, B is the set consisting of all ordered pairs of elements of A and B :

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

1.16 Notation. Given a set A by A^n we will denote the n -fold Cartesian product of A :

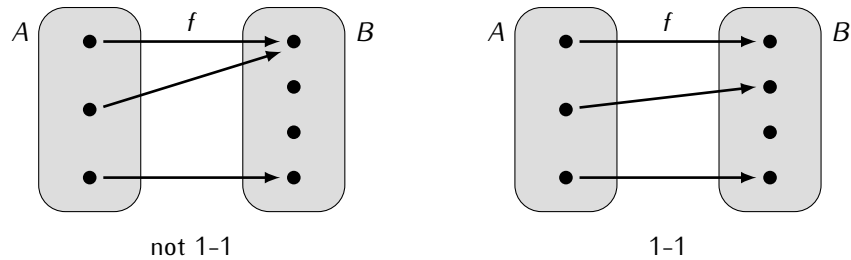
$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

1.18 Infinite products.

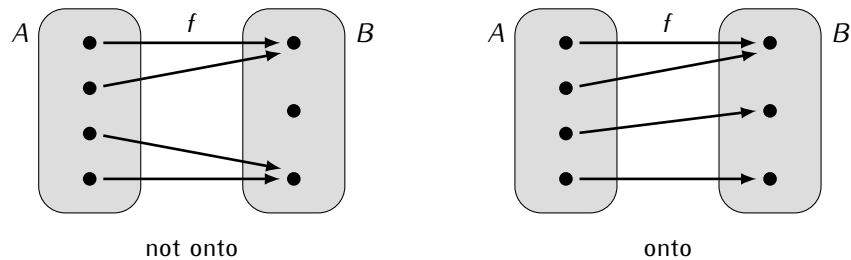
1.24 Definition. A *binary relation* on a set A is a subset $R \subseteq A \times A$. If $(a, b) \in R$ then we write aRb .

1.27 Definition. Let A, B be sets

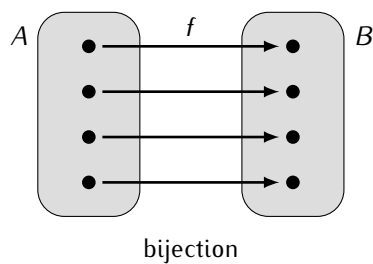
1) A function $f: A \rightarrow B$ is *1-1* if $f(x) = f(x')$ only if $x = x'$.



2) A function $f: A \rightarrow B$ is *onto* if for every $y \in B$ there is $x \in A$ such that $f(x) = y$.

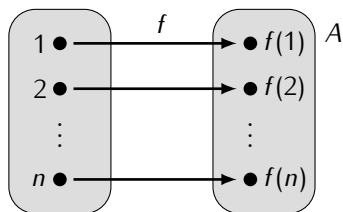


3) A function $f: A \rightarrow B$ is a *bijection* if f is both 1-1 and onto.

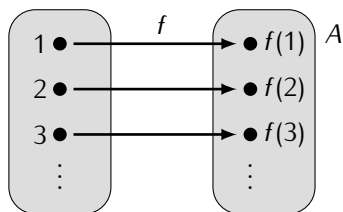


1.29 Definition. Sets A, B have the same cardinality if there exists a bijection $f: A \rightarrow B$. In such case we write $|A| = |B|$.

1.30 Definition. A set A is *finite* if either $A = \emptyset$ or A has the same cardinality as the set $\{1, \dots, n\}$ for some $n \geq 1$.



1.31 Definition. A set A is *infinitely countable* if it has the same cardinality as the set $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$.



1.32 Definition. A set A is *countable* if it is either finite or infinitely countable.

1.33 Example. The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is countable since we have a bijection $f: \mathbb{Z}^+ \rightarrow \mathbb{N}$ given by $f(k) = k - 1$.

1.34 Example. The set of integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is countable since we have a bijection $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}$ given by

$$f(k) = \begin{cases} k/2 & \text{if } k \text{ is even} \\ (1-k)/2 & \text{if } k \text{ is odd} \end{cases}$$

In other words:

$$f(1) = 0, \quad f(2) = 1, \quad f(3) = -1, \quad f(4) = 2, \quad f(5) = -2, \quad f(6) = 3, \quad \dots$$

1.35 Example. The set of rational numbers $\mathbb{Q}$ is countable. A bijection $f: \mathbb{Z}^+ \rightarrow \mathbb{Q}$ can be constructed as follows:

$0/1$	$1/1$	$-1/1$	$2/1$	$-2/1$	$3/1$	$-3/1$	$\dots$	$0/1 = f(1)$
$0/2$	$1/2$	$-1/2$	$2/2$	$-2/2$	$3/2$	$-3/2$	$\dots$	$0/2 = 0/1 = f(1)$
$0/3$	$1/3$	$-1/3$	$2/3$	$-2/3$	$3/3$	$-3/3$	$\dots$	$1/1 = f(2)$
$0/4$	$1/4$	$-1/4$	$2/4$	$-2/4$	$3/4$	$-3/4$	$\dots$	$-1/1 = f(3)$
$0/5$	$1/5$	$-1/5$	$2/5$	$-2/5$	$3/5$	$-3/5$	$\dots$	$1/2 = f(4)$
$0/6$	$1/6$	$-1/6$	$2/6$	$-2/6$	$3/6$	$-3/6$	$\dots$	$0/3 = 0/1 = f(1)$
$0/7$	$1/7$	$-1/7$	$2/7$	$-2/7$	$3/7$	$-3/7$	$\dots$	$0/4 = 0/1 = f(1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$1/3 = f(5)$
								$-1/2 = f(6)$
								$2/1 = f(7)$
								$\dots \quad \dots$

1.36 Theorem. 1) If A is a countable set and $B \subseteq A$ then B is countable.

2) If $\{A_1, A_2, \dots\}$ is a collection of countably many countable sets then the set $\bigcup_{i=1}^{\infty} A_i$ is countable.

3) If $\{A_1, A_2, \dots, A_n\}$ is a collection of finitely many countable sets then the set $A_1 \times \dots \times A_n$ is countable.

1.37 Example. The set of all real numbers in the interval $(0, 1)$ is not countable.

$$\begin{aligned} f(1) &= 0.31415\dots \\ f(2) &= 0.12345\dots \\ f(3) &= 0.75149\dots \\ f(4) &= 0.00032\dots \\ f(5) &= 0.11111\dots \\ &\dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

1.38 Example. The function $f: (0, 1) \rightarrow \mathbb{R}$ given by $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$ is a bijection. It follows that $|\mathbb{R}| = |(0, 1)|$. In particular $\mathbb{R}$ is an uncountable set.

1.39 Notation. 1) If A is a finite set of n elements then we write $|A| = n$.

2) If $|A| = |\mathbb{Z}^+|$ (i.e. A is an infinitely countable set) then we say that A has the *cardinality aleph naught* and we write $|A| = \aleph_0$.

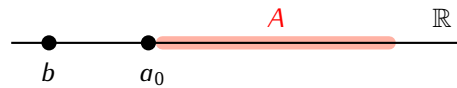
3) If $|A| = |\mathbb{R}|$ then we say that A has the *cardinality of the continuum* and we write $|A| = \mathfrak{c}$.

Infima and Suprema.

1.40 Definition. Let $A \subseteq \mathbb{R}$. The set A is *bounded below* if there exists a number b such that $b \leq x$ for all $x \in A$. The set A is *bounded above* if there exists a number c such that $x \leq c$ for all $x \in A$. The set A is *bounded* if it is both bounded below and bounded above.

1.41 Definition. Let $A \subseteq \mathbb{R}$. If the set A is bounded below then the *greatest lower bound* of A (or *infimum* of A) is a number $a_0 \in \mathbb{R}$ such that:

- 1) $a_0 \leq x$ for all $x \in A$
- 2) if $b \leq x$ for all $x \in A$ then $b \leq a_0$



We write: $a_0 = \inf A$.

If the set A is not bounded below then we set $\inf A := -\infty$.

1.43 Theorem. For any non-empty bounded below subset $A \subseteq \mathbb{R}$ the number $\inf A$ exists.

1.44 Definition. Let $A \subseteq \mathbb{R}$. If the set A is bounded above then the *least upper bound* of A (or *supremum* of A) is a number $a_0 \in \mathbb{R}$ such that:

- 1) $x \leq a_0$ for all $x \in A$
- 2) if $x \leq b$ for all $x \in A$ then $a_0 \leq b$



We write: $a_0 = \sup A$.

If the set A is not bounded above then we set $\sup A := +\infty$.

1.46 Theorem. For any non-empty bounded above subset $A \subseteq \mathbb{R}$ the number $\sup A$ exists.