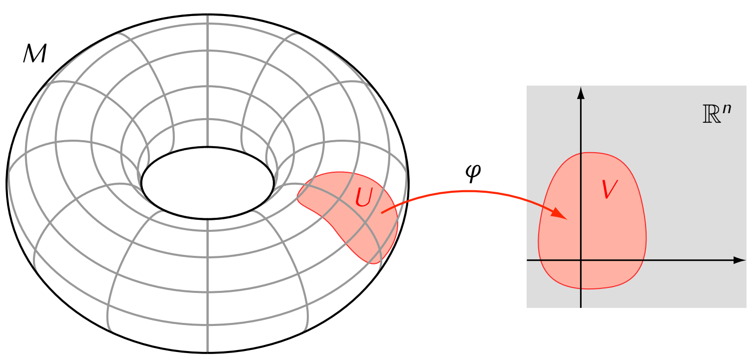
13 | Metrization of Manifolds

13.1 Definition. A *topological manifold of dimension n* is a topological space M which is a Hausdorff, second countable, and such that every point of M has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ (we say that M is *locally homeomorphic* to $\mathbb{R}^n$).

13.2 Note. Let M be a manifold of dimension n. If $U \subseteq M$ is an open set and $\varphi\colon U \to V$ is a homeomorphism of U with some open set $V \subseteq \mathbb{R}^n$ then we say that U is a *coordinate neighborhood* and φ is a *coordinate chart* on M.



13.3 Lemma. *If M is an n-dimensional manifold then:*

1) *for any point $x \in M$ there exists a coordinate chart $\varphi\colon U \to V$ such that $x \in U$, V is an open ball $V = B(y, r)$, and $\varphi(x) = y$;*
2) *for any point $x \in M$ there exists a coordinate chart $\psi\colon U \to V$ such that $x \in U$, $V = \mathbb{R}^n$, and $\psi(x) = 0$.*

Proof. Exercise. $\square$

13.4 Example. A space M is a manifold of dimension 0 if and only if M is a countable (finite or infinite) discrete space.

13.5 Example. If U is an open set in $\mathbb{R}^n$ then U is an n-dimensional manifold. The identity map $\mathrm{id}\colon U \to U$ is then a coordinate chart defined on the whole manifold U. In particular $\mathbb{R}^n$ is an n-dimensional manifold.

13.6 Example. The n-dimensional sphere

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is an n-dimensional manifold.

13.7 Proposition. *If M is an m-dimensional manifold and N is an n-dimensional manifold then $M \times N$ is an $m+n$-dimensional manifold.*

Proof. Exercise. $\square$

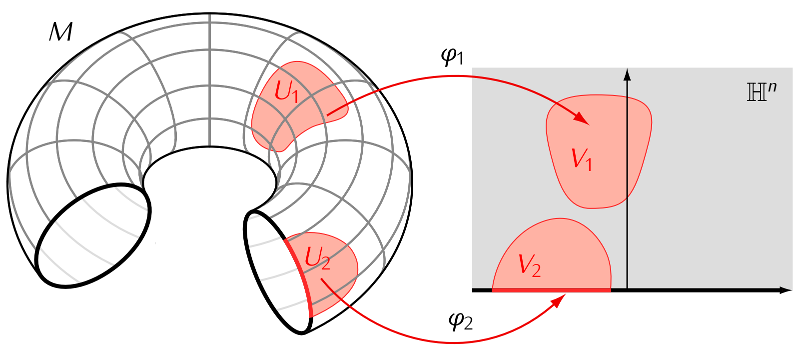
13.9 Note. There exist topological spaces that are locally homeomorphic to $\mathbb{R}^n$, but do not satisfy the the other conditions of the definition of a manifold (13.1).

13.10 Invariance of Dimension Theorem. *If M is a non-empty topological space such that M is a manifold of dimension m and M is also a manifold of dimension n then $m = n$.*

13.11 Definition. A *topological n-dimensional manifold with boundary* is a topological space M which is a Hausdorff, second countable, and such that every point of M has an open neighborhood homeomorphic to an open subset of $\mathbb{H}^n$.

13.12 Let $\partial\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{H}^n \mid x_n = 0\}$. If M is an n-dimensional manifold with boundary, $\varphi\colon U \to V$ is a coordinate chart, and $x \in U$ then there are two possibilities:

1) $\varphi(x) \in \partial\mathbb{H}^n$
2) $\varphi(x) \notin \partial\mathbb{H}^n$



In the first case we say that the point x is a *boundary point* of M, and in the the second case that x is an *interior point* of M.

13.13 Theorem. *Let M be an n-dimensional manifold with boundary, let $x_0 \in M$ and let $\varphi\colon U \to V$ be a local coordinate chart such that $x_0 \in U$. If $\varphi(x_0) \in \partial\mathbb{H}^n$ then for any other local coordinate chart $\psi\colon U' \to V'$ such that $x_0 \in U'$ we have $\psi(x_0) \in \partial\mathbb{H}^n$.*

13.14 Definition. Let M be a manifold with boundary. The subspace of M consisting of all boundary points of M is called *the boundary of M* and it is denoted by ∂M.

13.15 Example. The space $\mathbb{H}^n$ is trivially an n-dimensional manifold with boundary.

13.16 Example. For any n the closed n-dimensional ball

$$\overline{B}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

is an n-dimensional manifold with boundary (exercise). In this case we have $\partial\overline{B}^n = S^{n-1}$.

13.17 Example. If M is a manifold (without boundary) then we can consider it as a manifold with boundary. where $\partial M = \varnothing$.

13.19 Proposition. *If M is an n-dimensional manifold with boundary then:*

1) *$M \smallsetminus \partial M$ is an open subset of M and it is an n-dimensional manifold (without boundary);*
2) *∂M is a closed subset of M and it is an $(n-1)$-dimensional manifold (without boundary).*

Proof. Exercise. □

13.20 Theorem. *Every topological manifold (with or without boundary) is metrizable.*

13.21 Lemma. *Let M be an n-dimensional topological manifold, and let $\varphi\colon U \to V$ be a coordinate chart on M. If $\overline{B}(x, r)$ is a closed ball in $\mathbb{R}^n$ such that $\overline{B}(x, r) \subseteq V$ then the set $\varphi^{-1}(\overline{B}(x, r))$ is closed in M.*