16 | Compact Metric Spaces

16.1 Definition. A topological space X is *sequentially compact* if every sequence $\{x_n\} \subseteq X$ contains a convergent subsequence.

16.2 Theorem. *A metric space (X, ϱ) is compact if and only if it is sequentially compact.*

16.4 Lemma. *Let (X, ϱ) be a metric space. If a sequence $\{x_n\} \subseteq X$ does not contain any convergent subsequence then $\{x_n\}$ is a closed set in X.*

Proof. Exercise. $\square$

16.5 Lemma. *Let (X, ϱ) be a metric space. If a sequence $\{x_n\} \subseteq X$ does not contain any convergent subsequence then for each $k = 1, 2, \ldots$ there exists $\varepsilon_k > 0$ such that $B(x_k, \varepsilon_k) \cap \{x_n\} = x_k$.*

Proof. Exercise. $\square$

Proof of Theorem 16.2 ($\Rightarrow$).

$\square$

16.6 Definition. Let (X, ϱ) be a metric space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. A *Lebesgue number* for $\mathcal{U}$ is a number $\lambda_{\mathcal{U}} > 0$ such that for every $x \in X$ we have $B(x, \lambda_{\mathcal{U}}) \subseteq U_i$ for some $U_i \in \mathcal{U}$.

16.8 Lemma. *If (X, ϱ) is a sequentially compact metric space then for any open cover $\mathcal{U}$ of X there exists a Lebesgue number for $\mathcal{U}$.*

16.9 Definition. Let (X, ϱ) be a metric space. For $\varepsilon > 0$ an *ε-net* in X is a set of points $\{x_i\}_{i \in I} \subseteq X$ such that $X = \bigcup_{i \in I} B(x_i, \varepsilon)$.

16.11 Lemma. *Let (X, ϱ) be a sequentially compact metric space. For every $\varepsilon > 0$ there exists a finite ε-net in X.*

Proof of Theorem 16.2 (⇐) .

$\square$

16.12 Corollary. *If (X, ϱ) is a compact metric space then for any open cover $\mathfrak{U}$ of X there exists a Lebesgue number for $\mathfrak{U}$.*