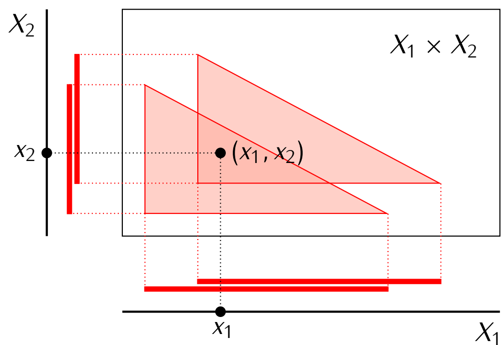
# 17 | Tychonoff Theorem

**17.1 Tychonoff Theorem.** *If $\{X_s\}_{s\in S}$ is a family of topological spaces and $X_s$ is compact for each $s \in S$ then the product space $\prod_{s\in S} X_s$ is compact.*

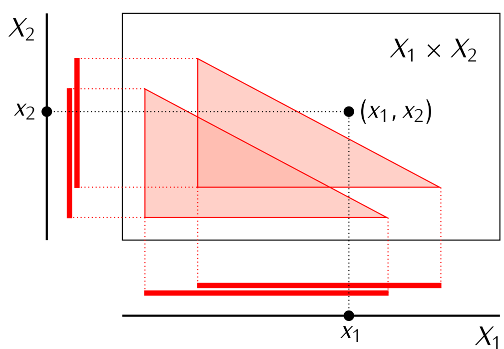
**17.2 Definition.** Let $\mathcal{A}$ be a family of subsets of a space $X$. The family $\mathcal{A}$ is *centered* if for any finite number of sets $A_1, \dots, A_n \in \mathcal{A}$ we have $A_1 \cap \dots \cap A_n \neq \varnothing$

**17.5 Lemma.** *Let $X$ be a topological space. The following conditions are equivalent:*

1) *The space $X$ is compact.*
2) *For any centered family $\mathcal{A}$ of closed subsets of $X$ we have $\bigcap_{A\in\mathcal{A}} A \neq \varnothing$.*



$X_2$
$X_1 \times X_2$
$x_2$
$(x_1, x_2)$
$x_1$
$X_1$

$X_2$
$X_1 \times X_2$
$x_2$
$(x_1, x_2)$
$x_1$
$X_1$

**17.6 Definition.** A *partially ordered set* (or *poset*) is a set $S$ equipped with a binary relation $\leq$ satisfying

(i) $x \leq x$ for all $x \in S$ (reflexivity)
(ii) if $x \leq y$ and $y \leq x$ then $y = x$ (antisymmetry)
(iii) if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity).

**17.7 Definition.** A *linearly ordered set* is a poset $(S, \leq)$ such that for any $x, y \in S$ we have either $x \leq y$ or $y \leq x$.

**17.9 Definition.** If $(S, \leq)$ is a poset then an element $x \in S$ is a *maximal element* of $S$ if we have $x \leq y$ only for $y = x$.

**17.13 Definition.** Let $(S, \leq)$ is a poset and let $T \subseteq S$. An *upper bound of* $T$ is an element $x \in S$ such that $y \leq x$ for all $y \in T$.

**17.14 Definition.** If $(S, \leq)$ is a poset. A *chain* in $S$ is a subset $T \subseteq S$ such that $T$ is linearly ordered.

**17.15 Zorn's Lemma.** *If* $(S, \leq)$ *is a non-empty poset such that every chain in* $S$ *has an upper bound in* $S$ *then* $S$ *contains a maximal element.*

*Proof.* See any book on set theory. $\square$

*Proof of Theorem 17.1.*

$\square$