18 | Compactification

18.1 Proposition. *Let X be a topological space. If there exists an embedding $j\colon X \to Y$ such that Y is a compact Hausdorff space then there exists an embedding $j_1\colon X \to Z$ such that Z is compact Hausdorff and $\overline{j_1(X)} = Z$.*

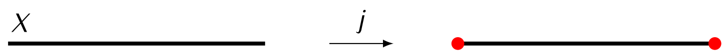
18.2 Definition. A space Z is a *compactification* of X if Z is compact Hausdorff and there exists an embedding $j\colon X \to Z$ such that $\overline{j(X)} = Z$.

18.3 Corollary. *Let X be a topological space. The following conditions are equivalent:*

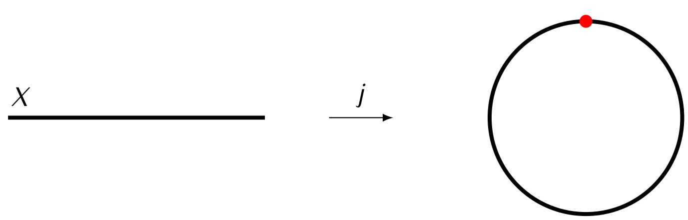
1) *There exists a compactification of X.*
2) *There exists an embedding $j\colon X \to Y$ where Y is a compact Hausdorff space.*

Proof. Follows from Proposition 18.1. □

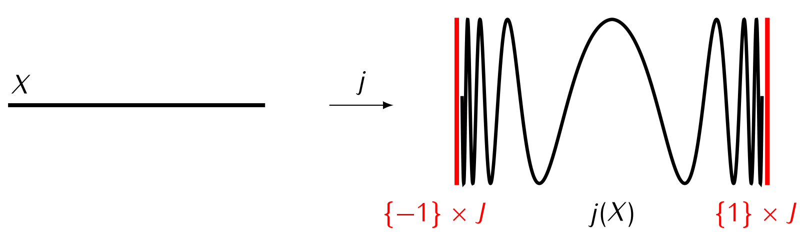
18.4 Example.



18.5 Example.



18.6 Example.



18.7 Theorem. *A space X has a compactification if and only if X is completely regular (i.e. it is a $T_{3^1/_2}$-space).*

18.9 Definition. Let X be a completely regular space and let $j_X \colon X \to \prod_{f \in C(X)}[0,1]$ be the embedding defined in the proof of Theorem 18.7 and let $\beta(X)$ be the closure of $j_X(X)$ in $\prod_{f \in C(X)}[0,1]$. The compactification $j_X \colon X \to \beta(X)$ is called the *Čech-Stone compactification* of X.

18.10 Definition. Let X be a space and let $i_1 \colon X \to Y_1$, $i_2 \colon X \to Y_2$ be compactifications of X. We will write $Y_1 \geq Y_2$ if there exists a continuous function $g \colon Y_1 \to Y_2$ such that $i_2 = g i_1$:

$$
\begin{tikzcd}
& X \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
Y_1 \arrow[rr, "g"'] & & Y_2
\end{tikzcd}
$$

18.11 Proposition. *Let $i_1 \colon X \to Y_1$, $i_2 \colon X \to Y_2$ be compactifications of a space X.*

1) If $Y_1 \geq Y_2$ then there exists only one map $g \colon Y_1 \to Y_2$ satisfying $i_2 = g i_1$. Moreover g is onto.

2) $Y_1 \geq Y_2$ and $Y_2 \geq Y_1$ if and only if the map $g \colon Y_1 \to Y_2$ is a homeomorphism.

Proof. Exercise. $\square$

18.12 Theorem. *Let X be a completely regular space and let $j_X\colon X \to \beta(X)$ be the Čech-Stone compactification of X. For any compactification $i\colon X \to Y$ of X we have $\beta(X) \geq Y$.*

18.13 Lemma. *If $f\colon X_1 \to X_2$ is a continuous map of compact Hausdorff spaces then $f(\overline{A}) = \overline{f(A)}$ for any $A \subseteq X_1$.*

Proof. Exercise. $\square$

18.14 Definition. A space Z is a *one-point compactification* of a space X if Z is a compactification of X with embedding $j\colon X \to Z$ such that the set $Z \smallsetminus j(X)$ consists of only one point.

18.16 Proposition. *If a space X has a one-point compactification $j\colon X \to Z$ then this compactification is unique up to homeomorphism. That is, if $j'\colon X \to Z'$ is another one-point compactification of X then there exists a homeomorphism $h\colon Z \to Z'$ such that $j' = hj$.*

Proof. Exercise. $\square$

18.17 Definition. A topological space X is *locally compact* if every point $x \in X$ has an open neighborhood $U_x \subseteq X$ such that the the closure $\overline{U}_x$ is compact.

18.19 Theorem. *Let X be a non-compact topological space. The following conditions are equivalent:*

1) *The space X is locally compact and Hausdorff.*
2) *There exists a one-point compactification of X.*

18.20 Corollary. *If X is a locally compact Hausdorff space then X is completely regular.*

Proof. Follows from Theorem 18.7 and Theorem 18.19. □

18.21 Corollary. *Let X be a topological space. The following conditions are equivalent:*

1) *The space X is locally compact and Hausdorff .*
2) *There exists an embedding $i\colon X \to Y$ where Y is compact Hausdorff space and $i(X)$ is an open set in Y.*

18.22 Proposition. *Let X be a non-compact, locally compact space and let $j\colon X \to X^+$ be the one-point compactification of X. For every compactification $i\colon X \to Y$ of X we have $Y \geq X^+$.*

Proof. Exercise. $\square$