

21 | Embeddings of Manifolds

21.1 Definition. Let X be a topological space and let $f: X \rightarrow \mathbb{R}$ be a continuous function. The *support* of f is the closure of the subset of X consisting of points with non-zero values:

$$\text{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$$

21.2 Definition. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X . A *partition of unity subordinate to \mathcal{U}* is a family of continuous functions $\{\lambda_i: X \rightarrow [0, 1]\}_{i \in I}$ such that

- (i) $\text{supp}(\lambda_i) \subseteq U_i$ for each $i \in I$;
- (ii) each point $x \in X$ has an open neighborhood U_x such that $U_x \cap \text{supp}(\lambda_i) \neq \emptyset$ for finitely many $i \in I$ only;
- (iii) for each $x \in X$ we have $\sum_{i \in I} \lambda_i(x) = 1$.

21.4 Lemma. Let X be a topological space, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and let $\{\lambda_i\}_{i \in I}$ be a partition of unity subordinate to \mathcal{U} .

1) Let $i \in I$ and let $f_i: U_i \rightarrow \mathbb{R}^n$ be a continuous function. Then the function $\tilde{f}_i: X \rightarrow \mathbb{R}^n$ given by

$$\tilde{f}_i(x) = \begin{cases} \lambda_i(x)f_i(x) & \text{for } x \in U_i \\ 0 & \text{for } x \in X \setminus U_i \end{cases}$$

is continuous.

2) Assume that for each $i \in I$ we have a continuous function $f_i: U_i \rightarrow \mathbb{R}^n$, and let $\tilde{f}_i: X \rightarrow \mathbb{R}^n$ be the function defined as above. Then the function $\tilde{f}: X \rightarrow \mathbb{R}^n$ given by

$$\tilde{f}(x) = \sum_{i \in I} \tilde{f}_i(x)$$

is continuous.

Proof. Exercise. □

21.5 Proposition. *Let X be a normal space. For any finite open cover $\{U_1, \dots, U_n\}$ of X there exists a partition of unity subordinate to this cover.*

21.6 Finite Shrinking Lemma. *Let X be a normal space and let $\{U_1, \dots, U_n\}$ be a finite open cover of X . There exists an open cover $\{V_1, \dots, V_n\}$ of X such that $\bar{V}_i \subseteq U_i$ for each $i \geq 1$.*

21.7 Shrinking Lemma. *Let X be a normal space and let $\{U_i\}_{i \in I}$ be an open cover of X such that each point of X belongs to finitely many sets U_i only. There exists an open cover $\{V_i\}_{i \in I}$ of X such that $\bar{V}_i \subseteq U_i$ for all $i \in I$.*

Proof. Exercise. □

Proof of Proposition 21.5.

□

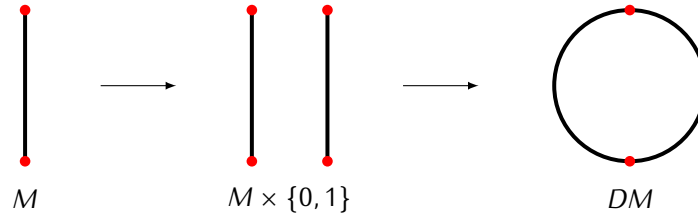
21.8 Corollary. *If X is a compact Hausdorff space then for every open cover \mathcal{U} of X there exists an partition of unity subordinate to \mathcal{U} .*

21.9 Theorem. *If M is a compact manifold without boundary then for some $N \geq 0$ there exists an embedding $j: M \rightarrow \mathbb{R}^N$.*

21.11 Definition. Let M be a manifold with boundary ∂M . The *double* of M is the topological space

$$DM = M \times \{0, 1\} / \sim$$

where $\{0, 1\}$ is the discrete space with two points and \sim is the equivalence relation on $M \times \{0, 1\}$ given by $(x, 0) \sim (x, 1)$ for all $x \in \partial M$.



21.12 Proposition. If M is an n -dimensional manifold with boundary then DM is an n -dimensional manifold without boundary. Moreover, if M is compact then so is DM .

Proof. Exercise. □

21.13 Corollary. If M is a compact manifold with boundary then for some $N > 0$ there exists an embedding $M \rightarrow \mathbb{R}^N$.