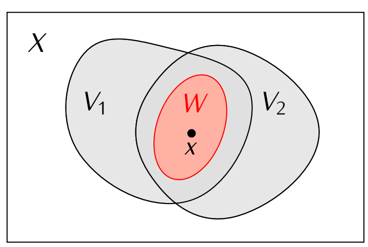
4 | Basis, Subbasis, Subspace

4.1 Definition. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. The collection $\mathcal{B}$ is a *basis* on X if it satisfies the following conditions:

1) $X = \bigcup_{V \in \mathcal{B}} V$;
2) for any $V_1, V_2 \in \mathcal{B}$ and $x \in V_1 \cap V_2$ there exists $W \in \mathcal{B}$ such that $x \in W$ and $W \subseteq V_1 \cap V_2$.



4.2 Example. If (X, ϱ) is a metric space then the set $\mathcal{B} = \{B(x, r) \mid x \in X, \ r > 0\}$ consisting of all open balls in X is a basis on X (exercise).

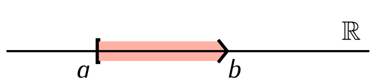
4.3 Proposition. *Let X be a set, and let $\mathcal{B}$ be a basis on X. Let $\mathcal{T}$ denote the collection of all subsets $U \subseteq X$ that can be obtained as the union of some elements of $\mathcal{B}$: $U = \bigcup_{V \in \mathcal{B}_1} V$ for some $\mathcal{B}_1 \subseteq \mathcal{B}$. Then $\mathcal{T}$ is a topology on X.*

Proof. Exercise. □

4.4 Definition. Let $\mathcal{B}$ be a basis on a set X and let $\mathcal{T}$ be the topology defined as in Proposition 4.3. In such case we will say that $\mathcal{B}$ is a *basis of the topology* $\mathcal{T}$ and that $\mathcal{T}$ is the *topology defined by the basis* $\mathcal{B}$.

4.6 Example. Consider $\mathbb{R}^n$ with the Euclidean metric d. Let $\mathcal{B}$ be the collection of all open balls $B(x, r) \subseteq \mathbb{R}^n$ such that $r \in \mathbb{Q}$ and $x = (x_1, x_2, \dots, x_n)$ where $x_1, \dots, x_n \in \mathbb{Q}$. Then $\mathcal{B}$ is a basis of the Euclidean topology on $\mathbb{R}^n$ (exercise).

4.8 Example. The set $\mathcal{B} = \{[a, b) \mid a, b \in \mathbb{R}\}$ is a basis of a certain topology on $\mathbb{R}$. We will call it the *arrow topology*.



4.9 Example. Let $\mathcal{B} = \{[a, b] \mid a, b \in \mathbb{R}\}$. The set $\mathcal{B}$ is a basis of the discrete topology on $\mathbb{R}$ (exercise).

4.10 Example. Let $X = \{a, b, c, d\}$ and let $\mathcal{B} = \{\{a, b, c\}, \{b, c, d\}\}$. The set $\mathcal{B}$ is not a basis of any topology on X since $b \in \{a, b, c\} \cap \{b, c, d\}$, and $\mathcal{B}$ does not contain any subset W such that $b \in W$ and $W \subseteq \{a, b, c\} \cap \{b, c, d\}$.

4.11 Proposition. *Let X be a set and let $\mathcal{S}$ be any collection of subsets of X such that $X = \bigcup_{V \in \mathcal{S}} V$. Let $\mathcal{T}$ denote the collection of all subsets of X that can be obtained using two operations:*

1) *taking finite intersections of sets in $\mathcal{S}$;*
2) *taking arbitrary unions of sets obtained in 1).*

Then $\mathcal{T}$ is a topology on X.

Proof. Exercise. □

4.12 Definition. Let X be a set and let $\mathcal{S}$ be any collection of subsets of X such that $X = \bigcup_{V \in \mathcal{S}} V$. The topology $\mathcal{T}$ defined by Proposition 4.11 is called the *topology generated by* $\mathcal{S}$, and the collection $\mathcal{S}$ is called a *subbasis* of $\mathcal{T}$.

4.13 Example. If $X = \{a, b, c, d\}$ and $\mathcal{S} = \{\{a, b, c\}, \{b, c, d\}\}$ then the topology generated by $\mathcal{S}$ is $\mathcal{T} = \{\{a, b, c\}, \{b, c, d\}, \{b.c\}, \{a, b, c, d\}, \varnothing\}$.

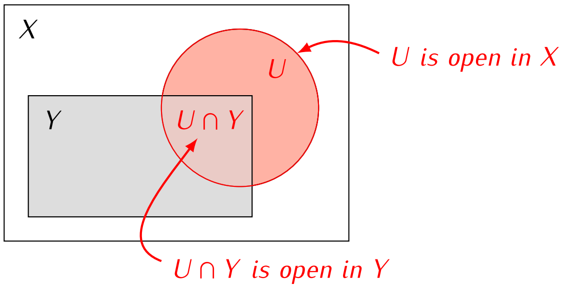
4.14 Proposition. *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{B}$ be a basis (or a subbasis) of $\mathcal{T}_Y$. A function $f \colon X \to Y$ is continuous if and only if $f^{-1}(V) \in \mathcal{T}_X$ for every $V \in \mathcal{B}$.*

Proof. Exercise. □

4.15 Definition. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. The collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

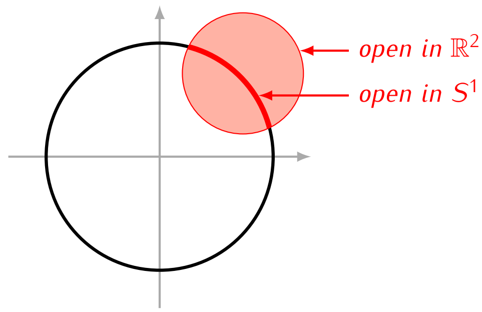
is a topology on Y called the *subspace topology*. We say that $(Y, \mathcal{T}_Y)$ is a *subspace* of the topological space $(X, \mathcal{T})$.



4.16 Example. The unit circle S^1 is defined by

$$S^1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$

The circle S^1 is a topological space considered as a subspace of $\mathbb{R}^2$.



In general the n-dimensional sphere S^n is defined by

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

It is a topological space considered as a subspace of $\mathbb{R}^{n+1}$.

4.17 Example. Consider $\mathbb{Z}$ as a subspace of $\mathbb{R}$. The subspace topology on $\mathbb{Z}$ is the same as the discrete topology (exercise).

4.18 Proposition. *Let X be a topological space and let Y be its subspace.*

1) The inclusion map $j\colon Y \to X$ is a continuous function.
2) If Z is a topological space then a function $f\colon Z \to Y$ is continuous if and only if the composition $jf\colon Z \to X$ is continuous.

Proof. Exercise. □

4.19 Proposition. *Let X be a topological space and let Y be its subspace. If $\mathcal{B}$ is a basis (or a subbasis) of X then the set $\mathcal{B}_Y = \{U \cap Y \mid U \in \mathcal{B}\}$ is a basis (resp. a subbasis) of Y.*

Proof. Exercise. □