5 | Closed Sets, Interior, Closure, Boundary

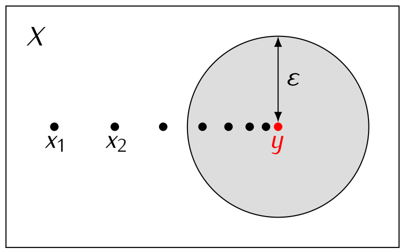
5.1 Definition. Let X be a topological space. A set $A \subseteq X$ is a *closed set* if the set $X \smallsetminus A$ is open.

5.5 Proposition. *Let X be a topological space.*

1) *The sets X, $\varnothing$ are closed.*
2) *If $A_i \subseteq X$ is a closed set for $i \in I$ then $\bigcap_{i\in I} A_i$ is closed.*
3) *If A_1, A_2 are closed sets then the set $A_1 \cup A_2$ is closed.*

5.7 Definition. Let (X, ϱ) be a metric space, and let $\{x_n\}$ be a sequence of points in X. We say that $\{x_n\}$ *converges* to a point $y \in X$ if for every $\varepsilon > 0$ there exists $N > 0$ such that $\varrho(y, x_n) < \varepsilon$ for all $n > N$. We write: $x_n \to y$.

Equivalently: $x_n \to y$ if for every $\varepsilon > 0$ there exists $N > 0$ such that $x_n \in B(y, \varepsilon)$ for all $n > N$.



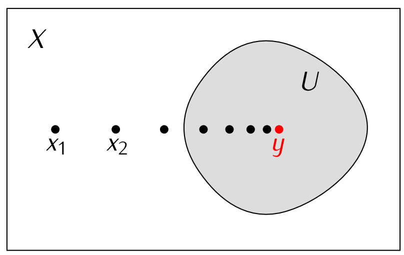
5.8 Proposition. *Let (X, ϱ) be a metric space and let $A \subseteq X$. The following conditions are equivalent:*

1) *The set A is closed in X.*
2) *If $\{x_n\} \subseteq A$ and $x_n \to y$ then $y \in A$.*

Proof. Exercise. $\square$

5.10 Definition. Let X be a topological space and $y \in X$. If $U \subseteq X$ is an open set such that $y \in U$ then we say that U is an *open neighborhood of* y.

5.11 Definition. Let X be a topological space. A sequence $\{x_n\} \subseteq X$ *converges* to $y \in X$ if for every open neighborhood U of y there exists $N > 0$ such that $x_n \in U$ for $n > N$.



5.12 Note. In general topological spaces a sequence may converge to many points at the same time.

5.13 Proposition. *Let (X, ϱ) be a metric space and let $\{x_n\}$ be a sequence in X. If $x_n \to y$ and $x_n \to z$ for some $y, z \in X$ then $y = z$.*

Proof. Exercise. $\square$

5.14 Proposition. *Let X be a topological space and let $A \subseteq X$ be a closed set. If $\{x_n\} \subseteq A$ and $x_n \to y$ then $y \in A$.*

Proof. Exercise. $\square$

5.16 Example. Let $X = \mathbb{R}$ with the following topology:

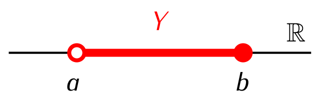
$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \varnothing \text{ or } U = (\mathbb{R} \smallsetminus S) \text{ for some countable set } S \subseteq \mathbb{R}\}$$

Closed sets in X are the whole space $\mathbb{R}$ and all countable subsets of $\mathbb{R}$. If $\{x_n\} \subseteq X$ is a sequence then $x_n \to y$ if and only if there exists $N > 0$ such that $x_n = y$ for all $n > N$ (exercise). It follows that if A is any (closed or not) subset of X, $\{x_n\} \subseteq A$, and $x_n \to y$ then $y \in A$.

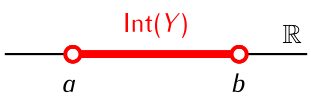
5.17 Definition. Let X be a topological space and let $Y \subseteq X$.

- The *interior of* Y is the set $\mathrm{Int}(Y) := \bigcup \{U \mid U \subseteq Y \text{ and } U \text{ is open in } X\}$.
- The *closure of* Y is the set $\overline{Y} := \bigcap \{A \mid Y \subseteq A \text{ and } A \text{ is closed in } X\}$.
- The *boundary of* Y is the set $\mathrm{Bd}(Y) := \overline{Y} \cap \overline{(X \smallsetminus Y)}$.

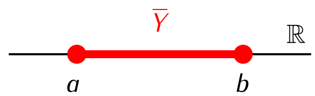
5.18 Example. Consider the set $Y = (a, b]$ in $\mathbb{R}$:



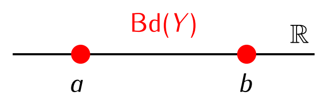
We have:



$\mathrm{Int}(Y) = (a, b)$

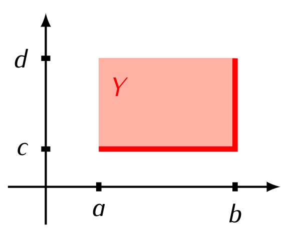


$\overline{Y} = [a, b]$

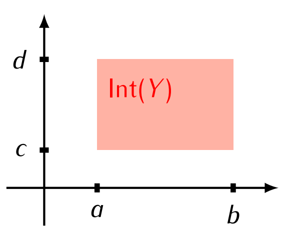


$\mathrm{Bd}(Y) = \{a, b\}$

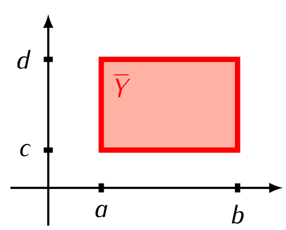
5.19 Example. Consider the set $Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid a < x_1 \leq b,\ c \leq x_2 < d\}$ in $\mathbb{R}^2$:



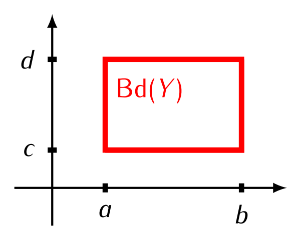
We have:



$\mathrm{Int}(Y) = (a, b) \times (c, d)$



$\overline{Y} = [a, b] \times [c, d]$



$$\mathrm{Bd}(Y) = [a, b] \times \{c, d\} \cup \{a, b\} \times [c, d]$$

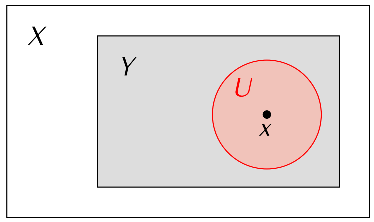
5.20 Proposition. *Let X be a topological space and let $Y \subseteq X$.*

1) *The set* $\operatorname{Int}(Y)$ *is open in X. It is the biggest open set contained in Y: if U is open and $U \subseteq Y$ then $U \subseteq \operatorname{Int}(Y)$.*
2) *The set $\overline{Y}$ is closed in X. It is the smallest closed set that contains Y: if A is closed and $Y \subseteq A$ then $\overline{Y} \subseteq A$.*

Proof. Exercise. $\square$

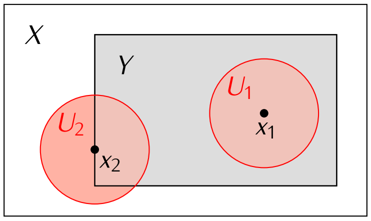
5.21 Proposition. *Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:*

1) $x \in \mathrm{Int}(Y)$
2) *There exists an open neighborhood U of x such that $U \subseteq Y$.*



5.22 Proposition. *Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:*

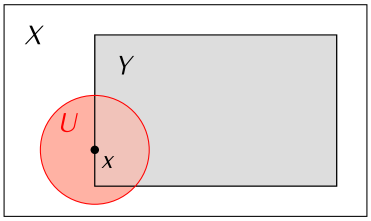
1) $x \in \overline{Y}$
2) *For every open neighborhood U of x we have $U \cap Y \neq \varnothing$.*



Proof. Exercise. □

5.23 Proposition. *Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:*

1) $x \in \mathrm{Bd}(Y)$
2) *For every open neighborhood U of x we have $U \cap Y \neq \varnothing$ and $U \cap (X \smallsetminus Y) \neq \varnothing$.*



5.24 Definition. Let X be a topological space. A set $Y \subseteq X$ is *dense in X* if $\overline{Y} = X$.

5.25 Proposition. *Let X be a topological space and let $Y \subseteq X$. The following conditions are equivalent:*

1) *Y is dense in X*
2) *If $U \subseteq X$ is an open set and $U \neq \varnothing$ then $U \cap Y \neq \varnothing$.*

5.26 Example. The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.