MTH 427/527

Introduction to Topology I

General Topology

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1 | Some Set Theory

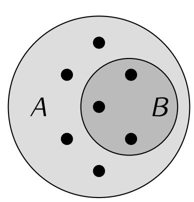
A topological space is a set equipped with some additional structure which, roughly speaking, specifies which elements of the set are close to each other. This lets us define what it means that a function between topological spaces is continuous: intuitively, such function maps elements which are close in one space to elements which are close in the other space. Before we start discussing topological spaces and continuous functions in detail it will be worth go over the basics notions related to sets and functions between sets. This chapter is intended as a quick review of this material. We will also fix here some notation and terminology.

Sets. In general sets will be denoted by capital letters: $A, B, C, \ldots$ We will also use the following notation for sets that will be of a particularly interest:

$\varnothing =$ the empty set (i.e. the set that contains no elements)
$\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers
$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ the set of positive integers
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ the set of integers
$\mathbb{Q} =$ the set of rational numbers
$\mathbb{R} =$ the set of real numbers

We will write $a \in B$ to denote that a is an element of the set B and $a \notin B$ to indicate that a is not an element of B. For example, $5 \in \mathbb{Z}$, $\frac{1}{3} \notin \mathbb{Z}$.

1.1 Definition. A set B is a *subset* of a set A if every element of B is in A. In such case we write $B \subseteq A$.



A set B is a *proper subset* of A if $B \subseteq A$ and $B \neq A$.

1.2 Example. $\varnothing \subseteq \mathbb{Z}^+ \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

1.3 Example. Here are some often used subsets of $\mathbb{R}$:

1) an open interval:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

2) a closed interval:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

3) a half open interval:

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

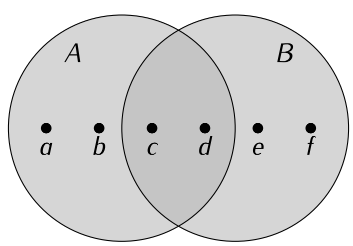
1.4 Definition. The *union* of sets A and B is the set $A \cup B$ that consists of all elements that belong to either A or B:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The *intersection* of sets A and B is the set $A \cap B$ that consists of all elements that belong to both A and B:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

1.5 Example. If $A = \{a, b, c, d\}$, $B = \{c, d, e, f\}$ then $A \cup B = \{a, b, c, d, e, f\}$ and $A \cap B = \{c, d\}$.



1.6 Example. If $A = \{a, b, c, d\}$ and $C = \{e, f, g\}$ then $A \cap C = \varnothing$.

1.7 Definition. If $A \cap B = \varnothing$ then we say that A and B are *disjoint sets*.

Definition 1.4 can be extended to unions and intersections of arbitrary families of sets. If $\{A_i\}_{\in I}$ is a family of sets then

$$\bigcup_{i\in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$
$$\bigcap_{i\in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

1.8 Example. For $n \in \mathbb{Z}$ let $A_n = [n, n+1]$. Then

$$\bigcup_{n\in\mathbb{Z}} A_n = \ldots \cup\ [-2,-1]\ \cup\ [-1,0]\ \cup\ [0,1]\ \cup\ [1,2]\ \cup \ldots = \mathbb{R}$$

1.9 Example. For $n = 1, 2, 3, \ldots$ let $B_n = (-\frac{1}{n}, \frac{1}{n})$. Then

$$\bigcap_{n} B_n = (-1,1)\ \cap\ (-\tfrac{1}{2}, \tfrac{1}{2})\ \cap\ (-\tfrac{1}{3}, \tfrac{1}{3})\ \cap \ldots = \{0\}$$

1.10 Definition. The *difference* of sets A and B is the set $A \smallsetminus B$ consisting of the elements of A that do not belong to B:

$$A \smallsetminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

1.11 Example. $A = \{a, b, c, d\}$, $B = \{c, d, e, f\}$

$$A \smallsetminus B = \{a, b\}$$
$$B \smallsetminus A = \{e, f\}$$

1.12 Definition. If $A \subseteq B$ then the set $B \smallsetminus A$ is called the *complement* of A in B.

1.13 Properties of the algebra of sets. Here are some basic formulas involving the operations of sets defined above. We will use them very often.

Distributivity:

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

De Morgan's Laws:

$$A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$$
$$A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$$

1.14 Definition. The *Cartesian product* of sets A, B is the set consisting of all ordered pairs of elements of A and B:

$$A \times B = \{(a, b) \mid a \in A, \quad b \in B\}$$

1.15 Example. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$

$$A \times B = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$$

1.16 Notation. Given a set A by A^n we will denote the n-fold Cartesian product of A:

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

1.17 Example.

$$\begin{aligned}\mathbb{R}^2 =& \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}\\ \mathbb{R}^3 =& \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}\end{aligned}$$

1.18 Infinite products. Let $A_1, ..., A_n$ be a collection of n sets. Notice that elements of the product $A_1 \times \cdots \times A_n$ can be identified with functions $f\colon \{1, 2, \dots, n\} \to \bigcup_{i=1}^n A_i$ such that $f(i) \in A_i$. Indeed, every such function defines an element $(f(1), f(2), \dots, f(n)) \in A_1 \times \cdots \times A_n$. Conversely, every element $(a_1, \dots, a_n) \in A_1 \times \cdots \times A_n$ defines a function $f\colon \{1, 2, \dots, n\} \to \bigcup_{i=1}^n A_i$ given by $f(i) = a_i$. We can use this observation to define products of an arbitrary (finite or infinite) families of sets. If $\{A_i\}_{i\in I}$ is a family of sets then $\prod_{i\in I} A_i$ is the set consisting of all functions $f\colon I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$.

1.19 Example. for $r \in \mathbb{R}$ let $A_r = [r, r + 1]$. Then $\prod_{r\in\mathbb{R}} A_r$ is the set consisting all functions $f\colon \mathbb{R} \to \bigcup_{r\in\mathbb{R}}[r, r + 1] = \mathbb{R}$ such that $f(r) \in [r, r + 1]$ for all $r \in \mathbb{R}$.

1.20 Note. We will usually denote elements of $\prod_{i\in I} A_i$ by $(a_i)_{i\in I}$. This notation indicates the element defined by the function $f\colon I \to \bigcup_{i\in I} A_i$ given by $f(i) = a_i$.

In many cases given a set A we are interested in describing a relation satisfied by some pairs of elements of the set. Here are some examples of such relations:

1.21 Example. In the set $\mathbb{R}$ of real numbers we can consider the relation "$<$". Numbers $a, b \in \mathbb{R}$ satisfy this relation if $b - a$ is a positive number. We write then $a < b$.

1.22 Example. In the set $\mathbb{Z}$ of integers we can consider the divisibility relation "$|$". Integers $a, b \in \mathbb{Z}$ satisfy this relation if $b = an$ for some $n \in \mathbb{Z}$. In such case we write $a|b$.

1.23 Example. In any set A we can define the equality relation "$=$" which is satisfied by elements $a, b \in A$ only if a and b are the same element.

Formally we define binary relations as follows:

1.24 Definition. A *binary relation* on a set A is a subset $R \subseteq A \times A$. If $(a, b) \in R$ then we write aRb.

1.25 Example. The divisibility relation on the set of integers is the subset $R \subseteq \mathbb{Z} \times \mathbb{Z}$ given by

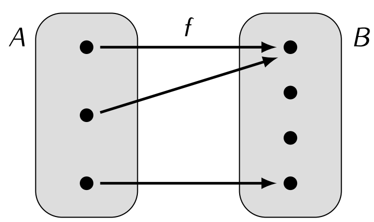
$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b = an \text{ for some } n \in \mathbb{Z}\}$$

1.26 Example. The equality relation on a set A is the subset of $R \subseteq A \times A$ where

$$R = \{(a, a) \in A \times A \mid a \in A\}$$

1.27 Definition. Let A, B be sets

1) A function $f\colon A \to B$ is *1-1* if $f(x) = f(x')$ only if $x = x'$.



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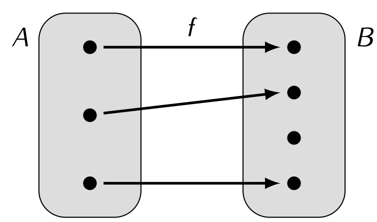
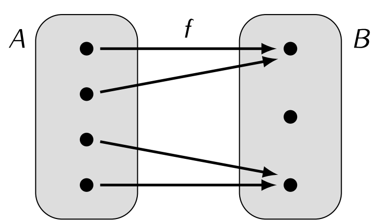
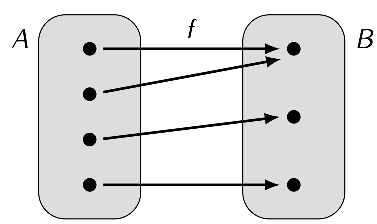


1-1

2) A function $f\colon A \to B$ is *onto* if for every $y \in B$ there is $x \in A$ such that $f(x) = y$

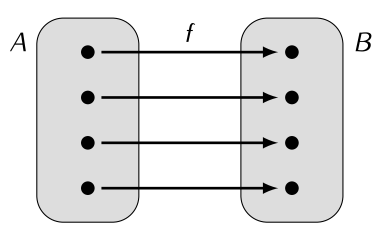


not onto



onto

3) A function $f\colon A \to B$ is a *bijection* if f is both 1-1 and onto.

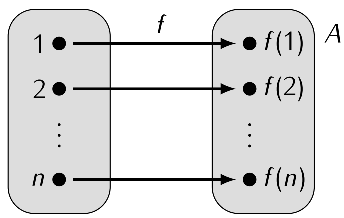


bijection

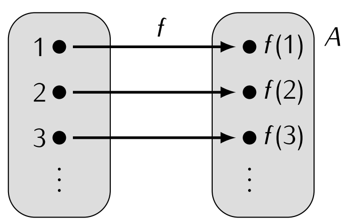
1.28 Note. 1) If $f\colon A \to B$ is a bijection then the inverse function $f^{-1}\colon B \to A$ exists and it is also a bijection.
2) If $f\colon A \to B$ and $g\colon B \to C$ are bijections then the function $gf\colon A \to C$ is also a bijection.

1.29 Definition. Sets A, B *have the same cardinality* if there exists a bijection $f\colon A \to B$. In such case we write $|A| = |B|$.

1.30 Definition. A set A is *finite* if either $A = \varnothing$ or A has the same cardinality as the set $\{1, \dots, n\}$ for some $n \geq 1$.



1.31 Definition. A set A is *infinitely countable* if it is has the same cardinality as the set $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$



1.32 Definition. A set A is *countable* if it is either finite or infinitely countable.

1.33 Example. The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is countable since we have a bijection $f\colon \mathbb{Z}^+ \to \mathbb{N}$ given by $f(k) = k - 1$.

1.34 Example. The set of integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is countable since we have a bijection $f\colon \mathbb{Z}^+ \to \mathbb{Z}$ given by

$$f(k) = \begin{cases} k/2 & \text{if } k \text{ is even} \\ (1-k)/2 & \text{if } k \text{ is odd} \end{cases}$$

In other words:

$$f(1) = 0,\ f(2) = 1,\ f(3) = -1,\ f(4) = 2,\ f(5) = -2,\ f(6) = 3,\ \dots$$

1.35 Example. The set of rational numbers $\mathbb{Q}$ is countable. A bijection $f\colon \mathbb{Z}^+ \to \mathbb{Q}$ can be constructed as follows:

$$\begin{array}{cccccccc}
{}^0/_1 & {}^1/_1 & -{}^1/_1 & {}^2/_1 & -{}^2/_1 & {}^3/_1 & -{}^3/_1 & \cdots \\
{}^0/_2 & {}^1/_2 & -{}^1/_2 & {}^2/_2 & -{}^2/_2 & {}^3/_2 & -{}^3/_2 & \cdots \\
{}^0/_3 & {}^1/_3 & -{}^1/_3 & {}^2/_3 & -{}^2/_3 & {}^3/_3 & -{}^3/_3 & \cdots \\
{}^0/_4 & {}^1/_4 & -{}^1/_4 & {}^2/_4 & -{}^2/_4 & {}^3/_4 & -{}^3/_4 & \cdots \\
{}^0/_5 & {}^1/_5 & -{}^1/_5 & {}^2/_5 & -{}^2/_5 & {}^3/_5 & -{}^3/_5 & \cdots \\
{}^0/_6 & {}^1/_6 & -{}^1/_6 & {}^2/_6 & -{}^2/_6 & {}^3/_6 & -{}^3/_6 & \cdots \\
{}^0/_7 & {}^1/_7 & -{}^1/_7 & {}^2/_7 & -{}^2/_7 & {}^3/_7 & -{}^3/_7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
\qquad
\begin{array}{rcrcl}
 & & {}^0/_1 & = & f(1) \\
{}^0/_2 & = & {}^0/_1 & = & f(1) \\
 & & {}^1/_1 & = & f(2) \\
 & & -{}^1/_1 & = & f(3) \\
 & & {}^1/_2 & = & f(4) \\
{}^0/_3 & = & {}^0/_1 & = & f(1) \\
{}^0/_4 & = & {}^0/_1 & = & f(1) \\
 & & {}^1/_3 & = & f(5) \\
 & & -{}^1/_2 & = & f(6) \\
 & & {}^2/_1 & = & f(7) \\
 & & \cdots & & \cdots
\end{array}$$

Here are some properties of countable sets:

1.36 Theorem. *1) If A is a countable set and $B \subseteq A$ then B is countable.*
2) If $\{A_1, A_2, \dots\}$ is a collection of countably many countable sets then the set $\bigcup_{i=1}^{\infty} A_i$ is countable.
3) If $\{A_1, A_2, \dots, A_n\}$ is a collection of finitely many countable sets then the set $A_1 \times \cdots \times A_n$ is countable.

1.37 Example. The set of all real numbers in the interval (0, 1) is not countable. Indeed, assume by contradiction that there exists a bijection $f\colon \mathbb{Z}^+ \to (0,1)$. Then we would have:

$$\begin{aligned}
f(1) &= 0.d^1_1 d^1_2 d^1_3 \dots \\
f(2) &= 0.d^2_1 d^2_2 d^2_3 \dots \\
f(3) &= 0.d^3_1 d^3_2 d^3_3 \dots \\
\dots \quad & \dots \quad \dots
\end{aligned}$$

where $d^k_1, d^k_2, d^k_3, \dots$ are digits in the decimal expansion of the number $f(k) \in (0,1)$. Let $x \in (0,1)$ be the number defined as follows:

$$x = 0.x_1 x_2 x_3 \dots$$

where

$$x_i = \begin{cases} 1 & \text{if } d^i_i \neq 1 \\ 2 & \text{if } d^i_i = 1 \end{cases}$$

For example, if we have

$$\begin{aligned}
f(1) &= 0.31415\dots\\
f(2) &= 0.12345\dots\\
f(3) &= 0.75149\dots\\
f(4) &= 0.00032\dots\\
f(5) &= 0.11111\dots\\
\dots\quad &\dots\ \dots\ \dots
\end{aligned}$$

then

$$x = 0.11212\dots$$

Notice that:

$$\begin{aligned}
x &\neq f(1) \quad \text{since } x_1 \neq d_1^1\\
x &\neq f(2) \quad \text{since } x_2 \neq d_2^2\\
x &\neq f(3) \quad \text{since } x_3 \neq d_3^3\\
\dots &\qquad\quad \dots \quad \dots
\end{aligned}$$

In general $x \neq f(k)$ for all $k \in \mathbb{Z}^+$, and so f is not onto.

1.38 Example. The function $f\colon (0,1) \to \mathbb{R}$ given by $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$ is a bijection. It follows that $|\mathbb{R}| = |(0,1)|$. In particular $\mathbb{R}$ is an uncountable set.

1.39 Notation. 1) If A is a finite set of n elements then we write $|A| = n$.

2) If $|A| = |\mathbb{Z}^+|$ (i.e. A is an infinitely countable set) then we say that A has the *cardinality aleph naught* and we write $|A| = \aleph_0$.

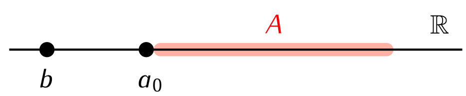
3) If $|A| = |\mathbb{R}|$ then we say that A has the *cardinality of the continuum* and we write $|A| = \mathfrak{c}$.

Infima and Suprema. In the following chapters we will often work with the set $\mathbb{R}$ of real numbers. In particular, we will often use suprema and infima of subsets of $\mathbb{R}$. We conclude this chapter with a quick review of these notions.

1.40 Definition. Let $A \subseteq \mathbb{R}$. The set A is *bounded below* if there exists a number b such that $b \leq x$ for all $x \in A$. The set A is *bounded above* if there exists a number c such that $x \leq c$ for all $x \in A$. The set A is *bounded* if it is both bounded below and bounded above.

1.41 Definition. Let $A \subseteq \mathbb{R}$. If the set A is bounded below then the *greatest lower bound* of A (or *infimum* of A) is a number $a_0 \in \mathbb{R}$ such that:

1) $a_0 \leq x$ for all $x \in A$
2) if $b \leq x$ for all $x \in A$ then $b \leq a_0$



We write: $a_0 = \inf A$.

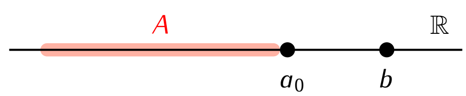
If the set A is not bounded below then we set $\inf A := -\infty$.

1.42 Example.
1) If $A = [0, 1]$ then $\inf A = 0$.
2) If $B = (0, 1)$ then $\inf B = 0$.
3) $\inf \mathbb{Z} = -\infty$

1.43 Theorem. *For any non-empty bounded below subset $A \subseteq \mathbb{R}$ the number $\inf A$ exists.*

1.44 Definition. Let $A \subseteq \mathbb{R}$. If the set A is bounded above then the *least upper bound* of A (or *supremum* of A) is a number $a_0 \in \mathbb{R}$ such that:

1) $x \leq a_0$ for all $x \in A$
2) if $x \leq b$ for all $x \in A$ then $a_0 \leq b$



We write: $a_0 = \sup A$.

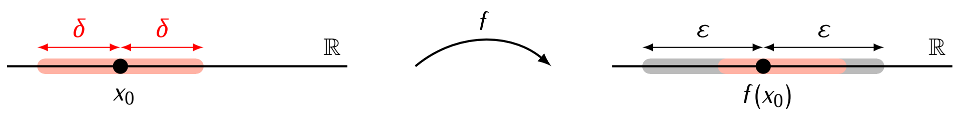
If the set A is not bounded above then we set $\sup A := +\infty$.

1.45 Example.
1) If $A = [0, 1]$ then $\sup A = 1$.
2) If $B = (0, 1)$ then $\sup B = 1$.
3) $\sup \mathbb{Z} = +\infty$

1.46 Theorem. *For any non-empty bounded above subset $A \subseteq \mathbb{R}$ the number $\sup A$ exists.*

2 | Metric Spaces

Recall that a function $f: \mathbb{R} \to \mathbb{R}$ is *continuous at a point* $x_0 \in \mathbb{R}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x_0 - x| < \delta$ then $|f(x_0) - f(x)| < \varepsilon$:

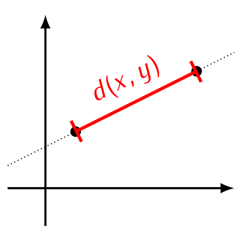


A function is *continuous* if it is continuous at every point $x_0 \in \mathbb{R}$.

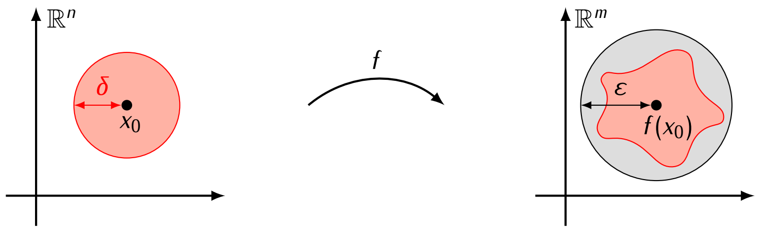
Continuity of functions of several variables $f: \mathbb{R}^n \to \mathbb{R}^m$ is defined in a similar way. Recall that $\mathbb{R}^n := \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are two points in $\mathbb{R}^n$ then the distance between x and y is given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

The number $d(x, y)$ is the length of the straight line segment joining the points x and y:



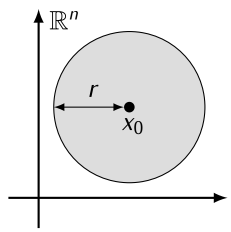
2.1 Definition. A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is *continuous at* $x_0 \in \mathbb{R}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x_0, x) < \delta$ then $d(f(x_0), f(x)) < \varepsilon$.



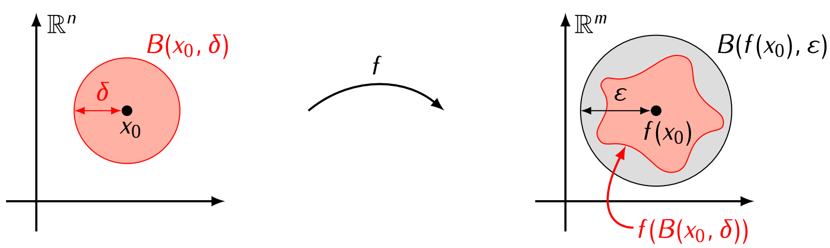
The above picture motivates the following, more geometric reformulation of continuity:

2.2 Definition. Let $x_0 \in \mathbb{R}^n$ and let $r > 0$. An *open ball* with radius r and with center at x_0 is the set

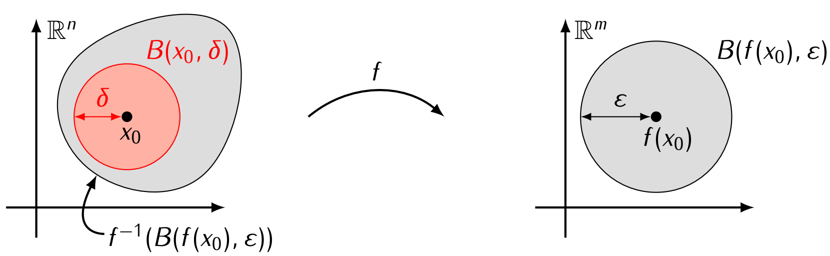
$$B(x_0, r) = \{x \in \mathbb{R}^n \mid d(x_0, x) < r\}$$



Using this terminology we can say that a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x_0 if for each $\varepsilon > 0$ there is a $\delta > 0$ such $f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$:



Here is one more way of rephrasing the definition of continuity: $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$:



Notice that in order to define continuity of functions $\mathbb{R}^n \to \mathbb{R}^m$ we used only the fact the for any two points in $\mathbb{R}^n$ or $\mathbb{R}^m$ we can compute the distance between these points. This suggests that we could define similarly what is means that a function $f\colon X \to Y$ is continuous where X and Y are any sets, provided that we have some way of measuring distances between points in these sets. This observation leads to the notion of a metric space:

2.3 Definition. A *metric space* is a pair (X, ϱ) where X is a set and ϱ is a function

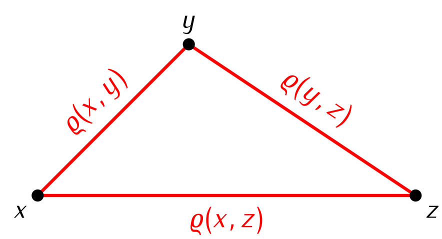
$$\varrho\colon X \times X \to \mathbb{R}$$

that satisfies the following conditions:

1) $\varrho(x, y) \geq 0$ and $\varrho(x, y) = 0$ if and only if $x = y$;
2) $\varrho(x, y) = \varrho(y, x)$;
3) for any $x, y, z \in X$ we have $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$.

The function ϱ is called a *metric* on the set X. For $x, y \in X$ the number $\varrho(x, y)$ is called the *distance* between x and y.

The first condition in Definition 2.3 says that distances between points of X are non-negative, and that the only point located within the distance zero from a point x is the point x itself. The second condition says that the distance from x to y is the same as the distance from y to x. The third condition is called the *triangle inequality*. It says that the distance between points x and z measured directly will never be bigger than the number we obtain by taking the distance from x to some intermediary point y and adding it to the distance between y and z:



We define continuity of functions between metric spaces the same way as for functions between $\mathbb{R}^n$ and $\mathbb{R}^m$:

2.4 Definition. Let (X, ϱ) and (Y, μ) be metric spaces. A function $f\colon X \to Y$ is *continuous at* $x_0 \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varrho(x_0, x) < \delta$ then $\mu(f(x_0), f(x)) < \epsilon$.

A function $f\colon X \to Y$ is *continuous* if it is continuous at every point $x_0 \in X$.

We can reformulate this definition in terms of open balls:

2.5 Definition. Let (X, ϱ) be a metric space. For $x_0 \in X$ and let $r > 0$ the *open ball* with radius r and with center at x_0 is the set

$$B_\varrho(x_0, r) = \{x \in X \mid \varrho(x_0, x) < r\}$$

We will often write $B(x_0, r)$ instead of $B_\varrho(x_0, r)$ when it will be clear from the context which metric is being used.

Notice that a function $f\colon X \to Y$ between metric spaces (X, ϱ) and (Y, μ) is continuous at $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $B_\varrho(x_0, \delta) \subseteq f^{-1}(B_\mu(f(x_0), \varepsilon))$.

Here are some examples of metric spaces:

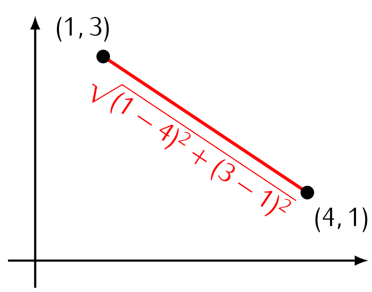
2.6 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ define:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

The metric d is called the *Euclidean metric* on $\mathbb{R}^n$.

For example, if $x = (1, 3)$ and $y = (4, 1)$ are points in $\mathbb{R}^2$ then

$$d(x, y) = \sqrt{(1 - 4)^2 + (3 - 1)^2} = \sqrt{13}$$



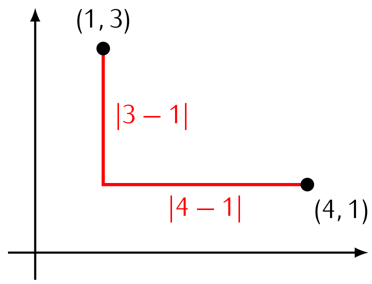
2.7 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ define:

$$\varrho_{ort}(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$$

The metric ϱ_{ort} is called the *orthogonal metric* on $\mathbb{R}^n$.

For example, if $x = (1, 3)$ and $y = (4, 1)$ are points in $\mathbb{R}^2$ then

$$\varrho_{ort}(x, y) = |1 - 4| + |3 - 1| = 5$$



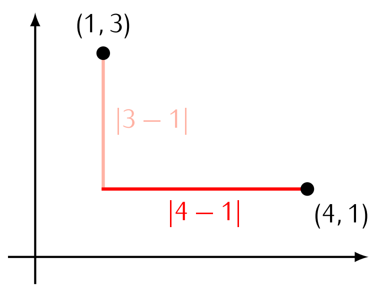
2.8 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define:

$$\varrho_{max}(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

The metric ϱ_{max} is called the *maximum metric* on $\mathbb{R}^n$.

For example, if $x = (1, 3)$ and $y = (4, 1)$ are points in $\mathbb{R}^2$ then

$$\varrho_{max}(x, y) = \max\{|1 - 4|, |3 - 1|\} = 3$$



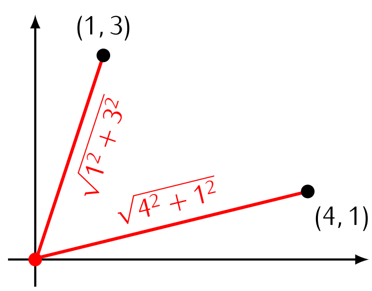
2.9 Example. Let $X = \mathbb{R}^n$. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ define $\varrho_h(x, y)$ as follows. If $x = y$ then $\varrho_h(x, y) = 0$. If $x \neq y$ then

$$\varrho_h(x, y) = \sqrt{x_1^2 + \cdots + x_n^2} + \sqrt{y_1^2 + \cdots + y_n^2}$$

The metric ϱ_h is called the *hub metric* on $\mathbb{R}^n$.

For example, if $x = (1, 3)$ and $y = (4, 1)$ are points in $\mathbb{R}^2$ then

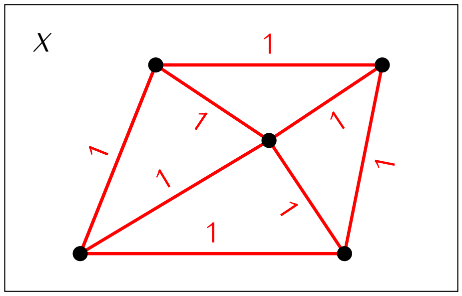
$$\varrho_h(x, y) = \sqrt{1^2 + 3^2} + \sqrt{4^2 + 1^2} = \sqrt{10} + \sqrt{17}$$



2.10 Example. Let X be any set. Define a metric ϱ_{disc} on X by

$$\varrho_{disc}(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The metric ϱ_{disc} is called the *discrete metric* on X.



2.11 Example. If (X, ϱ) is a metric space and $A \subseteq X$ then A is a metric space with the metric induced from X.

Exercises to Chapter 2

E2.1 Exercise. Verify the ϱ_{max} is a metric on $\mathbb{R}^n$.

E2.2 Exercise. For points $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ define

$$\varrho_{min}(x,y) = \min\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

Does this define a metric on $\mathbb{R}^n$? Justify your answer.

E2.3 Exercise. Let $\mathbb{Z}$ be a set of all integers, and let p be some fixed prime number. For $m, n \in \mathbb{Z}$ define

$$\varrho_p(m,n) := \begin{cases} 0 & \text{if } m = n \\ p^{-k} & \text{if } m - n = p^k r \text{ where } r \in \mathbb{Z},\ p \nmid r \end{cases}$$

Verify that ϱ_p is a metric on $\mathbb{Z}$. It is called the *p-adic metric*.

E2.4 Exercise. Let S be a set and let $\mathcal{F}(S)$ denote the set of all non-empty finite subsets of S. For $A, B \in \mathcal{F}(S)$ define

$$\varrho(A,B) = 1 - \frac{|A \cap B|}{|A \cup B|}$$

where $|A|$ denotes the number of elements of the set A. Show that ϱ is a metric on $\mathcal{F}(S)$.

E2.5 Exercise. Draw the following open balls in $\mathbb{R}^2$ defined by the specified metrics:

a) $B(x_0, 1)$ for $x_0 = (0, 0)$ and the orthogonal metric ϱ_{ort}.
b) $B(x_0, 1)$ for $x_0 = (0, 0)$ and the maximum metric ϱ_{max}.
c) $B(x_0, 1)$ for $x_0 = (0, 0)$ and the hub metric ϱ_h.
d) $B(x_0, 6)$ for $x_0 = (3, 4)$ and the hub metric ϱ_h.
e) $B(x_0, 1)$ for $x_0 = (3, 4)$ and the hub metric ϱ_h.

E2.6 Exercise. Let (X, ϱ) be a metric space, and let $x_0 \in X$, Show that if $x \in B(x_0, r)$ then exists $s > 0$ such that $B(x, s) \subseteq B(x_0, r)$.

E2.7 Exercise. a) Let (X, ϱ) be a metric space and let $B(x, r)$, $B(y, s)$ be open balls in X such that $B(y, s) \subseteq B(x, r)$ but $B(y, s) \neq B(x, r)$. Show that $s < 2r$.

b) Give an example of a metric space (X, ϱ) and open balls $B(x, r)$, $B(y, s)$ in X that satisfy the assumptions of part a) and such that $s > r$.

E2.8 Exercise. Let (X, ϱ_{disc}) be a discrete metric space and let (Y, μ) be some metric space. Show that every function $f\colon X \to Y$ is continuous.

E2.9 Exercise. Consider $\mathbb{R}^2$ as a metric space with the hub metric ϱ_h and $\mathbb{R}^1$ as a metric space with the Euclidean metric d.

a) Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) = (0, 0) \\ 1 & \text{otherwise} \end{cases}$$

is not continuous.

b) Show that the function $g\colon \mathbb{R}^2 \to \mathbb{R}^1$ given by

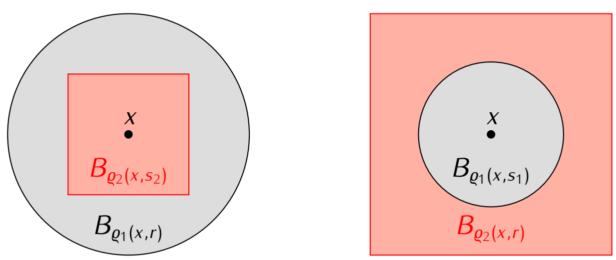
$$g(x_1, x_2) = \begin{cases} 0 & \text{if } x_1^2 + x_2^2 < 1 \\ 1 & \text{otherwise} \end{cases}$$

is continuous.

3 | Open Sets

We have seen that by equipping sets X, Y with metrics we can specify what it means that a function $f\colon X \to Y$ is continuous. In general continuity of functions depends on the choice of metrics: if we have two different metrics on X (or on Y) then a function $f\colon X \to Y$ that is continuous with respect to one of these metrics may be not continuous with respect to the other. This is however not always the case. Our first goal in this chapter will be to show that if two metrics on X or Y are *equivalent* then functions continuous with respect to one of them are continuous with respect to the other and vice versa.

3.1 Definition. Let ϱ_1 and ϱ_2 be two metrics on the same set X. We say that the metrics ϱ_1 and ϱ_2 are *equivalent* if for every $x \in X$ and for every $r > 0$ there exist $s_1, s_2 > 0$ such that $B_{\varrho_1}(x, s_1) \subseteq B_{\varrho_2}(x, r)$ and $B_{\varrho_2}(x, s_2) \subseteq B_{\varrho_1}(x, r)$.



3.2 Proposition. *Let ϱ_1, ϱ_2 be equivalent metrics on a set X, and let μ_1, μ_2 be equivalent metrics on a set Y. A function $f\colon X \to Y$ is continuous with respect to the metrics ϱ_1 and μ_1 if and only if it is continuous with respect to the metrics ϱ_2 and μ_2.*

Proof. Assume that f is continuous with respect to ϱ_1 and μ_1. We will show that it is also continuous with respect to ϱ_2 and μ_2 (the argument in the other direction is the same). Let $x \in X$ and let $\varepsilon > 0$. We need to show that there is $\delta > 0$ such that $B_{\varrho_2}(x, \delta) \subseteq f^{-1}(B_{\mu_2}(f(x), \varepsilon)))$. Since the metrics μ_1 and μ_2 are equivalent there exists $\varepsilon_1 > 0$ such that $B_{\mu_1}(f(x), \varepsilon_1)) \subseteq B_{\mu_2}(f(x), \varepsilon))$, and so $f^{-1}(B_{\mu_1}(f(x), \varepsilon_1))) \subseteq f^{-1}(B_{\mu_2}(f(x), \varepsilon)))$. Also, since by assumption f is continuous with respect to ϱ_1

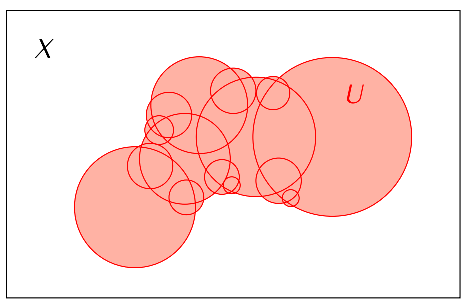
and μ_1, there is δ_1 such that $B_{\varrho_1}(x, \delta_1) \subseteq f^{-1}(B_{\mu_1}(f(x), \varepsilon_1)))$. Finally, using equivalence of metrics ϱ_1 and ϱ_2 we obtain that there exists $\delta > 0$ such that $B_{\varrho_2}(x, \delta) \subseteq B_{\varrho_1}(x, \delta_1)$. Combining these inclusions we get $B_{\varrho_2}(x, \delta) \subseteq f^{-1}(B_{\mu_2}(f(x), \varepsilon)))$. □

3.3 Example. The Euclidean metric d, the orthogonal metric ϱ_{ort} and the maximum metric ϱ_{max} are equivalent metrics on $\mathbb{R}^n$ (exercise).

3.4 Example. The following metrics on $\mathbb{R}^2$ are not equivalent to one another: the Euclidean metric d, the hub metric ϱ_h, and the discrete metric ϱ_{disc} (exercise).

Every metric defines open balls, but even if metrics are equivalent their open balls may look very differently (compare e.g. open balls in $\mathbb{R}^2$ taken with respect to d and ϱ_{ort}). It turns out, however, that each metric defines also a collection of so-called *open sets*, and that open sets defined by two metrics are the same precisely when these metrics are equivalent.

3.5 Definition. Let (X, ϱ) be a metric space. A subset $U \subseteq X$ is an *open set* if U is a union of (perhaps infinitely many) open balls in X: $U = \bigcup_{i \in I} B(x_i, r_i)$.



3.6 Proposition. *Let (X, ϱ) be a metric space and let $U \subseteq X$. The following conditions are equivalent:*

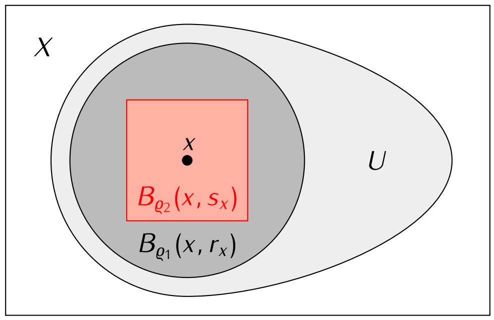
1) *The set U is open.*
2) *For every $x \in U$ there exists $r_x > 0$ such that $B(x, r_x) \subseteq U$.*

Proof. Exercise. □

3.7 Proposition. *Let X be a set and let ϱ_1, ϱ_2 be two metrics on X. The following conditions are equivalent:*

1) *The metrics ϱ_1 and ϱ_2 are equivalent.*
2) *A set $U \subseteq X$ is open with respect to the metric ϱ_1 if and only if it is open with respect to the metric ϱ_2.*

Proof. 1) $\Rightarrow$ 2) Assume that ϱ_1 and ϱ_2 are equivalent and that the set U is open with respect to ϱ_1. By Proposition 3.6 this means that for every $x \in U$ there exists $r_x > 0$ such that $B_{\varrho_1}(x, r_x) \subseteq U$. Since the metric ϱ_1 is equivalent to ϱ_2 we can find $s_x > 0$ such that $B_{\varrho_2}(x, s_x) \subseteq B_{\varrho_1}(x, r_x)$. As a consequence for every $x \in U$ we have $B(x, s_x) \subseteq U$.



Using Proposition 3.6 again we get that the set U is open with respect to ϱ_2. By the same argument we obtain that if U is open with respect to ϱ_2 then it is open with respect to ϱ_1.

2) $\Rightarrow$ 1) Exercise.

$\square$

Here are some basic properties of open sets in metric spaces:

3.8 Proposition. *Let (X, ϱ) be a metric space.*

1) *The sets X and $\varnothing$ are open sets.*
2) *If U_i is an open set for $i \in I$ then the set $\bigcup_{i \in I} U_i$ is open.*
3) *If U_1, U_2 are open sets then the set $U_1 \cap U_2$ is open.*

Proof. Exercise. $\square$

3.9 Note. From part 3) of Proposition 3.8 is follows that if $\{U_1, \dots, U_n\}$ is a finite family of open sets then $U_1 \cap \dots \cap U_n$ is open. However, if $\{U_i\}_{i \in I}$ is an infinite family of open sets then in general the set $\bigcap_{i \in I} U_i$ need not be open (exercise).

Our original definition of a continuous function between metric spaces stated that continuous functions behave well with respect to open balls. The next proposition says that in order to check if a function is continuous it is enough to know how it behaves with respect to open sets:

3.10 Proposition. *Let (X, ϱ), (Y, μ) be metric spaces and let $f\colon X \to Y$ be a function. The following conditions are equivalent:*

1) *The function f is continuous.*
2) *For every open set $U \subseteq Y$ the set $f^{-1}(U)$ is open in X.*

The proof of Proposition 3.10 will use the following fact:

3.11 Lemma. *Let (X, ϱ), (Y, μ) be metric spaces and let $f\colon X \to Y$ be a continuous function. If $B := B(y_0, r)$ is an open ball in Y then the set $f^{-1}(B)$ is open in X.*

Proof. Exercise. □

Proof of Proposition 3.10. 1) ⇒ 2) Assume that $f\colon X \to Y$ is a continuous function and that $U \subseteq Y$ is an open set. By definition this means that U is a union of some collection of open balls in Y:

$$U = \bigcup_{i \in I} B_\mu(y_i, r_i)$$

This gives:

$$f^{-1}(U) = f^{-1}\left(\bigcup_{i \in I} B_\mu(y_i, r_i)\right) = \bigcup_{i \in I} f^{-1}(B_\mu(y_i, r_i))$$

Since by Lemma 3.11 each of the sets $f^{-1}(B_\mu(y_i, r_i)$ is open in X and by Proposition 3.8 a union of open sets is open we obtain that the set $f^{-1}(U)$ is open in X.

2) ⇒ 1) Assume that $f^{-1}(U)$ is open in X for every open set $U \subseteq Y$. Given $x \in X$ and $\varepsilon > 0$ take $U = B_\mu(f(x), \varepsilon)$. By assumption the set $f^{-1}(B_\mu(f(x), \varepsilon)) \subseteq X$ is open. Since $x \in f^{-1}(B_\mu(f(x), \varepsilon))$ this implies that there exists $\delta > 0$ such that $B_\varrho(x, \delta) \subseteq f^{-1}(B_\mu(f(x), \varepsilon))$. This shows that f is a continuous function. □

Recall that we introduced metric spaces in order to be able to define continuity of functions. Proposition 3.10 says however that to define continuity we don't really need to use metrics, it is enough to know which sets are open. This observation leads to the following generalization of the notion of a metric space:

3.12 Definition. Let X be a set. A *topology* on X is a collection $\mathcal{T}$ of subsets of X satisfying the following conditions:

1) $X, \varnothing \in \mathcal{T}$;
2) If $U_i \in \mathcal{T}$ for $i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$;
3) If $U_1, U_2 \in \mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called *open sets*.

A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X.

In the setting of topological spaces we can define continuous functions as follows:

3.13 Definition. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f\colon X \to Y$ is *continuous* if for every $U \in \mathcal{T}_Y$ we have $f^{-1}(U) \in \mathcal{T}_X$.

3.14 Example. If (X, ϱ) is a metric space then X is a topological space with the topology

$$\mathcal{T} = \{U \subseteq X \mid U \text{ is a union of open balls}\}$$

We say that the topology $\mathcal{T}$ is *induced by the metric* ϱ.

3.15 Note. From now on, unless indicated otherwise, we will consider $\mathbb{R}^n$ as a topological space with the topology induced by the Euclidean metric.

3.16 Example. Let X be an arbitrary set and let

$$\mathcal{T} = \{\text{all subsets of } X\}$$

The topology $\mathcal{T}$ is called the *discrete topology* on X. If X is equipped with this topology then we say that it is a *discrete topological space*.

Note that the discrete topology is induced by the discrete metric ϱ_{disc} on X. Indeed, for $x \in X$ we have

$$B_{\varrho_{disc}}\left(x, \tfrac{1}{2}\right) = \{x\}$$

so for any subset $U \subseteq X$ we get

$$U = \bigcup_{x \in U} B_{\varrho_{disc}}\left(x, \tfrac{1}{2}\right)$$

3.17 Example. Let X be an arbitrary set and let

$$\mathcal{T} = \{X, \varnothing\}$$

The topology $\mathcal{T}$ is called the *antidiscrete topology* on X.

3.18 Example. Let $X = \mathbb{R}$ and let

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \varnothing \text{ or } U = (\mathbb{R} \smallsetminus S) \text{ for some finite set } S \subseteq \mathbb{R}\}$$

The topology $\mathcal{T}$ is called the *Zariski topology* on $\mathbb{R}$.

One can ask whether for every topological space $(X, \mathcal{T})$ we can find a metric ϱ on X such that the topology $\mathcal{T}$ is induced by ϱ. Our next goal is to show that this is not the case: some topologies do not come from any metric. Thus, the notion of a topological space is more general than that of a metric space.

3.19 Definition. A topological space $(X, \mathcal{T})$ is *metrizable* if there exists a metric ϱ on X such that $\mathcal{T}$ is the topology induced by ϱ.

3.20 Lemma. *If $(X, \mathcal{T})$ is a metrizable topological space and $x, y \in X$ are points such that $x \neq y$ then there exists an open set $U \subseteq X$ such that $x \in U$ and $y \notin U$.*

Proof. Exercise. □

3.21 Proposition. *If X is a set containing more than one point then the antidiscrete topology on X is not metrizable.*

Proof. This follows directly from Lemma 3.20. □

Exercises to Chapter 3

E3.1 Exercise. Verify the statement of Example 3.3.

E3.2 Exercise. Verify the statement of Example 3.4.

E3.3 Exercise. The goal of this exercise is to show that the converse of Proposition 3.2 is also true. Let X be a set and let ϱ_1, ϱ_2 be two metrics on X.

a) Assume that for each metric space (Y, μ) and for each function $f\colon X \to Y$ the function f is continuous with respect to ϱ_1 and μ if and only if it is continuous respect to ϱ_2 and μ. Show that ϱ_1 and ϱ_2 must be equivalent metrics.

b) Assume that for each metric space (Y, μ) and for each function $g\colon Y \to X$ the function g is continuous with respect to μ and ϱ_1 if and only if it is continuous respect to μ and ϱ_2. Show that ϱ_1 and ϱ_2 must be equivalent metrics.

E3.4 Exercise. Prove Proposition 3.6.

E3.5 Exercise. Consider the set $\mathbb{R}^2$ with the Euclidean metric.

a) Show that the open half plane $H = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ is an open set in $\mathbb{R}^2$

b) Show that the closed half plane $\overline{H} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$ is not an open set in $\mathbb{R}^2$

E3.6 Exercise. Consider the set $\mathbb{R}^2$ with the hub metric ϱ_h. Show that the set

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq -1\}$$

is an open set in $\mathbb{R}^2$

E3.7 Exercise. Consider the set $\mathbb{R}$ with the Euclidean metric. Show that every open set in $\mathbb{R}$ is a *disjoint* union of open intervals (a, b) (where possibly $a = -\infty$ or $b = +\infty$).

E3.8 Exercise. Prove the implication 2) ⇒ 1) of Proposition 3.7.

E3.9 Exercise. Prove Proposition 3.8.

E3.10 Exercise. Consider the set $\mathbb{R}^2$ with the Euclidean metric. Give an example of open sets $U_1, U_2, \ldots,$ in $\mathbb{R}^2$ such that the set $\bigcap_{n=1}^{\infty} U_i$ is not open.

E3.11 Exercise. Prove Lemma 3.11.

E3.12 Exercise. Prove Lemma 3.20.

E3.13 Exercise. Show that the set $\mathbb{R}$ with the Zariski topology is not metrizable.

E3.14 Exercise. Let X be a topological space consisting of a finite number of points. Show that if X is metrizable then it is a discrete space.

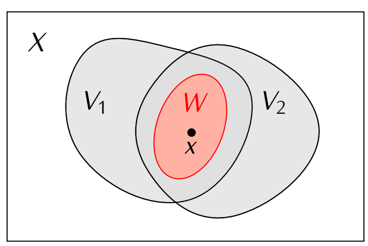
E3.15 Exercise. Let $\mathbb{R}_{Eu}$ denote the set $\mathbb{R}$ with the Euclidean topology and $\mathbb{R}_{Za}$ the set $\mathbb{R}$ with the Zariski topology. Show that every continuous function $f : \mathbb{R}_{Za} \to \mathbb{R}_{Eu}$ must be constant.

4 | Basis, Subbasis, Subspace

Our main goal in this chapter is to develop some tools that make it easier to construct examples of topological spaces. By Definition 3.12 in order to define a topology on a set X we need to specify which subsets of X are open sets. It can difficult to describe all open sets explicitly, so topological spaces are often defined by giving either a *basis* or a *subbasis* of a topology. Interesting topological spaces can be also obtained by considering *subspaces* of topological spaces. We explain these notions below.

4.1 Definition. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. The collection $\mathcal{B}$ is a *basis* on X if it satisfies the following conditions:

1) $X = \bigcup_{V \in \mathcal{B}} V$;
2) for any $V_1, V_2 \in \mathcal{B}$ and $x \in V_1 \cap V_2$ there exists $W \in \mathcal{B}$ such that $x \in W$ and $W \subseteq V_1 \cap V_2$.



4.2 Example. If (X, ϱ) is a metric space then the set $\mathcal{B} = \{B(x, r) \mid x \in X,\ r > 0\}$ consisting of all open balls in X is a basis on X (exercise).

4.3 Proposition. *Let X be a set, and let $\mathcal{B}$ be a basis on X. Let $\mathcal{T}$ denote the collection of all subsets $U \subseteq X$ that can be obtained as the union of some elements of $\mathcal{B}$: $U = \bigcup_{V \in \mathcal{B}_1} V$ for some $\mathcal{B}_1 \subseteq \mathcal{B}$. Then $\mathcal{T}$ is a topology on X.*

Proof. Exercise. □

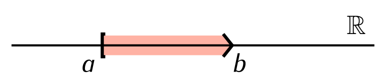
4.4 Definition. Let $\mathcal{B}$ be a basis on a set X and let $\mathcal{T}$ be the topology defined as in Proposition 4.3. In such case we will say that $\mathcal{B}$ is a *basis of the topology* $\mathcal{T}$ and that $\mathcal{T}$ is the *topology defined by the basis* $\mathcal{B}$.

4.5 Example. Let (X, ϱ) be a metric space, let $\mathcal{T}$ be the topology on X induced by ϱ, and let $\mathcal{B}$ be the collection of all open balls in X. Directly from the definition of the topology $\mathcal{T}$ (3.14) it follows that $\mathcal{B}$ is a basis of $\mathcal{T}$.

4.6 Example. Consider $\mathbb{R}^n$ with the Euclidean metric d. Let $\mathcal{B}$ be the collection of all open balls $B(x, r) \subseteq \mathbb{R}^n$ such that $r \in \mathbb{Q}$ and $x = (x_1, x_2, \dots, x_n)$ where $x_1, \dots, x_n \in \mathbb{Q}$. Then $\mathcal{B}$ is a basis of the Euclidean topology on $\mathbb{R}^n$ (exercise).

4.7 Note. If a topological space X has a basis consisting of countably many sets then we say that X satisfies the *2^{nd} countability axiom* or that X is *second countable*. Since the set of rational numbers is countable it follows that the basis of the Euclidean topology given in Example 4.6 is countable. Thus, $\mathbb{R}^n$ with the Euclidean topology is a second countable space. Second countable spaces have some interesting properties, some of which we will encounter later on.

4.8 Example. The set $\mathcal{B} = \{[a, b) \mid a, b \in \mathbb{R}\}$ is a basis of a certain topology on $\mathbb{R}$. We will call it the *arrow topology*.



4.9 Example. Let $\mathcal{B} = \{[a, b] \mid a, b \in \mathbb{R}\}$. The set $\mathcal{B}$ is a basis of the discrete topology on $\mathbb{R}$ (exercise).

4.10 Example. Let $X = \{a, b, c, d\}$ and let $\mathcal{B} = \{\{a, b, c\}, \{b, c, d\}\}$. The set $\mathcal{B}$ is not a basis of any topology on X since $b \in \{a, b, c\} \cap \{b, c, d\}$, and $\mathcal{B}$ does not contain any subset W such that $b \in W$ and $W \subseteq \{a, b, c\} \cap \{b, c, d\}$.

4.11 Proposition. *Let X be a set and let $\mathcal{S}$ be any collection of subsets of X such that $X = \bigcup_{V \in \mathcal{S}} V$. Let $\mathcal{T}$ denote the collection of all subsets of X that can be obtained using two operations:*

1) *taking finite intersections of sets in $\mathcal{S}$;*
2) *taking arbitrary unions of sets obtained in 1).*

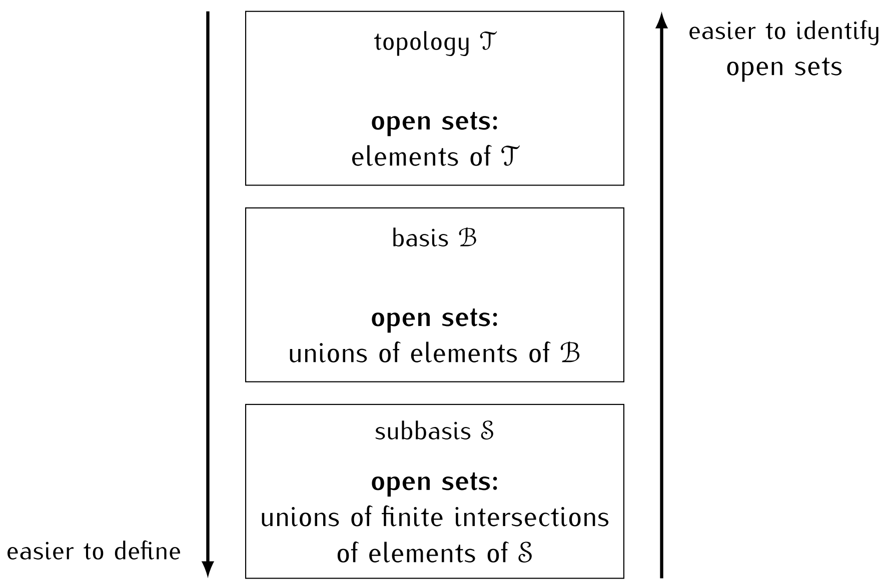
Then $\mathcal{T}$ is a topology on X.

Proof. Exercise. □

4.12 Definition. Let X be a set and let $\mathcal{S}$ be any collection of subsets of X such that $X = \bigcup_{V \in \mathcal{S}} V$. The topology $\mathcal{T}$ defined by Proposition 4.11 is called the *topology generated by* $\mathcal{S}$, and the collection $\mathcal{S}$ is called a *subbasis* of $\mathcal{T}$.

4.13 Example. If $X = \{a, b, c, d\}$ and $\mathcal{S} = \{\{a, b, c\}, \{b, c, d\}\}$ then the topology generated by $\mathcal{S}$ is $\mathcal{T} = \{\{a, b, c\}, \{b, c, d\}, \{b.c\}, \{a, b, c, d\}, \varnothing\}$.

The notions of a basis and a subbasis provide shortcuts for defining topologies: it is easier to specify a basis of a topology than to define explicitly the whole topology (i.e. to describe all open sets). Specifying a subbasis is even easier. The price we pay for this convenience is that it is more difficult to identify which sets are open if we know only a basis or a subbasis of a topology:



The next proposition often simplifies checking if a function between topological spaces is continuous:

4.14 Proposition. *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{B}$ be a basis (or a subbasis) of $\mathcal{T}_Y$. A function $f\colon X \to Y$ is continuous if and only if $f^{-1}(V) \in \mathcal{T}_X$ for every $V \in \mathcal{B}$.*

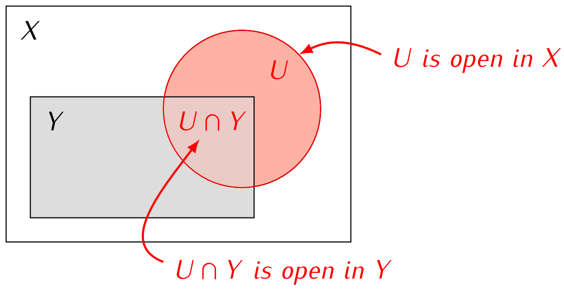
Proof. Exercise. □

A useful way of obtaining new examples of topological spaces is by considering subspaces of existing spaces:

4.15 Definition. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. The collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

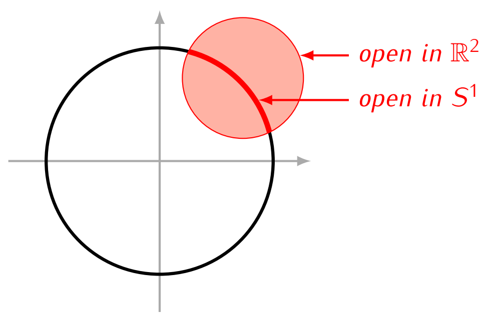
is a topology on Y called the *subspace topology*. We say that $(Y, \mathcal{T}_Y)$ is a *subspace* of the topological space $(X, \mathcal{T})$.



4.16 Example. The unit circle S^1 is defined by

$$S^1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$

The circle S^1 is a topological space considered as a subspace of $\mathbb{R}^2$.



In general the n-dimensional sphere S^n is defined by

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

It is a topological space considered as a subspace of $\mathbb{R}^{n+1}$.

4.17 Example. Consider $\mathbb{Z}$ as a subspace of $\mathbb{R}$. The subspace topology on $\mathbb{Z}$ is the same as the discrete topology (exercise).

4.18 Proposition. *Let X be a topological space and let Y be its subspace.*

1) The inclusion map $j\colon Y \to X$ is a continuous function.
2) If Z is a topological space then a function $f\colon Z \to Y$ is continuous if and only if the composition $jf\colon Z \to X$ is continuous.

Proof. Exercise. □

4.19 Proposition. *Let X be a topological space and let Y be its subspace. If $\mathcal{B}$ is a basis (or a subbasis) of X then the set $\mathcal{B}_Y = \{U \cap Y \mid U \in \mathcal{B}\}$ is a basis (resp. a subbasis) of Y.*

Proof. Exercise. □

Exercises to Chapter 4

E4.1 Exercise. Prove Proposition 4.3

E4.2 Exercise. Verify the statement of Example 4.6.

E4.3 Exercise. Verify the statement of Example 4.9.

E4.4 Exercise. Prove Proposition 4.11.

E4.5 Exercise. Prove Proposition 4.14.

E4.6 Exercise. Consider the interval $[0, 1]$ as a subspace of $\mathbb{R}$. Determine which of the following sets are open in $[0, 1]$. Justify your answers.

a) $\left(\frac{1}{2}, 1\right)$ b) $\left(\frac{1}{2}, 1\right]$ c) $\left(\frac{1}{3}, \frac{2}{3}\right)$ d) $\left(\frac{1}{3}, \frac{2}{3}\right]$

E4.7 Exercise. Verify the statement of Example 4.17.

E4.8 Exercise. Prove Proposition 4.18.

E4.9 Exercise. The goal of this exercise if to show subspace topology is uniquely determined by the properties listed in Proposition 4.18. Let X be a topological space, let $Y \subseteq X$ and let $j\colon Y \to X$ be the inclusion map. Let $\mathcal{T}$ be a topology, and let $Y_{\mathcal{T}}$ denote Y considered as a topological spaces with respect to the topology $\mathcal{T}$. Assume that $Y_{\mathcal{T}}$ satisfies the following conditions:

1) The map $j\colon Y_{\mathcal{T}} \to X$ is a continuous function.
2) If Z is a topological space then a function $f\colon Z \to Y_{\mathcal{T}}$ is continuous if and only if the composition $jf\colon Z \to X$ is continuous.

Show that $\mathcal{T}$ is the subspace topology on Y. That is, $U \in \mathcal{T}$ if and only if $U = Y \cap U'$ where U' is some open set in X.

E4.10 Exercise. Recall that a topological space X is second countable if the topology on X has a countable basis. Show that the discrete topology on a set X is second countable if and only if X is a countable set.

E4.11 Exercise. Show that $\mathbb{R}$ with the arrow topology is not second countable. (Hint: Assume by contradiction that $\mathcal{B} = \{V_1, V_2, \dots\}$ is a countable basis of the arrow topology. Let $\alpha_i = \inf V_i$. Take $\alpha_0 \in \mathbb{R} \smallsetminus \{\alpha_1, \alpha_2, \dots\}$. Show that the set $[\alpha_0, \alpha_0 + 1)$ is not a union of sets from $\mathcal{B}$).

E4.12 Exercise. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same set X. We say that the topology $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ (e.i. if every open set in $\mathcal{T}_1$ is also open in $\mathcal{T}_2$). Let $\mathcal{T}_{Ar}$ be the arrow topology on $\mathbb{R}$ and let $\mathcal{T}_{Eu}$ be the Euclidean topology on $\mathbb{R}$. Show that $\mathcal{T}_{Ar}$ is finer than $\mathcal{T}_{Eu}$.

5 | Closed Sets, Interior, Closure, Boundary

5.1 Definition. Let X be a topological space. A set $A \subseteq X$ is a *closed set* if the set $X \smallsetminus A$ is open.

5.2 Example. A closed interval $[a, b] \subseteq \mathbb{R}$ is a closed set since the set $\mathbb{R} \smallsetminus [a, b] = (-\infty, a) \cup (b, +\infty)$ is open in $\mathbb{R}$.

5.3 Example. Let $\mathcal{T}_{Za}$ be the Zariski topology on $\mathbb{R}$. Recall that $U \in \mathcal{T}_{Za}$ if either $U = \varnothing$ or $U = \mathbb{R} \smallsetminus S$ where $S \subset \mathbb{R}$ is a finite set. As a consequence closed sets in the Zariski topology are the whole space $\mathbb{R}$ and all finite subsets of $\mathbb{R}$.

5.4 Example. If X is a topological space with the discrete topology then every subset $A \subseteq X$ is closed in X since every set $X \smallsetminus A$ is open in X.

5.5 Proposition. *Let X be a topological space.*

1) *The sets X, $\varnothing$ are closed.*
2) *If $A_i \subseteq X$ is a closed set for $i \in I$ then $\bigcap_{i \in I} A_i$ is closed.*
3) *If A_1, A_2 are closed sets then the set $A_1 \cup A_2$ is closed.*

Proof. 1) The set X is closed since the set $X \smallsetminus X = \varnothing$ is open. Similarly, the set $\varnothing$ is closed since the set $X \smallsetminus \varnothing = X$ is open.

2) We need to show that the set $X \smallsetminus \bigcap_{i \in I} A_i$ is open. By De Morgan's Laws (1.13) we have:

$$X \smallsetminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \smallsetminus A_i)$$

By assumption the sets A_i are closed, so the sets $X \smallsetminus A_i$ are open. Since any union of open sets is open we get that $X \smallsetminus \bigcap_{i\in I} A_i$ is an open set.

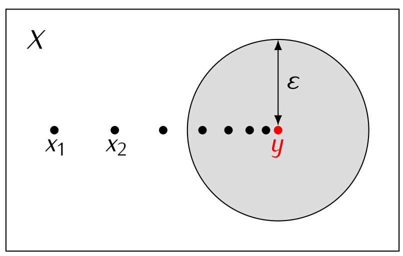
3) Exercise. □

5.6 Note. By induction we obtain that if $\{A_1, \dots, A_n\}$ is a finite collection of closed sets then the set $A_1 \cup \dots \cup A_n$ is closed. It is not true though that an infinite union of closed sets must be closed. For example, the sets $A_n = [\frac{1}{n}, 1]$ are closed in $\mathbb{R}$, but the set $\bigcup_{n=1}^{\infty} A_n = (0, 1]$ is not closed.

In metric spaces closed sets can be characterized using the notion of convergence of sequences:

5.7 Definition. Let (X, ϱ) be a metric space, and let $\{x_n\}$ be a sequence of points in X. We say that $\{x_n\}$ *converges* to a point $y \in X$ if for every $\varepsilon > 0$ there exists $N > 0$ such that $\varrho(y, x_n) < \varepsilon$ for all $n > N$. We write: $x_n \to y$.

Equivalently: $x_n \to y$ if for every $\varepsilon > 0$ there exists $N > 0$ such that $x_n \in B(y, \varepsilon)$ for all $n > N$.



5.8 Proposition. *Let (X, ϱ) be a metric space and let $A \subseteq X$. The following conditions are equivalent:*

1) *The set A is closed in X.*
2) *If $\{x_n\} \subseteq A$ and $x_n \to y$ then $y \in A$.*

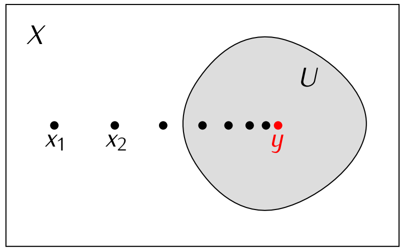
Proof. Exercise. □

5.9 Example. Take $\mathbb{R}$ with the Euclidean metric, and let $A = (0, 1]$. Let $x_n = \frac{1}{n}$. Then $\{x_n\} \subseteq A$, but $x_n \to 0 \notin A$. This shows that A is not a closed set in $\mathbb{R}$.

The notion of convergence of a sequence can be extended from metric spaces to general topological spaces by replacing open balls with center at a point y with open neighborhoods of y:

5.10 Definition. Let X be a topological space and $y \in X$. If $U \subseteq X$ is an open set such that $y \in U$ then we say that U is an *open neighborhood of y*.

5.11 Definition. Let X be a topological space. A sequence $\{x_n\} \subseteq X$ *converges* to $y \in X$ if for every open neighborhood U of y there exists $N > 0$ such that $x_n \in U$ for $n > N$.



5.12 Note. In general topological spaces a sequence may converge to many points at the same time. For example let $(X, \mathcal{T})$ be a space with the antidiscrete topology $\mathcal{T} = \{X, \varnothing\}$. Any sequence $\{x_n\} \subseteq X$ converges to any point $y \in X$ since the only open neighborhood of y is whole space X, and $x_n \in X$ for all n. The next proposition says that such situation cannot happen in metric spaces:

5.13 Proposition. *Let (X, ϱ) be a metric space and let $\{x_n\}$ be a sequence in X. If $x_n \to y$ and $x_n \to z$ for some $y, z \in X$ then $y = z$.*

Proof. Exercise. □

5.14 Proposition. *Let X be a topological space and let $A \subseteq X$ be a closed set. If $\{x_n\} \subseteq A$ and $x_n \to y$ then $y \in A$.*

Proof. Exercise. □

5.15 Note. For a general topological space X the converse of Proposition 5.14 is not true. That is, assume that $A \subseteq X$ is a set with the property that if $\{x_n\} \subseteq A$ and $x_n \to y$ then $y \in A$. The next example shows that this does not imply that the set A must be closed in X.

5.16 Example. Let $X = \mathbb{R}$ with the following topology:

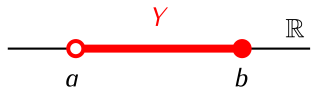
$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \varnothing \text{ or } U = (\mathbb{R} \smallsetminus S) \text{ for some countable set } S \subseteq \mathbb{R}\}$$

Closed sets in X are the whole space $\mathbb{R}$ and all countable subsets of $\mathbb{R}$. If $\{x_n\} \subseteq X$ is a sequence then $x_n \to y$ if and only if there exists $N > 0$ such that $x_n = y$ for all $n > N$ (exercise). It follows that if A is any (closed or not) subset of X, $\{x_n\} \subseteq A$, and $x_n \to y$ then $y \in A$.

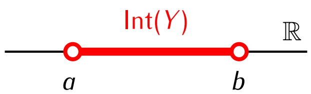
5.17 Definition. Let X be a topological space and let $Y \subseteq X$.

- The *interior of* Y is the set $\mathrm{Int}(Y) := \bigcup \{U \mid U \subseteq Y \text{ and } U \text{ is open in } X\}$.
- The *closure of* Y is the set $\overline{Y} := \bigcap \{A \mid Y \subseteq A \text{ and } A \text{ is closed in } X\}$.
- The *boundary of* Y is the set $\mathrm{Bd}(Y) := \overline{Y} \cap \overline{(X \smallsetminus Y)}$.

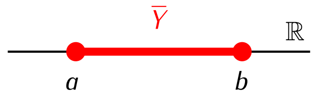
5.18 Example. Consider the set $Y = (a, b]$ in $\mathbb{R}$:



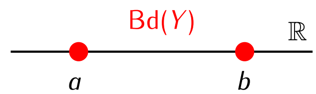
We have:



$\operatorname{Int}(Y) = (a, b)$

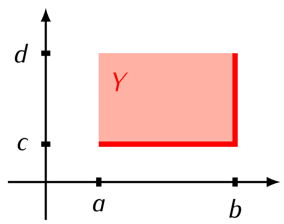


$\overline{Y} = [a, b]$

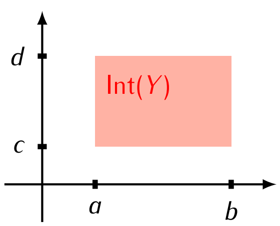


$\operatorname{Bd}(Y) = \{a, b\}$

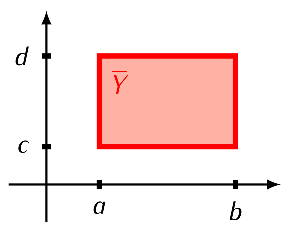
5.19 Example. Consider the set $Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid a < x_1 \leq b,\ c \leq x_2 < d\}$ in $\mathbb{R}^2$:



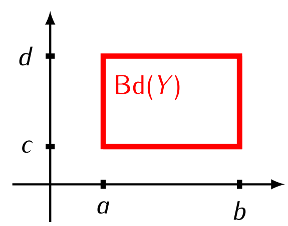
We have:



$\operatorname{Int}(Y) = (a, b) \times (c, d)$



$\overline{Y} = [a, b] \times [c, d]$



$$\operatorname{Bd}(Y) = [a, b] \times \{c, d\} \\ \cup \{a, b\} \times [c, d]$$

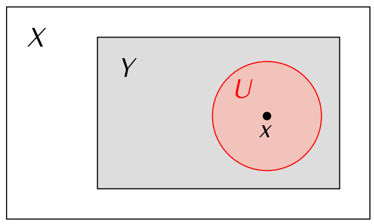
5.20 Proposition. *Let X be a topological space and let $Y \subseteq X$.*

1) *The set $\operatorname{Int}(Y)$ is open in X. It is the biggest open set contained in Y: if U is open and $U \subseteq Y$ then $U \subseteq \operatorname{Int}(Y)$.*
2) *The set $\overline{Y}$ is closed in X. It is the smallest closed set that contains Y: if A is closed and $Y \subseteq A$ then $\overline{Y} \subseteq A$.*

Proof. Exercise. □

5.21 Proposition. *Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:*

1) $x \in \operatorname{Int}(Y)$
2) *There exists an open neighborhood U of x such that $U \subseteq Y$.*

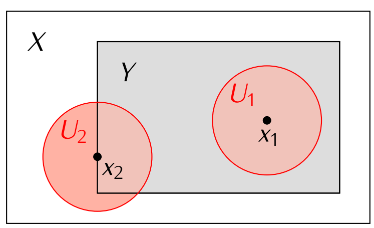


Proof. 1) $\Rightarrow$ 2) Assume that $x \in \operatorname{Int}(Y)$. Since $\operatorname{Int}(Y)$ is an open set and $\operatorname{Int}(Y) \subseteq Y$ we can take $U = \operatorname{Int}(Y)$.

2) $\Rightarrow$ 1) Assume that $x \in U$ for some open set U such that $U \subseteq Y$. Since $\operatorname{Int}(Y)$ is the union of all open sets contained in Y thus $U \subseteq \operatorname{Int}(Y)$ and so $x \in \operatorname{Int}(Y)$. □

5.22 Proposition. *Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:*

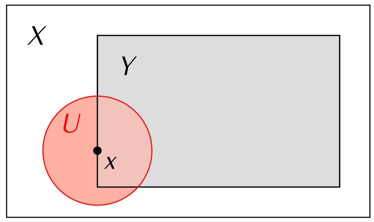
1) $x \in \overline{Y}$
2) *For every open neighborhood U of x we have $U \cap Y \neq \varnothing$.*



Proof. Exercise. □

5.23 Proposition. *Let X be a topological space, let $Y \subseteq X$, and let $x \in X$. The following conditions are equivalent:*

1) $x \in \operatorname{Bd}(Y)$
2) *For every open neighborhood U of x we have $U \cap Y \neq \varnothing$ and $U \cap (X \smallsetminus Y) \neq \varnothing$.*



Proof. This follows from the definition of $\mathrm{Bd}(Y)$ and Proposition 5.22. □

5.24 Definition. Let X be a topological space. A set $Y \subseteq X$ is *dense in* X if $\overline{Y} = X$.

5.25 Proposition. *Let X be a topological space and let $Y \subseteq X$. The following conditions are equivalent:*

1) *Y is dense in X*
2) *If $U \subseteq X$ is an open set and $U \neq \varnothing$ then $U \cap Y \neq \varnothing$.*

Proof. This follows directly from Proposition 5.22. □

5.26 Example. The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercises to Chapter 5

E5.1 Exercise. Prove Proposition 5.8

E5.2 Exercise. Prove Proposition 5.13

E5.3 Exercise. Let (X, ϱ) be a metric space. A sequence $\{x_n\}$ is called a *Cauchy sequence* if for any $N > 0$ there exists $\varepsilon > 0$ such that if $n, m > N$ then $\varrho(x_m, x_n) < \varepsilon$. Show that if $\{x_n\}$ is a sequence in X that converges to some point $x_0 \in X$ then $\{x_n\}$ is a Cauchy sequence.

E5.4 Exercise. Prove Proposition 5.14

E5.5 Exercise. Let X be the topological space defined in Example 5.16 and let $\{x_n\}$ be a sequence in X. Show that $x_n \to y$ for some $y \in X$ iff there exists $N > 0$ such that $x_n = y$ for all $n > N$.

E5.6 Exercise. Prove Proposition 5.22

E5.7 Exercise. Let X be a topological space and let Y be a subspace of X. Show that a set $A \subseteq Y$ is closed in Y if and only if there exists a set B closed in X such that $Y \cap B = A$.

E5.8 Exercise. Let X be a topological space and let $Y \subseteq X$ be a subspace.

a) Assume that Y is open in X. Show that if $U \subseteq Y$ is open in Y then U is open in X.

b) Assume that Y is closed in X. Show that if $A \subseteq Y$ is closed in Y then A is closed in X.

E5.9 Exercise. Let (X, ϱ) be a metric space. The *closed ball* with center at a point $x_0 \in X$ and radius $r > 0$ is the set

$$\overline{B}(x_0, r) = \{x \in X \mid \varrho(x_0, x) \leq r\}$$

a) Show that for any $x_0 \in X$ and any $r > 0$ the closed ball $\overline{B}(x_0, r)$ is a closed set.

b) Consider $\mathbb{R}^n$ with the Euclidean metric d. Show that for any $x_0 \in \mathbb{R}^n$ and any $r > 0$ the closed ball $\overline{B}(x_0, r)$ is the closure of the open ball $B(x_0, r)$ (i.e. $\overline{B}(x_0, r) = \overline{B(x_0, r)}$).

c) Give an example showing that in a general metric space (X, ϱ) the closed ball $\overline{B}(x_0, r)$ need not be the closure of the open ball $B(x_0, r)$.

E5.10 Exercise. Consider the following subset of $\mathbb{R}$:

$$Y = \left\{ -\frac{1}{n} \;\middle|\; n \in \mathbb{Z},\ n \geq 1 \right\}$$

Describe $\mathrm{Int}(Y)$, $\overline{Y}$, and $\mathrm{Bd}(Y)$ in the following topological spaces:

a) $\mathbb{R}$ with the Euclidean topology.
b) $\mathbb{R}$ with the Zariski topology.
c) $\mathbb{R}$ with the arrow topology.
d) $\mathbb{R}$ with the discrete topology.
e) $\mathbb{R}$ with the antidiscrete topology.
f) $\mathbb{R}$ with the topology defined in Example 5.16.

E5.11 Exercise. Let (X, ϱ) be a metric space. We say that a set $Y \subseteq X$ is *bounded* if there exists an open ball $B(x, r) \subseteq X$ such that $Y \subseteq B(x, r)$. Show that if Y is a bounded set then $\overline{Y}$ is also bounded.

E5.12 Exercise. Let X be a topological space and let $Y_1, Y_2 \subseteq X$.

a) Show $\overline{Y}_1 \cup \overline{Y}_2 = \overline{Y_1 \cup Y_2}$
b) Is it true always true that $\overline{Y}_1 \cap \overline{Y}_2 = \overline{Y_1 \cap Y_2}$? Justify your answer.

E5.13 Exercise. Let X be a topological space and let $Y \subseteq X$ be a dense subset of X. Show that if $f, g\colon X \to \mathbb{R}$ are continuous functions such that $f(x) = g(x)$ for all $x \in Y$ then $f(x) = g(x)$ for all $x \in X$.

E5.14 Exercise. Let X be a topological space, and let $A, B \subseteq X$. Show that if $\overline{B} \subseteq \mathrm{Int}(A)$ then $X = \mathrm{Int}(X \smallsetminus B) \cup \mathrm{Int}(A)$.

E5.15 Exercise. Let $\mathbb{R}_{Ar}$ denote the set of real numbers with the arrow topology (4.8). The goal of this exercise is to show that this space is not metrizable.

a) Recall that a space X is second countable if it has a countable basis. We say that a space X is *separable* if there is a set $Y \subseteq X$ such that Y is countable and dense in X. Show that if X is a metrizable space then X is separable if and only if X is second countable.

b) Show that $\mathbb{R}_{Ar}$ is a separable space.

Since by Exercise 4.11 $\mathbb{R}_{Ar}$ is not second countable this implies that $\mathbb{R}_{Ar}$ is not metrizable.

6 | Continuous Functions

Let X, Y be topological spaces. Recall that a function $f\colon X \to Y$ is continuous if for every open set $U \subseteq Y$ the set $f^{-1}(U) \subseteq X$ is open. In this chapter we study some properties of continuous functions. We also introduce the notion of a *homeomorphism* that plays a central role in topology: from the topological perspective interesting properties of spaces are the properties that are preserved by homeomorphisms.

6.1 Proposition. *Let X, Y be topological spaces. A function $f\colon X \to Y$ is continuous if and only if for every closed set $A \subseteq Y$ the set $f^{-1}(A) \subseteq X$ is closed.*

Proof. Assume that $f\colon X \to Y$ is a continuous function and let $A \subseteq Y$ be a closed set. We have

$$f^{-1}(A) = X \smallsetminus f^{-1}(Y \smallsetminus A)$$

The set $Y \smallsetminus A$ is open in Y so by continuity of f the set $f^{-1}(Y \smallsetminus A) \subseteq X$ is open in X. It follows that $f^{-1}(A)$ is closed in X.

Conversely, assume that $f\colon X \to Y$ is a function such that for every closed set $A \subseteq Y$ the set $f^{-1}(A) \subseteq X$ is closed. Let $U \subseteq Y$ be an open set. We have

$$f^{-1}(U) = X \smallsetminus f^{-1}(Y \smallsetminus U)$$

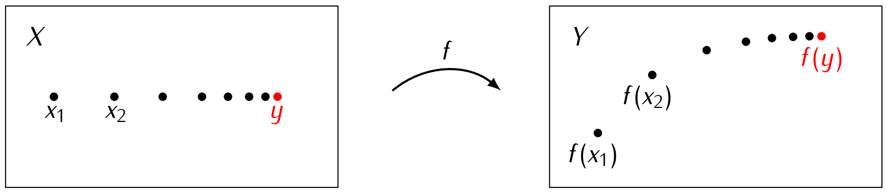
The set $Y \smallsetminus U$ is closed in Y so by assumption the set $f^{-1}(Y \smallsetminus U)$ is closed in X. If follows that $f^{-1}(U)$ is open in X. Therefore f is a continuous function. $\square$

For metric spaces continuous functions are precisely the functions that preserve convergence of sequences:

6.2 Proposition. *Let (X, ϱ) be a metric space, let Y be a topological space, and let $f\colon X \to Y$ be a function. The following conditions are equivalent:*

1) *f is continuous.*

2) For any sequence $\{x_n\} \subseteq X$ if $x_n \to y$ for some $y \in X$ then $f(x_n) \to f(y)$.



Proof. 1) $\Rightarrow$ 2) Exercise.

2) $\Rightarrow$ 1) Let $A \subseteq Y$ be a closed set. We will show that the set $f^{-1}(A)$ is closed in X. By Proposition 5.8 it suffices to show that if $\{x_n\} \subseteq f^{-1}(A)$ is a sequence and $x_n \to x$ then $x \in f^{-1}(A)$.

If $x_n \to x$ then by assumption we have $f(x_n) \to f(x)$. Since $\{f(x_n)\} \subseteq A$ and A is a closed set, thus by Proposition 5.8 we obtain that $f(x) \in A$, and so $x \in f^{-1}(A)$. □

The implication 1) $\Rightarrow$ 2) in Proposition 6.2 holds for maps between general topological spaces:

6.3 Proposition. *Let $f\colon X \to Y$ be a continuous function of topological spaces. If $\{x_n\} \subseteq X$ is a sequence and $x_n \to x$ for some $x \in X$ then $f(x_n) \to f(x)$.*

Proof. Exercise. □

6.4 Example. We will show that the implication 2) $\Rightarrow$ 1) in Proposition 6.2 is not true if X is a general topological space. Let X be the space defined in Example 5.16: $X = \mathbb{R}$ with the topology

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \varnothing \text{ or } U = (\mathbb{R} \smallsetminus S) \text{ for some countable set } S \subseteq \mathbb{R}\}$$

Recall that if $\{x_n\}$ is a sequence in X then $x_n \to x$ if and only if there exists $N > 0$ such that $x_n = x$ for all $n > N$. Let $f\colon X \to X$ be a function given by

$$f(x) = \begin{cases} 0 & \text{if } x \in (0,1) \\ 1 & \text{if } x \notin (0,1) \end{cases}$$

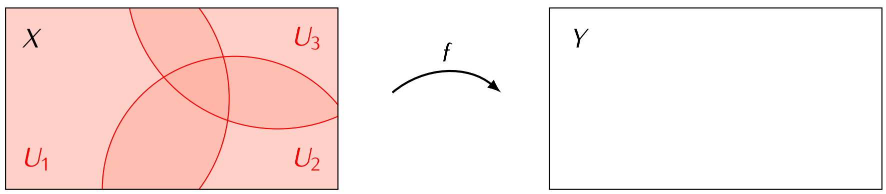
This function is not continuous since the set $\{0\}$ is closed in X and the set $(0,1) = f^{-1}(\{0\})$ is not closed in X. On the other hand let $\{x_n\} \subseteq X$ be a sequence and let $x_n \to x$. There is $N > 0$ such that $x_n = x$ for $n > N$, so $f(x_n) = f(x)$ for all $n > N$ and so $f(x_n) \to f(x)$.

6.5 Proposition. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions then the function $gf\colon X \to Z$ is also continuous.*

Proof. Exercise. □

Frequently functions $f\colon X \to Y$ are constructed by gluing together several functions defined on subspaces of X. The next two facts are useful for verifying that functions obtained in this way are continuous.

6.6 Open Pasting Lemma. *Let X, Y be topological spaces and let $\{U_i\}_{i\in I}$ be a family of open sets in X such that $\bigcup_{i\in I} U_i = X$. Assume that for $i \in I$ we have a continuous function $f_i\colon U_i \to Y$ such that $f_i(x) = f_j(x)$ if $x \in U_i \cap U_j$. Then the function $f\colon X \to Y$ given by $f(x) = f_i(x)$ for $x \in U_i$ is continuous.*



Proof. Let $V \subseteq Y$ be an open set. We will show that the set $f^{-1}(V) \subseteq X$ is open. Since $\bigcup_{i\in I} U_i = X$ we have

$$f^{-1}(V) = \bigcup_{i\in I} f^{-1}(V) \cap U_i = \bigcup_{i\in I} f_i^{-1}(V)$$

Since $f_i\colon U_i \to Y$ is a continuous function the set $f_i^{-1}(V)$ is open in U_i. Also, since U_i is open in X by Exercise 5.8 we obtain that the set $f_i^{-1}(V)$ is open in X. Thus $f^{-1}(V)$ is an open set. □

6.7 Closed Pasting Lemma. *Let X, Y be topological spaces and let $A_1, A_2 \subseteq X$ be closed sets such that $A_1 \cup A_2 = X$. Assume that for $i = 1, 2$ we have a continuous function $f_i\colon A_i \to Y$ such that $f_1(x) = f_2(x)$ if $x \in A_1 \cap A_2$. Then the function $f\colon X \to Y$ given by $f(x) = f_i(x)$ for $x \in A_i$ is continuous.*

Proof. Exercise. □

6.8 Example. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the absolute value function, $f(x) = |x|$. On the set $A_1 = (-\infty, 0]$ this function is given by $f|_{A_1}(x) = -x$, and on $A_2 = [0, +\infty)$ it is given by $f|_{A_2}(x) = x$. Since both $f|_{A_1}$ and $f|_{A_2}$ are continuous functions and A_1, A_2 are closed sets in $\mathbb{R}$ by the Closed Pasting Lemma 6.7 we obtain that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.

6.9 Note. Lemma 6.7 holds if instead of two closed sets we take any finite number of sets $A_1, \dots, A_n$ such that $\bigcup_{i=1}^n A_i = X$. On the other hand the statement of the lemma does not hold in general if the collection of sets $\{A_i\}$ is infinite.

6.10 Definition. A *homeomorphism* is a continuous function $f\colon X \to Y$ such that f is a bijection and the inverse function $f^{-1}\colon Y \to X$ is continuous.

6.11 Proposition. *1) For any topological space the identify function $\mathrm{id}_X\colon X \to X$ given by $\mathrm{id}_X(x) = x$ is a homeomorphism.*

2) If $f\colon X \to Y$ and $g\colon Y \to Z$ are homeomorphisms then the function $gf\colon X \to Z$ is also a homeomorphism.

3) If $f\colon X \to Y$ is a homeomorphism then the inverse function $f^{-1}\colon Y \to X$ is also a homeomorphism.

4) If $f\colon X \to Y$ is a homeomorphism and $Z \subseteq X$ then the function $f|_Z\colon Z \to f(Z)$ is also a homeomorphism.

Proof. Exercise. □

6.12 Note. If $f\colon X \to Y$ is a continuous bijection then f need not be a homeomorphism since the inverse function f^{-1} may be not continuous. For example, let $X = \{x_1, x_2\}$ be a space with the discrete topology and let $Y = \{y_1, y_2\}$ be a space with the antidiscrete topology. Let $f\colon X \to Y$ be given by $f(x_i) = y_i$. The function f is continuous but f^{-1} is not continuous since the set $\{x_1\}$ is open in X, but the set $(f^{-1})^{-1}(\{x_1\}) = \{y_1\}$ is not open in Y.

6.13 Proposition. *Let $f\colon X \to Y$ be a continuous bijection. The following conditions are equivalent:*

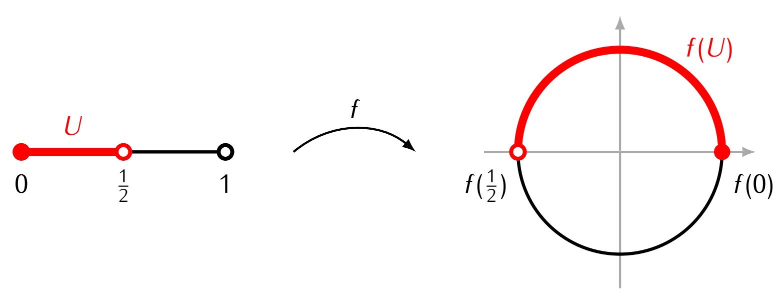
(i) The function f is a homeomorphism.
(ii) For each open set $U \subseteq X$ the set $f(U) \subseteq Y$ is open.
(iii) For each closed set $A \subseteq X$ the set $f(A) \subseteq Y$ is closed.

Proof. Exercise. □

6.14 Example. Recall that S^1 denotes the unit circle:

$$S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$

The function $f\colon [0, 1) \to S^1$ given by $f(x) = (\cos 2\pi x, \sin 2\pi x)$ is a continuous bijection, but it is not a homeomorphism since the set $U = [0, \frac{1}{2})$ is open in $[0, 1)$, but $f(U)$ is not open in S^1.

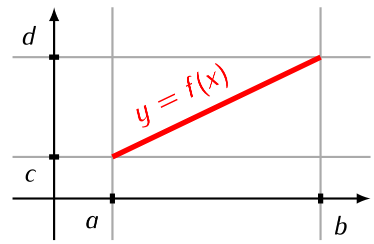


6.15 Definition. We say that topological spaces X, Y are *homeomorphic* if there exists a homeomorphism $f\colon X \to Y$. In such case we write: $X \cong Y$.

6.16 Note. Notice that if $X \cong Y$ and $Y \cong Z$ then $X \cong Z$.

6.17 Example. For any $a < b$ and $c < d$ the open intervals $(a, b), (c, d) \subseteq \mathbb{R}$ are homeomorphic. To see this take e.g. the function $f\colon (a, b) \to (c, d)$ defined by

$$f(x) = \left(\frac{c-d}{a-b}\right)x + \left(\frac{ad-bc}{a-b}\right)$$



This function is a continuous bijection. Its inverse function $f^{-1}\colon (c, d) \to (a, b)$ is given by

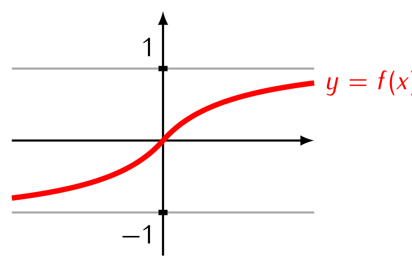
$$f^{-1}(x) = \left(\frac{a-b}{c-d}\right)x + \left(\frac{cb-da}{c-d}\right)$$

so it is also continuous. By the same argument for any $a < b$ and $c < d$ the closed intervals $[a, b], [c, d] \subseteq \mathbb{R}$ are homeomorphic.

6.18 Note. In Chapter 7 we will show that an open interval (a, b) is not homeomorphic to a closed interval $[c, d]$.

6.19 Example. We will show that for any $a < b$ the open interval (a, b) is homeomorphic to $\mathbb{R}$. Since $(a, b) \cong (-1, 1)$ it will be enough to check that $\mathbb{R} \cong (-1, 1)$. Take the function $f\colon \mathbb{R} \to (-1, 1)$ given by

$$f(x) = \frac{x}{1+|x|}$$



This function is a continuous bijection with the inverse function $f^{-1}\colon (-1, 1) \to \mathbb{R}$ is given by

$$f^{-1}(x) = \frac{x}{1-|x|}$$

Since f^{-1} is continuous we obtain that f is a homeomorphism.

6.20 Note. If spaces X and Y are homeomorphic then usually there are many homeomorphisms $X \to Y$. For example, the function $g\colon (-1, 1) \to \mathbb{R}$ given by

$$g(x) = \tan\left(\tfrac{\pi}{2}x\right)$$

is another homeomorphism between the spaces $(-1, 1)$ and $\mathbb{R}$.

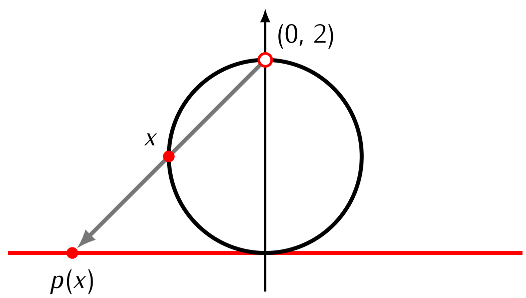
6.21 Example. We will show that for any point $x_0 \in S^1$ there is a homeomorphism $S^1 \smallsetminus \{x_0\} \cong \mathbb{R}$. Denote by $S^1_{(0,1)} \subseteq \mathbb{R}$ the circle of radius 1 with the center at the point $(0, 1) \in \mathbb{R}^2$:

$$S^1_{(0,1)} := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + (x_2 - 1)^2 = 1\}$$

It is easy to check that for $x_0 \in S^1$ the space $S^1 \smallsetminus \{x_0\}$ is homeomorphic to the space $X = S^1_{(0,1)} \smallsetminus \{(0, 2)\}$. Likewise, it is easy to check that $\mathbb{R}$ is homeomorphic to the subspace $Y \subseteq \mathbb{R}^2$ that consists of all points of the x-axis:

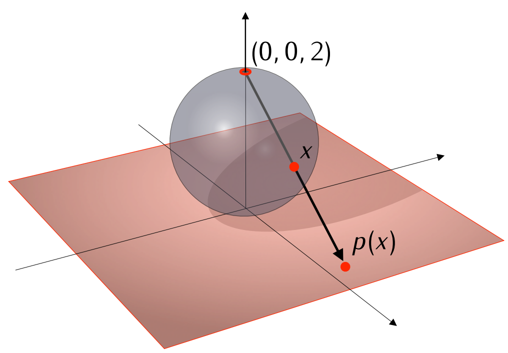
$$Y := \{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}\}$$

It is then enough to show that $X \cong Y$. A homeomorphism $p\colon X \to Y$ can be constructed as follows. For any point $x \in X$ there is a unique line in $\mathbb{R}^2$ that passes through x and though the point $(0, 2) \in \mathbb{R}^2$. We define $p(x)$ to be the point of intersection of this line with the x-axis:



The function p is called the *stereographic projection*.

In a similar way we can construct a stereographic projection in any dimension $n \geq 1$ that gives a homeomorphism between the space $S^n \smallsetminus \{x_0\}$ (i.e. the n-dimensional sphere with one point deleted) and the space $\mathbb{R}^n$:



Exercises to Chapter 6

E6.1 Exercise. Consider the set of rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Show that $\mathbb{Q}$ is not homeomorphic to a space with the discrete topology.

E6.2 Exercise. Prove Proposition 6.3.

E6.3 Exercise. Prove Proposition 6.5.

E6.4 Exercise. Prove Lemma 6.7.

E6.5 Exercise. Prove Proposition 6.13.

E6.6 Exercise. Let X be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous functions.

a) Show that the set
$$A = \{x \in X \mid f(x) \geq g(x)\}$$
is closed in X.

b) Let $h\colon X \to \mathbb{R}$ be a function given by $h(x) = \max\{f(x), g(x)\}$. Show that h is continuous.

E6.7 Exercise. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x) > g(x)$ for all $x \in \mathbb{R}$. Define subspaces X, Y of $\mathbb{R}^2$ as follows.

$$X := \{(x, y) \in \mathbb{R}^2 \mid g(x) \leq y \leq f(x)\} \qquad Y := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$$

Show that $X \cong Y$.

E6.8 Exercise. Let $x_0 = (0, 0) \in \mathbb{R}^2$ and let $\overline{B}(x_0, 1) \subseteq \mathbb{R}^2$ be a closed ball defined by the Euclidean metric d:
$$\overline{B}(x_0, 1) = \{x \in \mathbb{R}^2 \mid d(x, x_0) \leq 1\}$$
Define subspaces $X, Y \subseteq \mathbb{R}^2$ as follows:

$$X := \mathbb{R}^2 \smallsetminus \{x_0\} \qquad Y := \mathbb{R}^2 \smallsetminus \overline{B}(x_0, 1)$$

Show that $X \cong Y$.

E6.9 Exercise. Let (X, ϱ) be a metric space. A subspace $Y \subseteq X$ is a *retract* of X if there exists a continuous function $r\colon X \to Y$ such that $r(x) = x$ for all $x \in Y$. Show that if $Y \subseteq X$ is a retract of X then Y is a closed in X.

7 | Connectedness

7.1 Let $[a, b] \subseteq \mathbb{R}$ be a closed interval and let $(a, b) \subseteq \mathbb{R}$ be an open interval. We would like to show that $[a, b]$ and (a, b) are non-homeomorphic topological spaces. The idea of a proof of this fact is as follows. Assume that there exists a homeomorphism

$$f\colon [a, b] \to (a, b)$$

Recall that by Proposition 6.11 for any $Y \subseteq [a, b]$ the function $f|_Y\colon Y \to f(Y)$ also would be a homeomorphism. If we take $Y = [a, b] \smallsetminus \{a\} = (a, b]$ then

$$f(Y) = f([a, b]) \smallsetminus \{f(a)\} = (a, b) \smallsetminus \{f(a)\}$$

Intuitively the spaces Y and $f(Y)$ are different in an essential way since Y comes in one piece while $f(Y)$ is split into two pieces by removal of the point $f(a)$:

$Y = [a, b] \smallsetminus \{a\}$ $\qquad$ $f(Y) = (a, b) \smallsetminus \{f(a)\}$

For this reason we can expect that the spaces are Y and $f(Y)$ are not homeomorphic, and that, as a consequence, $[a, b]$ and (a, b) are not homeomorphic as well.

In order to make this intuitive argument into a rigorous proof we need to define precisely what it means that a topological space is "in one piece" and then show that this feature is preserved by homeomorphisms. The property of being "in one piece" is captured by the definition of a connected space:

7.2 Definition. A topological space X is *connected* if for any two open sets $U, V \subseteq X$ such that $U \cup V = X$ and $U, V \neq \varnothing$ we have $U \cap V \neq \varnothing$.

7.3 Definition. If X is a topological space and $U, V \subseteq X$ are non-empty open sets such that $U \cap V = \varnothing$ and $U \cup V = X$ then we say that $\{U, V\}$ is a *separation* of X.

Thus, a space X is connected if there does not exists a separation of X.

7.4 Example. For $a < b$ take $(a, b) \subseteq \mathbb{R}$ and let $c \in (a, b)$. The space $X = (a, b) \smallsetminus \{c\}$ is not connected. Indeed, the sets $U = (a, c)$ and $V = (c, b)$ form a separation of X.

7.5 Proposition. *Let $a < b$. The intervals (a, b), $[a, b]$, $(a, b]$, and $[a, b)$ are connected topological spaces.*

Proof. Assume first that $[a, b]$ is a closed interval and that that $U, V \subseteq [a, b]$ are open sets such $a \in U$, $b \in V$, and $U \cup V = [a, b]$. We will show that $U \cap V \neq \varnothing$. Let $x_0 = \inf V$. There are two possibilities: either $x_0 \notin U$ or $x_0 \in U$. In the first case $x_0 \in V$ and $x_0 > a$. Since V is an open set there exists $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq V$. This implies that there is $x \in V$ such that $x < x_0$ which is impossible by the definition of x_0.

Thus the only possible option is $x_0 \in U$. Since U is an open set there exists $\varepsilon' > 0$ such that $[x_0, x_0 + \varepsilon') \subseteq U$. On the other hand, by the definition of x_0 we have $[x_0, x_0 + \varepsilon') \cap V \neq \varnothing$. Therefore $U \cap V \neq \varnothing$.

Assume now that I is an interval (either closed, open, or half-open) and that $U, V \subseteq I$ are non-empty open sets such that $U \cup V = I$. We will show that $U \cap V \neq \varnothing$. Let $c, d \in I$ be points such that $c \in U$ and $d \in V$. We can assume that $c < d$. Take $U' = U \cap [c, d]$ and $V' = V \cap [c, d]$. The sets U', V' are open in $[c, d]$, $c \in U'$, $d \in V'$, and $U' \cup V' = [c, d]$. By the observation above we have $U' \cap V' \neq \varnothing$, and so $U \cap V \neq \varnothing$.

□

One can show that intervals are in fact the only subspaces of $\mathbb{R}$ that are connected:

7.6 Proposition. *If X is a connected subspace of $\mathbb{R}$ then X is an interval (either open, closed, or half-closed, finite or infinite).*

Proof. Exercise. □

7.7 Going back to the argument outlined in 7.1, by Proposition 7.5 we get that the space $Y = (a, b]$ is connected, and the space $f(Y) = (a, b) \smallsetminus f(a)$ is not connected by Example 7.4. We still need to show however that a connected space cannot be homeomorphic to one that is not connected. In fact a stronger statement is true:

7.8 Proposition. *Let $f: X \to Y$ be a continuous function. If f is onto and the space X is connected then Y is also connected.*

Proof. Assume that Y is not connected and let $U, V \subseteq Y$ be a separation of X. Then the sets $f^{-1}(U), f^{-1}(V)$ form a separation of X which contradicts the assumption that X is connected. □

7.9 Corollary. *If $f: X \to Y$ is a continuous function and X is a connected space then $f(X)$ is connected.*

Proof. By restricting the range of f we obtain a function $f: X \to f(X)$ which is continuous and onto, and so it we can apply Proposition 7.8. □

A very useful consequence of Corollary 7.9 is the following fact:

7.10 Intermediate Value Theorem. *Let X be a connected topological space and let $f: X \to \mathbb{R}$ be a continuous function. If $a < b$ are points in $\mathbb{R}$ such that $a = f(x)$ and $b = f(y)$ for some $x, y \in X$ then for each $c \in [a, b]$ there exists $z \in X$ such that $c = f(z)$.*

Proof. By Corollary 7.9 the set $f(X)$ is connected, and so by Proposition 7.6 $f(X)$ is an interval. It follows that for any $a, b \in f(X)$ we have $[a, b] \subseteq f(X)$. □

Since every homeomorphism $f: X \to Y$ is onto directly from Corollary 7.9 we get:

7.11 Corollary. *If $X \cong Y$ and X is a connected space then Y is also connected.*

7.12 Corollary. *The space $\mathbb{R}$ is connected.*

Proof. This follows from Corollary 7.11 and Proposition 7.5 since $\mathbb{R} \cong (a, b)$ for any $a < b$. □

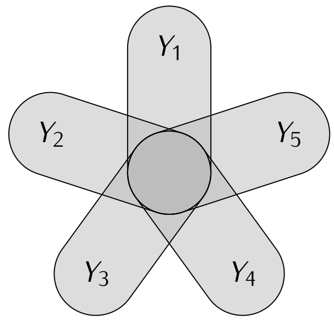
7.13 Note. A *topological invariant* is a property of topological spaces such that if a space X has this property and $X \cong Y$ then Y also has this property. By Corollary 7.11 connectedness is a topological invariant.

7.14 Proposition. *Let X be a topological space. The following conditions are equivalent :*

1) *X is connected*
2) *For any closed sets $A, B \subseteq X$ such that $A, B \neq X$ and $A \cap B = \varnothing$ we have $A \cup B \neq X$.*
3) *If $A \subseteq X$ is a set that is both open and closed then either $A = X$ or $A = \varnothing$.*
4) *If $D = \{0, 1\}$ is a space with the discrete topology then any continuous function $f: X \to D$ is a constant function.*

Proof. Exercise. □

7.15 Proposition. *Let X be a topological space and for $i \in I$ let Y_i be a subspace of X. Assume that $\bigcup_{i\in I} Y_i = X$ and $\bigcap_{i\in I} Y_i \neq \varnothing$. If Y_i is connected for each $i \in I$ then X is also connected.*



Proof. Let $D = \{0, 1\}$ be a space with the discrete topology and let $f\colon X \to D$ be a continuous function. By Proposition 7.14 it is enough to show that f is a constant function. Let $x_0 \in \bigcap_{i\in I} Y_i$. We can assume that $f(x_0) = 0$. For any $i \in I$ the function $f|_{Y_i}\colon Y_i \to D$ is constant since Y_i is connected. Since $x_0 \in Y_i$ and $f(x_0) = 0$ we get that $f(x) = 0$ for all $x \in Y_i$. Since this applies to all subspaces Y_i we obtain that $f(x) = 0$ for all $x \in \bigcup_{i\in I} Y_i = X$. □

7.16 Corollary. *The space $\mathbb{R}^n$ is connected for all $n \geq 1$.*

Proof. For $0 \neq x \in \mathbb{R}^n$ let $L_x \subseteq \mathbb{R}^n$ be the line passing through x and the origin:

$$L_x = \{tx \in \mathbb{R}^n \mid t \in \mathbb{R}\}$$

For every $x \in \mathbb{R}^n$ consider the continuous function $f_x\colon \mathbb{R} \to \mathbb{R}^n$ given by $f_x(t) = tx$. Since $\mathbb{R}$ is connected and $f_x(\mathbb{R}) = L_x$ if follows that L_x is connected. We have $\mathbb{R}^n = \bigcup_{x\in\mathbb{R}^n} L_x$ and $\bigcap_{x\in\mathbb{R}^n} L_x = \{0\}$. Therefore by Proposition 7.15 the space $\mathbb{R}^n$ is connected. □

7.17 Definition. Let X be a topological space. A *connected component* of X is a subspace $Y \subseteq X$ such that

1) Y is connected
2) if $Y \subseteq Z \subseteq X$ and Z is connected then $Y = Z$.

7.18 Proposition. *Let X be a topological space.*

1) *For every point $x_0 \in X$ there exist a connected component $Y \subseteq X$ such that $x_0 \in Y$.*
2) *If Y, Y' are connected components of X then either $Y \cap Y' = \varnothing$ or $Y = Y'$.*

Proof. 1) Given a point $x_0 \in X$ let $\{C_i\}_{i\in I}$ be the collection of all subspaces of X such that $x_0 \in C_i$ and C_i is connected. Define $Y := \bigcup_{i\in I} C_i$. We have $x_0 \in Y$. Also, since $x_0 \in \bigcap_{i\in I} C_i$ by Proposition

7.15 we obtain that Y is connected. If $Y \subseteq Z \subseteq X$ and Z is connected then $Z = C_{i_0}$ for some $i_0 \in I$, and so $Z = Y$. Therefore Y is a connected component of X.

2) Let Y, Y' be two connected components of X. Assume that $Y \cap Y' \neq \varnothing$. By Proposition 7.15 we get then that $Y \cup Y'$ is connected. Since $Y \subseteq Y \cup Y'$ we must have $Y = Y \cup Y'$. By the same argument we obtain that $Y' = Y \cup Y'$. Therefore $Y = Y'$

□

7.19 Corollary. *Let X be a topological space. If $Z \subseteq X$ is a connected subspace then there exists a connected component $Y \subseteq X$ such that $Z \subseteq Y$.*

Proof. Exercise. □

7.20 Corollary. *Let $f\colon X \to Y$ be a continuous function. If X is a connected space then there exists a connected component $Z \subseteq Y$ such that $f(X) \subseteq Z$.*

Proof. Exercise. □

Exercises to Chapter 7

E7.1 Exercise. Let X be a topological space and let $Y \subseteq X$ be a subspace. Show that if Y is a connected space and Y is dense in X then X is connected.

E7.2 Exercise. Prove Proposition 7.6.

E7.3 Exercise. Show that the sphere S^n is connected for all $n \geq 1$.

E7.4 Exercise. Let $a < b$. Show that the closed interval $[a, b] \subseteq \mathbb{R}$ is not homeomorphic to the half-closed interval $(a, b]$.

E7.5 Exercise. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *strictly increasing* is for all $x, y \in \mathbb{R}$ such that $x > y$ we have $f(x) > f(y)$, and is it *strictly decreasing* is for all $x, y \in \mathbb{R}$ such that $x > y$ we have $f(x) < f(y)$. Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous 1-1 function then f is either strictly increasing or strictly decreasing.

E7.6 Exercise. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) \cdot f(f(x)) = 1$ for all $x \in \mathbb{R}$ and that $f(10) = 9$. Find the value of $f(5)$. Justify your answer.

E7.7 Exercise. Let $f\colon S^n \to \mathbb{R}$ be a continuous function. Show that there exists a point $x \in S^n$ such that $f(x) = f(-x)$. Here if $x = (x_1, \dots, x_n) \in S^n$ then $-x = (-x_1, \dots, -x_n)$.

E7.8 Exercise. Let $a < b$. Show that there does not exist a continuous bijection $f\colon (a, b) \to [a, b]$. Remember that a continuous bijection need not be a homeomorphism since the inverse function may be not continuous (see 6.12).

E7.9 Exercise. Prove Proposition 7.14.

E7.10 Exercise. Let X be a topological space. Show that the following conditions are equivalent:

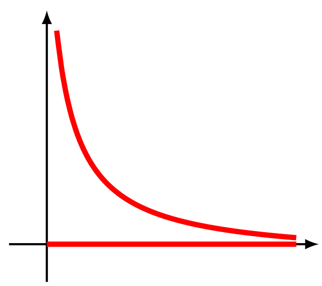
1) X is connected
2) if $A \subseteq X$ is any set such that $A \neq X$ and $A \neq \varnothing$ then $\mathrm{Bd}(A) \neq \varnothing$.

E7.11 Exercise. Let X be a topological space. Show that every connected component of X is closed in X.

E7.12 Exercise. Let (X, ϱ) be a metric space. Assume for some $x_0 \in X$ and $r > 0$ the open ball $B(x_0, r)$ consists of countably many points. Show that X is not connected.

E7.13 Exercise. Let X be the subspace of $\mathbb{R}^2$ consisting of the positive x-axis and of the graph of the function $f(x) = \frac{1}{x}$ for $x > 0$:

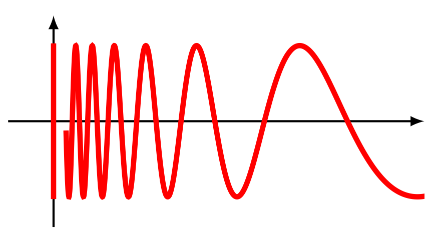
$$X := \{(x, 0) \in \mathbb{R}^2 \mid x \geq 0\} \cup \{(x, \tfrac{1}{x}) \in \mathbb{R}^2 \mid x > 0\}$$



Show that X is not connected.

E7.14 Exercise. The *topologist's sine curve* is the subspace Y of $\mathbb{R}^2$ that consists of a segment of the y-axis and of the graph of the function $f(x) = \sin(\frac{1}{x})$:

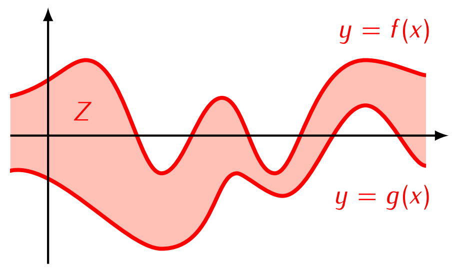
$$Y := \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\} \cup \{(x, \sin(\tfrac{1}{x})) \in \mathbb{R}^2 \mid x > 0\}$$



Show that Y is connected.

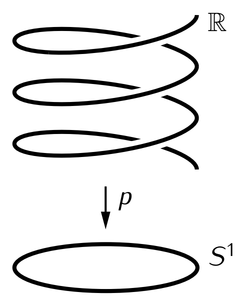
E7.15 Exercise. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions such that $g(x) < f(x)$ for all $x \in \mathbb{R}$. Let Z be the subspace of $\mathbb{R}^2$ given by

$$Z = \{(x, y) \mid g(x) \leq y \leq f(x)\}$$



Show that Z is connected.

E7.16 Exercise. Consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $p\colon \mathbb{R} \to S^1$ denote the function given by $p(t) = (\sin 2\pi t, \cos 2\pi t)$. Geometrically speaking this function wraps $\mathbb{R}$ infinitely many times around the circle:



Show that there does not exist a continuous function $g\colon S^1 \to \mathbb{R}$ such that $pg = \mathrm{id}_{S^1}$.

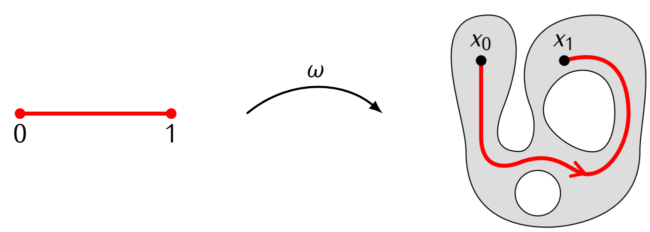
E7.17 Exercise. A space X is *totally disconnected* if every connected component of X consists of a single point. Obviously every discrete topological space is totally disconnected. Consider the set ot rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Show that $\mathbb{Q}$ is totally disconnected. Note that by Exercise 6.1 $\mathbb{Q}$ is not a discrete space.

E7.18 Exercise. Show that metrizability is a topological invariant. That is, it X and Y are homeomorphic spaces, and X metrizable then so is Y.

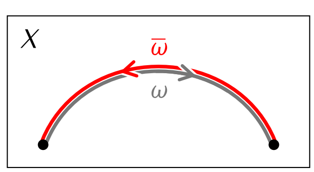
8 | Path Connectedness

The notion of connectedness of a space was invented to define rigorously what it means that a space is "in one piece". In this chapter we introduce path connectedness which is designed to capture a similar property but in a different way. It turns out that these two notions are not the same: while every path connected space is connected, the opposite is not true. In effect path connectedness gives us a new topological invariant of spaces. Additional related invariants are obtained by considering local connectedness and local path connectedness of spaces.

8.1 Definition. Let X be a topological space. A *path* in X is a continuous function $\omega\colon [0,1] \to X$. If $\omega(0) = x_0$ and $\omega(1) = x_1$ then we say that ω joins x_0 with x_1.

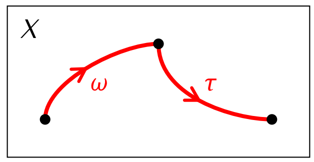


8.2 Definition. 1) If $\omega\colon [0,1] \to X$ is a path in X then the *inverse* of ω is the path $\overline{\omega}$ given by $\overline{\omega}(t) = \omega(1-t)$ for $t \in [0,1]$.



2) If $\omega.\tau\colon [0,1] \to X$ are paths such that $\omega(1) = \tau(0)$ then the *concatenation* of ω and τ if the path $\omega * \tau$ given by

$$(\omega * \tau)(t) = \begin{cases} \omega(2t) & \text{for } t \in [0, 1/2] \\ \tau(2t-1) & \text{for } t \in [1/2, 1] \end{cases}$$



8.3 Definition. A space X is *path connected* if for every $x_0, x_1 \in X$ there is a path joining x_0 with x_1.

8.4 Example. For any $n \geq 1$ the space $\mathbb{R}^n$ is path connected. Indeed, if $x_0, x_1 \in \mathbb{R}^n$ then define $\omega\colon [0,1] \to \mathbb{R}^n$ by

$$\omega(t) = (1-t)x_0 + tx_1$$

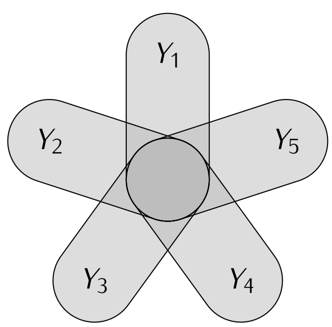
We have $\omega(0) = x_0$ and $\omega(1) = x_1$.

8.5 Proposition. *Every path connected space is connected.*

Proof. Exercise. □

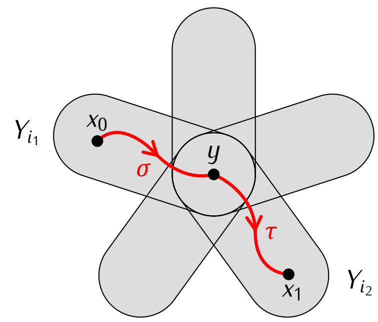
8.6 Note. It is not true that a connected space must be path connected. For example, let Y be the topologist's sine curve (7.14). This is a connected space. On the other hand Y is not path connected (exercise).

8.7 Proposition. *Let X be a topological space and for $i \in I$ let Y_i be a subspace of X. Assume that $\bigcup_{i\in I} Y_i = X$ and $\bigcap_{i \in I} Y_i \neq \varnothing$. If Y_i is path connected for each $i \in I$ then X is also path connected.*



Proof. Let $x_0, x_1 \in X$. We need to show that there exists a path $\omega\colon [0,1] \to X$ such that $\omega(0) = x_0$ and $\omega(1) = x_1$.

Since $X = \bigcup_{i\in I} Y_i$ we have $x_0 \in Y_{i_0}$ and $x_1 \in Y_{i_1}$ for some $i_0, i_1 \in I$. Let $y \in \bigcap_{i\in I} Y_i$. Since Y_{i_0} is path connected and $x_0, y \in Y_{i_0}$ there is a path $\sigma\colon [0,1] \to Y_{i_0}$ such that $\sigma(0) = x_0$ and $\sigma(1) = y$. Also, since Y_{i_1} is path connected and $x_1, y \in Y_{i_1}$ there is a path $\tau\colon [0,1] \to Y_{i_1}$ such that $\tau(0) = y$ and $\tau(1) = x_1$. The concatenation $\sigma * \tau$ gives a path joining x_0 with x_1.



□

8.8 Definition. Let X be a topological space. A *path connected component* of X is a subspace $Y \subseteq X$ such that

1) Y is path connected
2) if $Y \subseteq Z \subseteq X$ and Z is path connected then $Y = Z$.

8.9 Proposition. *Let X be a topological space.*

1) For every point $x_0 \in X$ there exist a path connected component $Y \subseteq X$ such that $x_0 \in Y$.
2) If Y, Y' are path connected components of X then either $Y \cap Y' = \varnothing$ or $Y = Y'$.

Proof. Similar to the proof of Proposition 7.18. □

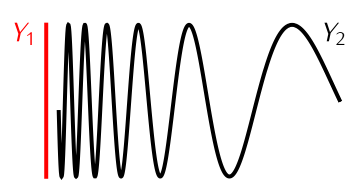
8.10 Proposition. *Let $x_0 \in X$ The path connected component $Y \subseteq X$ that contains x_0 is given by:*

$$Y = \{x \in X \mid \text{there exists a path joining } x \text{ with } x_0\}$$

Proof. Exercise. □

8.11 Example. Let Y be the topologist's sine curve. The space Y has only one connected component (since Y is connected). On the other hand it has two path connected components:

$$Y_1 = \{(0, y) \mid -1 \leq y \leq 1\} \quad \text{and} \quad Y_2 = \{(x, \sin(\tfrac{1}{x})) \mid x > 0\}$$



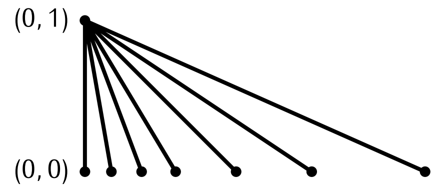
8.12 Definition. Let X be a topological space.

1) X is *locally connected* if for any $x \in X$ and any open neighborhood U of x there is an open neighborhood V of x such that $V \subseteq U$ and V is connected.

2) X is *locally path connected* if for any $x \in X$ and any open neighborhood U of x there is an open neighborhood V of x such that $V \subseteq U$ and V is path connected.

8.13 Example. Let $X = (0,1) \cup (2,3) \subseteq \mathbb{R}$. The space X is neither connected nor path connected but it is both locally connected and locally path connected.

8.14 Example. Let X be the subspace of $\mathbb{R}^2$ consisting of the intervals joining points $(0,0)$ and $(1/n, 0)$ for $n = 1, 2, \dots$ with the point $(0,1)$:



The space X is called the *harmonic broom*. This space is connected and path connected. It is neither locally connected nor locally path connected since any neighborhood of the point $(0,0)$ that does not contain the point $(0,1)$ is not connected.

8.15 Proposition. *If X is locally path connected then it is locally connected.*

Proof. Exercise. □

8.16 Proposition. *If a space X is locally connected then connected components of X are open in X.*

Proof. Exercise. □

8.17 Proposition. *If a space X is locally path connected then path connected components of X are open in X.*

Proof. Exercise. □

8.18 Proposition. *If X is a connected and locally path connected space then X is path connected.*

Proof. It is enough to show that X has only one path connected component. Assume, by contradiction, that X has at least two distinct path connected components. Let Y be some path connected component of X and let Y' be the union of all other path connected components. By Proposition 8.17 both Y and Y' are open sets. Also $Y \cap Y' = \varnothing$ and $Y \cup Y' = X$. This contradicts the assumption that X is connected. □

Exercises to Chapter 8

E8.1 Exercise. Prove Proposition 8.5.

E8.2 Exercise. Prove Proposition 8.10.

E8.3 Exercise. The goal of this exercise is to verify that the statement of Note 8.6 holds. Show that the topologist sine curve (Exercise 7.14) is not path connected.

E8.4 Exercise. Let X be a topological space whose elements are integers, and such that $U \subseteq X$ is open if either $U = \varnothing$ or $U = X \smallsetminus S$ for some finite set S. Show that X is locally connected but not locally path connected.

E8.5 Exercise. Prove Proposition 8.16.

E8.6 Exercise. Prove Proposition 8.17.

E8.7 Exercise. Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with coefficients in $\mathbb{R}$. Since each matrix consists of n^2 real numbers the set $M_n(\mathbb{R})$ can be identified with $\mathbb{R}^{n^2}$. Using this identification we can consider $M_n(\mathbb{R})$ as a metric space. Let $GL_n(\mathbb{R})$ be the subspace of $M_n(\mathbb{R})$ consisting of all invertible matrices. Equivalently:

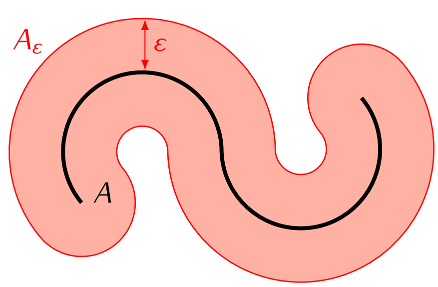
$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$$

where $\det A$ is the determinant of A. Show that $GL_n(R)$ has exactly two path connected components: $GL_n^+(\mathbb{R})$ and $GL_n^-(\mathbb{R})$ where

$$GL_n^+(\mathbb{R}) = \{A \in GL_n(R) \mid \det A > 0\}, \qquad GL_n^-(\mathbb{R}) = \{A \in GL_n(R) \mid \det A < 0\}$$

E8.8 Exercise. Let X be a subspace of $\mathbb{R}^n$. Show that if X is connected and it is open in $\mathbb{R}^n$ then X is path connected.

E8.9 Exercise. For $A \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ define $A_\varepsilon := \{x \in \mathbb{R}^n \mid d(x, y) < \varepsilon \text{ for some } y \in A\}$.



Show that if A is connected then A_ε is path connected for any $\varepsilon > 0$.

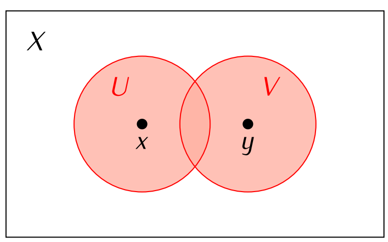
E8.10 Exercise. Let A be a countable set of points in $\mathbb{R}^2$. Show that the space $\mathbb{R}^2 \smallsetminus A$ is path connected.

E8.11 Exercise. Let X be a topological space and let $U, V \subseteq X$ be open sets such that $U \cup V$ and $U \cap V$ are path connected. Show that U and V are path connected.

9 | Separation Axioms

Separation axioms are a family of topological invariants that give us new ways of distinguishing between various spaces. The idea is to look how open sets in a space can be used to create "buffer zones" separating pairs of points and closed sets. Separations axioms are denoted by T_1, T_2, etc., where T comes from the German word *Trennungsaxiom*, which just means "separation axiom". Separation axioms can be also seen as a tool for identifying how close a topological space is to being metrizable: spaces that satisfy an axiom T_i can be considered as being closer to metrizable spaces than spaces that do not satisfy T_i.

9.1 Definition. A topological space X *satisfies the axiom* T_1 if for every points $x, y \in X$ such that $x \neq y$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.



9.2 Example. If X is a space with the antidiscrete topology and X consists of more than one point then X does not satisfy T_1.

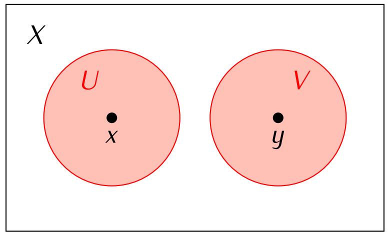
9.3 Proposition. *Let X be a topological space. The following conditions are equivalent:*

1) *X satisfies T_1.*
2) *For every point $x \in X$ the set $\{x\} \subseteq X$ is closed.*

Proof. Exercise. ☐

9.4 Definition. A topological space X *satisfies the axiom* T_2 if for any points $x, y \in X$ such that $x \neq y$

there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$.



A space that satisfies the axiom T_2 is called a *Hausdorff space*.

9.5 Note. Any metric space satisfies T_2. Indeed, for $x, y \in X$, $x \neq y$ take $U = B(x, \varepsilon)$, $V = B(y, \varepsilon)$ where $\varepsilon < \frac{1}{2}\varrho(x, y)$. Then U, V are open sets, $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

9.6 Note. If X satisfies T_2 then it satisfies T_1.

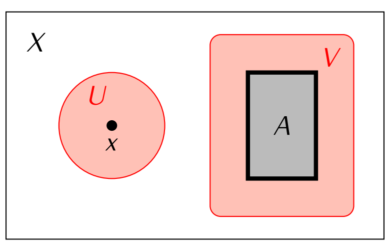
9.7 Example. The real line $\mathbb{R}$ with the Zariski topology satisfies T_1 but not T_2.

The following is a generalization of Proposition 5.13

9.8 Proposition. *Let X be a Hausdorff space and let $\{x_n\}$ be a sequence in X. If $x_n \to y$ and $x_n \to z$ for some then $y = z$.*

Proof. Exercise. $\square$

9.9 Definition. A topological space X *satisfies the axiom* T_3 if X satisfies T_1 and if for each point $x \in X$ and each closed set $A \subseteq X$ such that $x \notin A$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $A \subseteq V$, and $U \cap V = \varnothing$.



A space that satisfies the axiom T_3 is called a *regular space*.

9.10 Note. Since in spaces satisfying T_1 sets consisting of a single point are closed (9.3) it follows that if a space satisfies T_3 then it satisfies T_2.

9.11 Example. Here is an example of a space X that satisfies T_2 but not T_3. Take the set

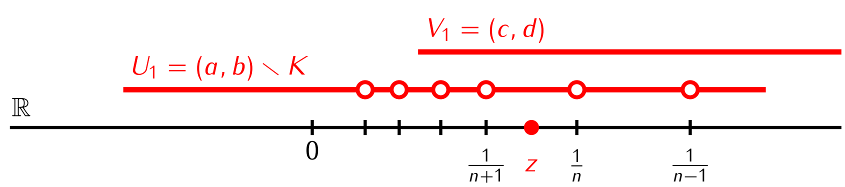
$$K = \{\tfrac{1}{n} \mid n = 1, 2, \dots\} \subseteq \mathbb{R}$$

Define a topological space X as follows. As a set $X = \mathbb{R}$. A basis $\mathcal{B}$ of the topology on X is given by

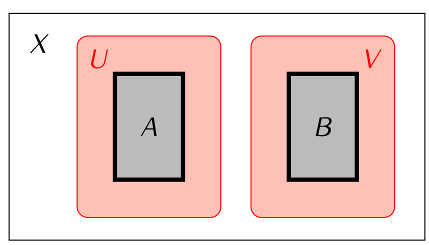
$$\mathcal{B} = \{U \subseteq \mathbb{R} \mid U = (a, b) \text{ or } U = (a, b) \smallsetminus K \text{ for some } a < b\}$$

Notice that the set $X \smallsetminus K$ is open in X, so K is a closed set.

The space X satisfies T_2 since any two points can be separated by some open intervals. On the other hand we will see that X does not satisfy T_3. Take $x = 0 \in X$ and let $U, V \subseteq X$ be open sets such that $x \in U$ and $K \subseteq V$. We will show that $U \cap V \neq \varnothing$. Since $x \in U$ and U is open there exists a basis element $U_1 \in \mathcal{B}$ such that $x \in U_1$ and $U_1 \subseteq U$. By assumption $U_1 \cap K = \varnothing$, so $U_1 = (a, b) \smallsetminus K$ for some $a < 0 < b$. Take n such that $\frac{1}{n} < b$. Since $\frac{1}{n} \in V$ and V is open there is a basis element $V_1 \in \mathcal{B}$ such that $\frac{1}{n} \in V_1$ and $V_1 \subseteq V$. Since $V_1 \cap K \neq \varnothing$ we have $V_1 = (c, d)$ for some $c < \frac{1}{n} < d$. For any $z \in \mathbb{R}$ such that $c < z < \frac{1}{n}$ and $z \notin K$ we have $z \in U_1 \cap V_1$, and so $z \in U \cap V$.



9.12 Definition. A topological space X *satisfies the axiom* T_4 if X satisfies T_1 and if for any closed sets $A, B \subseteq X$ such that $A \cap B = \varnothing$ there exist open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$.



A space that satisfies the axiom T_4 is called a *normal space*.

9.13 Note. If X satisfies T_4 then it satisfies T_3.

9.14 Theorem. *Every metric space is normal.*

The proof of this theorem will rely on the following fact:

9.15 Proposition. *Let X be a topological space satisfying T_1. If for any pair of closed sets $A, B \subseteq X$ satisfying $A \cap B = \varnothing$ there exists a continuous function $f\colon X \to [0, 1]$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$ then X is a normal space.*

Proof. Exercise. □

9.16 Definition. Let (X, ϱ) be a metric space. The *distance between a point $x \in X$ and a set $A \subseteq X$* is the number

$$\varrho(x, A) := \inf\{\varrho(x, a) \mid a \in A\}$$

9.17 Lemma. *If (X, ϱ) is a metric space and $A \subseteq X$ is a closed set then $\varrho(x, A) = 0$ if and only if $x \in A$.*

Proof. Exercise. □

9.18 Lemma. *Let (X, ϱ) be a metric space and $A \subseteq X$. The function $\varphi\colon X \to \mathbb{R}$ given by*

$$\varphi(x) = \varrho(x, A)$$

is continuous.

Proof. Let $x \in X$. We need to check that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varrho(x, x') < \delta$ then $|\varphi(x) - \varphi(x')| < \epsilon$. It will be enough to show that

$$|\varphi(x) - \varphi(x')| \leq \varrho(x, x')$$

for all $x, x' \in X$ since then we can take $\delta = \varepsilon$.

For $a \in A$ we have

$$\varrho(x, A) \leq \varrho(x, a) \leq \varrho(x, x') + \varrho(x', a)$$

This gives

$$\varrho(x, A) \leq \varrho(x, x') + \varrho(x', A)$$

and so

$$\varphi(x) - \varphi(x') = \varrho(x, A) - \varrho(x', A) \leq \varrho(x, x')$$

In the same way we obtain $\varphi(x') - \varphi(x) \leq \varrho(x', x)$, and so $|\varphi(x) - \varphi(x')| \leq \varrho(x, x')$.

□

Proof of Theorem 9.14. Let (X, ϱ) be a metric space and let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$. By Proposition 9.15 it will suffice to show that there exists a continuous function $f\colon X \to [0, 1]$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$. Take f to be the function given by

$$f(x) = \frac{\varrho(x, A)}{\varrho(x, A) + \varrho(x, B)}$$

By Lemma 9.17 $\varrho(x, A) = 0$ only if $x \in A$, and $\varrho(x, B) = 0$ only if $x \in B$. Since $A \cap B = \varnothing$ we have $\varrho(x, A) + \varrho(x, B) \neq 0$ for all $x \in X$, so f is well defined. From Lemma 9.18 it follows that f is a continuous function. Finally, for any $x \in A$ we have

$$f(x) = \frac{\varrho(x, A)}{\varrho(x, A) + \varrho(x, B)} = \frac{0}{0 + \varrho(x, B)} = 0$$

and for any $x \in B$ we have

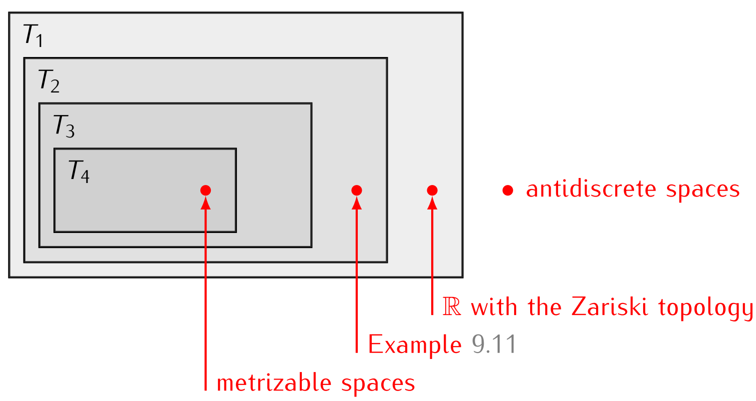
$$f(x) = \frac{\varrho(x, A)}{\varrho(x, A) + \varrho(x, B)} = \frac{\varrho(x, A)}{\varrho(x, A) + 0} = 1$$

□

Notice that the function f constructed in the proof of Theorem 9.14 satisfies a condition that is stronger than the assumption of Proposition 9.15: we have $f(x) = 0$ if and only if $x \in A$ and $f(x) = 1$ if and only if $x \in B$. Thus we obtain:

9.19 Corollary. *If (X, ϱ) is a metric space and $A, B \subseteq X$ are closed sets such that $A \cap B = \varnothing$ then there exists a continuous function $f\colon X \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.*

9.20 Note. The results described above can be summarized by the following picture:



Each rectangle represents the class of topological spaces satisfying the corresponding separation axiom. No area of this diagram is empty. Even though we have not seen here an example of a space that satisfies T_3 but not T_4 such spaces do exist.

Exercises to Chapter 9

E9.1 Exercise. Prove Proposition 9.3.

E9.2 Exercise. Prove Proposition 9.8.

E9.3 Exercise. Prove Proposition 9.15.

E9.4 Exercise. Prove Lemma 9.17.

E9.5 Exercise. Let X be a topological space and let Y be a subspace of X.

a) Show that if X satisfies T_1 then Y satisfies T_1.
b) Show that if X satisfies T_2 then Y satisfies T_2.
c) Show that if X satisfies T_3 then Y satisfies T_3.

Note: It may happen that X satisfies T_4 but Y does not.

E9.6 Exercise. Show that if X is a normal space and Y is a closed subspace of X then Y is a normal space.

E9.7 Exercise. Let X be a Hausdorff space. Show that the following conditions are equivalent:

(i) every subspace of X is a normal space.
(ii) for any two sets $A, B \subseteq X$ such that $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$ there exists open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \varnothing$.

E9.8 Exercise. This is a generalization of Exercise 6.9. Recall that a retract of a topological space X is a subspace $Y \subseteq X$ for which there exists a continuous function $r\colon X \to Y$ such that $r(x) = x$ for all $x \in Y$. Show that if X is a Hausdorff space and $Y \subseteq X$ is a retract of X then Y is a closed in X.

E9.9 Exercise. Let X be a space satisfying T_1. Show that the following conditions are equivalent:

(i) X is a normal space.
(ii) For any two disjoint closed sets $A, B \subseteq X$ there exist closed sets $A', B' \subseteq X$ such that $A \cap A' = \varnothing$, $B \cap B' = \varnothing$ and $A' \cup B' = X$.

E9.10 Exercise. Let X be a topological space and Y be a Hausdorff space. Let $f, g\colon X \to Y$ be continuous functions and let $A \subseteq X$ be given by

$$A = \{x \in X \mid f(x) = g(x)\}$$

Show that A is closed in X.

E9.11 Exercise. Let X be a topological space, Y be a Hausdorff space, and let A be a set dense in X. Let $f, g\colon X \to Y$ be continuous functions. Show that if $f(x) = g(x)$ for all $x \in A$ then $f(x) = g(x)$ for all $x \in X$

E9.12 Exercise. Let $f\colon X \to Y$ be a continuous function. Assume that f is onto and that for any closed set $A \subseteq X$ the set $f(A) \subseteq Y$ is closed. Show that if X is a normal space then Y is also normal.

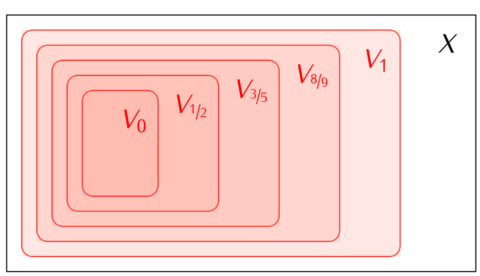
10 | Urysohn Lemma

The separation axioms introduced in the last chapter can be seen as a tool to constructing closer and closer approximations of the class of metrizable spaces. However, even normal spaces, i.e. spaces that satisfy the of the strongest of these axioms need not be metrizable. For example, take the real line $\mathbb{R}$ with the arrow topology (4.8). One can show that it is a normal space (exercise), but by Exercise 5.15 this space is not metrizable. The Urysohn Lemma, which is the main result of this chapter, shows however that normal spaces retain some useful properties of metrizable spaces. Recall that in the last chapter we have seen that for any metric space X, and any pair of disjoint closed sets in X we can find is a continuous function $f: X \to [0, 1]$ which maps one set to 0 and the other set to 1. The Urysohn lemma says that the same property holds for any normal space:

10.1 Urysohn Lemma. *Let X be a normal space and let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$. There exists a continuous function $f: X \to [0, 1]$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$.*

The proof of this fact will use a couple of lemmas:

10.2 Lemma. *Let X be a topological space. Assume that for each $r \in [0, 1] \cap \mathbb{Q}$ we are given an open set $V_r \subseteq X$ such that $\overline{V}_r \subseteq V_{r'}$ if $r < r'$. There exists a continuous function $f: X \to [0, 1]$ such that if $x \in V_r$ then $f(x) \leq r$ and if $x \notin V_1$ then $f(x) = 1$.*



Proof. Define the function $f\colon X \to [0,1]$ by:

$$f(x) = \begin{cases} 1 & \text{if } x \notin V_1 \\ \inf\{r \mid x \in V_r\} & \text{if } x \in V_1 \end{cases}$$

We need to show that f is continuous. Notice that the set

$$\mathcal{S} = \{U \subseteq [0,1] \mid U = [0,a) \text{ or } U = (a,1] \text{ for some } a \in [0,1]\}$$

is a subbasis of the topology on $[0,1]$, so it will suffice to show that for any $a \in [0,1]$ the sets $f^{-1}([0,a))$ and $f^{-1}((a,1])$ are open in X.

We have:

$$f^{-1}([0,a)) = \bigcup_{r<a} V_r$$

so $f^{-1}([0,a))$ is an open set.

Next, we claim that

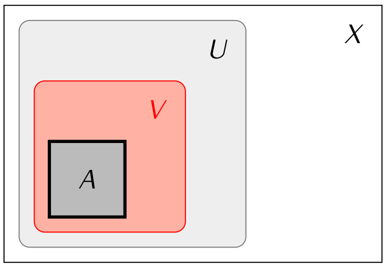
$$f^{-1}((a,1]) = \bigcup_{r>a} (X \smallsetminus \overline{V}_r)$$

Indeed, if $x \in X \smallsetminus \overline{V}_r$ for some $r > a$ then $x \notin V_r$. This gives $f(x) \geq r > a$, and so $x \in f^{-1}((a,1])$. Conversely, assume that $x \in f^{-1}((a,1])$. Then $f(x) > a$ so there exist $r > a$ such that $x \notin V_r$ Take $r' \in [0,1] \cap \mathbb{Q}$ such that $a < r' < r$. Since $\overline{V}_{r'} \subseteq V_r$ we get that $x \notin \overline{V}_{r'}$, or equivalently $x \in X \smallsetminus \overline{V}_{r'}$. Therefore $x \in \bigcup_{r>a} X \smallsetminus \overline{V}_r$.

Since the sets $X \smallsetminus \overline{V}_r$ are open it follows that $f^{-1}((a,1])$ is an open set.

$\square$

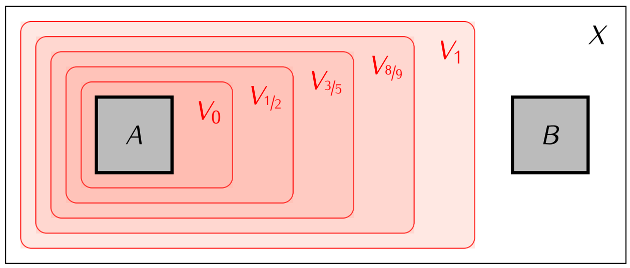
10.3 Lemma. *Let X be a normal space, let $A \subseteq X$ be a closed set and let $U \subseteq X$ be an open set such that $A \subseteq U$. There exists an open set V such that $A \subseteq V$ and $\overline{V} \subseteq U$.*



Proof. Exercise. $\square$

Proof of Urysohn Lemma 10.1. We will show that for each $r \in [0,1] \cap \mathbb{Q}$ there exists an open set $V_r \subseteq X$ such that

1) $A \subseteq V_0$
2) $B \subseteq X \smallsetminus V_1$
3) if $r < r'$ then $\overline{V}_r \subseteq V_{r'}$.



By Lemma 10.2 this will give a continuous function $f\colon X \to [0,1]$ such that $f(x) \leq r$ for all $x \in V_r$ and $f(x) = 1$ for all $x \notin V_1$. By 1) we will get then that $f(x) = 0$ for all $x \in A$ and by 2) that $f(x) = 1$ for all $x \in B$.

Construction of sets V_r proceeds as follows. Since the set $[0,1] \cap \mathbb{Q}$ is countable we can arrange its elements into a sequence:

$$[0,1] \cap \mathbb{Q} = \{r_0, r_1, r_2, \dots\}$$

We can assume that $r_0 = 0$ and $r_1 = 1$. We will construct the sets V_{r_k} by induction with respect to k.

Take $V_{r_1} = X \smallsetminus B$. Since V_{r_1} is open and $A \subseteq V_{r_1}$ by Lemma 10.3 there exists an open set V such that $A \subseteq V$ and $\overline{V} \subseteq V_{r_0}$. Define $V_{r_1} = V$.

Next, assume that we have already constructed sets $V_{r_0}, \dots, V_{r_n}$. We obtain the set $V_{r_{n+1}}$ as follows. Let r_p be the biggest number in the set $\{r_0, \dots, r_n\}$ satisfying $r_p < r_{n+1}$, and let r_q be the smallest number in $\{r_0, \dots, r_n\}$ satisfying $r_{n+1} < r_q$. Since $r_p < r_q$ we have $\overline{V}_{r_p} \subseteq V_{r_q}$. By Lemma 10.3 there exists an open set V such that $\overline{V}_{r_p} \subseteq V$ and $\overline{V} \subseteq V_{r_q}$. We set $V_{r_{n+1}} := V$.

□

One can ask whether an analog of Urysohn Lemma holds for regular spaces: given a regular space X, a point $x \in X$, and a closed set $A \subseteq X$ such that $x \notin A$ is there a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(A) \subseteq \{1\}$? It turns out that this is not true, but it provides motivation for one more separation axiom:

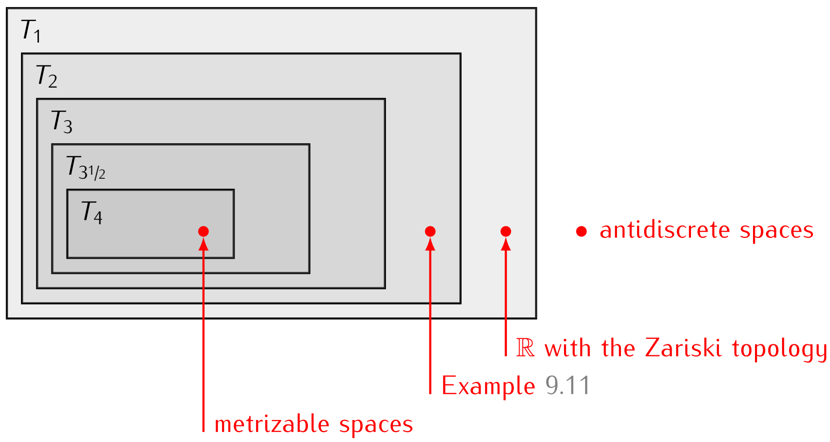
10.4 Definition. A topological space X *satisfies the axiom* $T_{3^1/_2}$ if X satisfies T_1 and if for each point $x \in X$ and each closed set $A \subseteq X$ such that $x \notin A$ there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f|_A = 0$.

A space that satisfies the axiom $T_{3^1/_2}$ is called a *completely regular space* or a *Tychonoff space*.

While Definition 10.4 may seem a bit artificial at the moment, there is a different context which makes the class of completely regular spaces interesting. We will get back to this in Chapter 18.

10.5 Note. By Urysohn Lemma every normal space is completely regular. Also, if X is a completely regular space then X is regular. Indeed, for a point $x \in X$ and a closed set $A \subseteq X$ such that $x \notin A$ let $f\colon X \to [0, 1]$ be a function as in Definition 10.4. Let $U = f^{-1}([0, \frac{1}{2}))$ and let $V = f^{-1}((\frac{1}{2}, 1])$. Then the sets U, V are open in X, $A \subseteq U$, $x \in V$, and $U \cap V = \varnothing$.

The diagram in Note 9.20 can be now extended as follows:



No area of this diagram is empty: there exist regular spaces that are not completely regular and there exist completely regular spaces that are not normal.

Exercises to Chapter 10

E10.1 Exercise. Let $\mathbb{R}_{Ar}$ denote the set of real numbers with the arrow topology (4.8). Show that this space is normal.

E10.2 Exercise. Prove Lemma 10.3.

E10.3 Exercise. By Corollary 9.19 metric spaces satisfy a stronger version of the Urysohn Lemma 10.2: for any pair of disjoint, closed subsets A, B in a metric space X there exists a continuous function $f\colon X \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$. One can ask if the same is true for all normal spaces. The goal of this exercise is to resolve this question.

a) A set $A \subseteq X$ is called a G_δ*-set* if there exists a countable family of open sets $U_1, U_2, \dots$ such that $A = \bigcap_{n=1}^{\infty} U_n$. Let X be a topological space and let $A, B \subseteq X$ be disjoint, closed subsets such that there exists a function $f\colon X \to [0, 1]$ with $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$. Show that both A and B are G_δ-sets.

Note: One can also show that the converse holds: if X is a normal space and A, B are closed, disjoint G_δ-sets in X then such function f exists (see Exercise 11.4).

b) Let X be be a topological space defined as follows. As a set $X = \mathbb{R} \cup \{\infty\}$ where ∞ is an extra point. Any set $U \subseteq X$ such that $\infty \notin U$ is open in X. If $\infty \in U$ then U is open if $X \smallsetminus U$ is a finite set. Show that X is a normal space, but that not every closed set in X is a G_δ-set. Thus the stronger version of Urysohn Lemma does not hold in X.

Notice that as a consequence the space X described in part b) gives another example of a space which is normal but not metrizable.

11 | Tietze Extension Theorem

The main goal of this chapter is to prove the following fact which describes one of the most useful properties of normal spaces:

11.1 Tietze Extension Theorem (v.1). *Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f\colon A \to [a, b]$ be a continuous function for some $[a, b] \subseteq \mathbb{R}$. There exits a continuous function $\bar{f}\colon X \to [a, b]$ such that $\bar{f}|_A = f$.*

The main idea of the proof is to use Urysohn Lemma 10.1 to construct functions $\bar{f}_n\colon X \to [a, b]$ for $n = 1, 2, \dots$ such that as n increases $\bar{f}_n|_A$ gives ever closer approximations of f. Then we take $\bar{f}$ to be the limit of the sequence $\{\bar{f}_n\}$. We start by looking at sequences of functions and their convergence.

11.2 Definition. Let X, Y be a topological spaces and let $\{f_n\colon X \to Y\}$ be a sequence of functions. We say that the sequence $\{f_n\}$ *converges pointwise* to a function $f\colon X \to Y$ if for each $x \in X$ the sequence $\{f_n(x)\} \subseteq Y$ converges to the point $f(x)$.

11.3 Note. If $\{f_n\colon X \to Y\}$ is a sequence of continuous functions that converges pointwise to $f\colon X \to Y$ then f need not be continuous. For example, let $f_n\colon [0, 1] \to \mathbb{R}$ be the function given by $f_n(x) = x^n$. Notice that $f_n(x) \to 0$ for all $x \in [0, 1)$ and that $f_n(1) \to 1$. Thus the sequence $\{f_n\}$ converges pointwise to the function $f\colon [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases}$$

The functions f_n are continuous but f is not.

11.4 Definition. Let X be a topological space, let (Y, ϱ) be a metric space, and let $\{f_n\colon X \to Y\}$ be a sequence of functions. We say that the sequence $\{f_n\}$ *converges uniformly* to a function $f\colon X \to Y$ if for every $\varepsilon > 0$ there exists $N > 0$ such that

$$\varrho(f(x), f_n(x)) < \varepsilon$$

for all $x \in X$ and for all $n > N$.

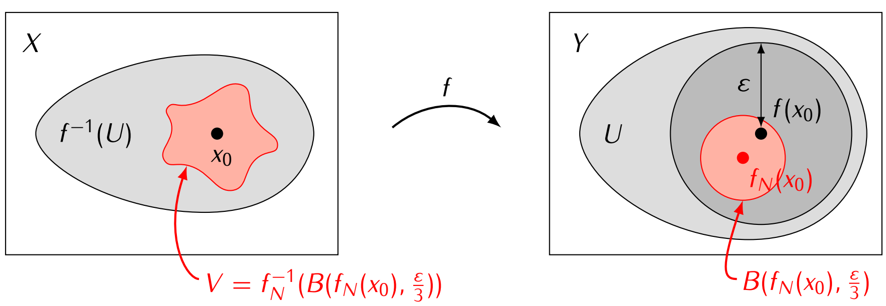
11.5 Note. If a sequence $\{f_n\}$ converges uniformly to f then it also converges pointwise to f, but the converse is not true in general.

11.6 Proposition. *Let X be a topological space and let (Y, ϱ) be a metric space. Assume that $\{f_n \colon X \to Y\}$ is a sequence of functions that converges uniformly to $f \colon X \to Y$. If all functions f_n are continuous then f is also a continuous function.*

Proof. Let $U \subseteq Y$ be an open set. We need to show that the set $f^{-1}(U) \subseteq X$ is open. If suffices to check that each point $x_0 \in f^{-1}(U)$ has an open neighborhood V such that $V \subseteq f^{-1}(U)$. Since U is an open set there exists $\varepsilon > 0$ such $B(f(x_0), \varepsilon) \subseteq U$. Choose $N > 0$ such that $\varrho(f(x), f_N(x)) < \frac{\varepsilon}{3}$ for all $x \in X$, and take $V = f_N^{-1}(B(f_N(x_0), \frac{\varepsilon}{3}))$. Since f_N is a continuous function the set V is an open neighborhood of x_0 in X. It remains to show that $V \subseteq f^{-1}(U)$. For $x \in V$ we have:

$$\varrho(f(x), f(x_0)) \leq \varrho(f(x), f_N(x)) + \varrho(f_N(x), f_N(x_0)) + \varrho(f_N(x_0), f(x_0)) < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon$$

This means that $f(x) \in B(f(x_0), \varepsilon) \subseteq U$, and so $x \in f^{-1}(U)$.



$\square$

11.7 Lemma. *Let X be a normal space, $A \subseteq X$ be a closed subspace, and let $f \colon A \to \mathbb{R}$ be a continuous function such that for some $C > 0$ we have $|f(x)| \leq C$ for all $x \in A$. There exists a continuous function $g \colon X \to \mathbb{R}$ such that $|g(x)| \leq \frac{1}{3}C$ for all $x \in X$ and $|f(x) - g(x)| \leq \frac{2}{3}C$ for all $x \in A$.*

Proof. Define $Y := f^{-1}([-C, -\frac{1}{3}C])$, $Z := f^{-1}([\frac{1}{3}C, C])$. Since $f \colon A \to \mathbb{R}$ is a continuous function these sets are closed in A, but since A is closed in X the sets Y and Z are also closed in X. Since $Y \cap Z = \varnothing$ by the Urysohn Lemma 10.1 there is a continuous function $h \colon X \to [0, 1]$ such that $h(Y) \subseteq \{0\}$ and $h(Z) \subseteq \{1\}$. Define $g \colon X \to \mathbb{R}$ by

$$g(x) := \frac{2C}{3}\left(h(x) - \frac{1}{2}\right)$$

□

Proof of Theorem 11.1. Without loss of generality we can assume that $[a, b] = [0, 1]$. For $n = 1, 2, \dots$ we will construct continuous functions $g_n \colon X \to \mathbb{R}$ such that

(i) $|g_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1}$ for all $x \in X$;
(ii) $\left|f(x) - \sum_{i=1}^{n} g_i(x)\right| \leq \left(\frac{2}{3}\right)^{n}$ for all $x \in A$.

We argue by induction. Existence of g_1 follows directly from Lemma 11.7. Assume that for some $n \geq 1$ we already have functions $g_1, \dots, g_n$ satisfying (i) and (ii). In Lemma 11.7 take f to be the function $f - \sum_{i=1}^{n} g_i$ and take $C = \left(\frac{2}{3}\right)^{n}$. Then we can take $g_{n+1} := g$ where g is the function given by the lemma.

Let $\bar{f}_n := \sum_{i=1}^{n} g_n$ and let $\bar{f} := \sum_{i=1}^{\infty} g_n$. Using condition (i) we obtain that the sequence $\{\bar{f}_n\}$ converges uniformly to $\bar{f}$ (exercise). Since each of the functions $\bar{f}_n$ is continuous, thus by Proposition 11.6 we obtain that $\bar{f}$ is a continuous function. Also, using (ii) be obtain that $\bar{f}(x) = f(x)$ for all $x \in A$ (exercise). □

Here is another useful reformulation of Tietze Extension Theorem:

11.8 Tietze Extension Theorem (v.2). *Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f \colon A \to \mathbb{R}$ be a continuous function. There exits a continuous function $\bar{f} \colon X \to \mathbb{R}$ such that $\bar{f}|_A = f$.*

Proof. It is enough to show that for any continuous function $g \colon A \to (-1, 1)$ we can find a continuous function $\bar{g} \colon X \to (-1, 1)$ such that $\bar{g}|_A = g$. Indeed, if this holds then given a function $f \colon A \to \mathbb{R}$ let $g = hf$ where $h \colon \mathbb{R} \to (-1, 1)$ is an arbitrary homeomorphism. Then we can take $\bar{f} = h^{-1}\bar{g}$.

Assume then that $g \colon A \to (-1, 1)$ is a continuous function. By Theorem 11.1 there is a function $g_1 \colon X \to [-1, 1]$ such that $g_1|_A = g$. Let $B := g_1^{-1}(\{-1, 1\})$. The set B is closed in X and $A \cap B = \varnothing$ since $g_1(A) = g(A) \subseteq (-1, 1)$. By Urysohn Lemma 10.1 there is a continuous function $k \colon X \to [0, 1]$ such that $B \subseteq k^{-1}(\{0\})$ and $A \subseteq k^{-1}(\{1\})$. Let $\bar{g}(x) := k(x) \cdot g_1(x)$. We have:

1) if $g_1(x) \in (-1, 1)$ then $\bar{g}(x) \in (-1, 1)$
2) if $g_1(x) \in \{-1, 1\}$ then $x \in B$ so $\bar{g}(x) = 0 \cdot g_1(x) = 0$

It follows that $\bar{g} \colon X \to (-1, 1)$. Also, $\bar{g}$ is a continuous function since k and g_1 are continuous. Finally, if $x \in A$ then $\bar{g}(x) = 1 \cdot g_1(x) = g(x)$, so $\bar{g}|_A = g$. □

Tietze Extension Theorem holds for functions defined on normal spaces. It turns out the function extension property is actually equivalent to the notion of normality of a space:

11.9 Theorem. *Let X be a space satisfying T_1. The following conditions are equivalent:*

1) X is a normal space.

2) For any closed sets $A, B \subseteq X$ such that $A \cap B = \varnothing$ there is a continuous function $f\colon X \to [0,1]$ such that such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$.

3) If $A \subseteq X$ is a closed set then any continuous function $f\colon A \to \mathbb{R}$ can be extended to a continuous function $\bar{f}\colon X \to \mathbb{R}$.

Proof. The implication 1) $\Rightarrow$ 2) is the Urysohn Lemma 10.1 and 2) $\Rightarrow$ 1) is Proposition 9.15. The implication 1) $\Rightarrow$ 3) is the Tietze Extension Theorem 11.8. The proof of implication 3) $\Rightarrow$ 1) is an exercise.

$\square$

Exercises to Chapter 11

E11.1 Exercise. Prove implication 3) $\Rightarrow$ 1) of Theorem 11.9.

E11.2 Exercise. Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f\colon A \to \mathbb{R}$ be a continuous function.

a) Assume that $g\colon X \to \mathbb{R}$ is a continuous function such that $f(x) \leq g(x)$ for all $x \in A$. Show that there exists a continuous function $F\colon X \to \mathbb{R}$ satisfying $F|_A = f$ and $F(x) \leq g(x)$ for all $x \in X$.

b) Assume that $g, h\colon X \to \mathbb{R}$ are a continuous function such that $h(x) \leq f(x) \leq g(x)$ for all $x \in A$ and $h(x) \leq g(x)$ for all $x \in X$. Show that there exists a continuous function $F'\colon X \to \mathbb{R}$ satisfying $F'|_A = f$ and $h(x) \leq F'(x) \leq g(x)$ for all $x \in X$.

E11.3 Exercise. Recall that if X is a topological space then a subspace $Y \subseteq X$ is a called a retract of X if there exists a continuous function $r\colon X \to Y$ such that $r(x) = x$ for all $x \in Y$. Let X be a normal space and let $Y \subseteq X$ be a closed subspace of X such that $Y \cong \mathbb{R}$. Show that Y is a retract of X.

E11.4 Exercise. Let X be topological space. Recall from Exercise 10.3 that a set $A \subseteq X$ is a G_δ-set if there exists a countable family of open sets $U_1, U_2, \dots$ such that $A = \bigcap_{n=1}^{\infty} U_n$.

a) Show that if X is a normal space and $A \subseteq X$ is a closed G_δ-set then there exists a continuous function $f\colon X \to [0,1]$ such that $A = f^{-1}(\{0\})$.

b) Show that if X is a normal space and $A, B \subseteq X$ are closed G_δ-sets such that $A \cap B = \varnothing$ then there exists a continuous function $f\colon X \to [0,1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.

12 | Urysohn Metrization Theorem

In this chapter we return to the problem of determining which topological spaces are metrizable i.e. can be equipped with a metric which is compatible with their topology. We have seen already that any metrizable space must be normal, but that not every normal space is metrizable. We will show, however, that if a normal space space satisfies one extra condition then it is metrizable. Recall that a space X is second countable if it has a countable basis. We have:

12.1 Urysohn Metrization Theorem. *Every second countable normal space is metrizable.*

The main idea of the proof is to show that any space as in the theorem can be identified with a subspace of some metric space. To make this more precise we need the following:

12.2 Definition. A continuous function $i\colon X \to Y$ is an *embedding* if its restriction $i\colon X \to i(X)$ is a homeomorphism (where $i(X)$ has the topology of a subspace of Y).

12.3 Example. The function $i\colon (0,1) \to \mathbb{R}$ given by $i(x) = x$ is an embedding. The function $j\colon (0,1) \to \mathbb{R}$ given by $j(x) = 2x$ is another embedding of the interval $(0,1)$ into $\mathbb{R}$.

12.4 Note. 1) If $j\colon X \to Y$ is an embedding then j must be 1-1.

2) Not every continuous 1-1 function is an embedding. For example, take $\mathbb{N} = \{0, 1, 2, \dots\}$ with the discrete topology, and let $f\colon \mathbb{N} \to \mathbb{R}$ be given

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{n} & \text{if } n > 0 \end{cases}$$

The function f is continuous and it is 1-1, but it is not an embedding since $f\colon \mathbb{N} \to f(\mathbb{N})$ is not a homeomorphism.

12.5 Lemma. *If $j\colon X \to Y$ is an embedding and Y is a metrizable space then X is also metrizable.*

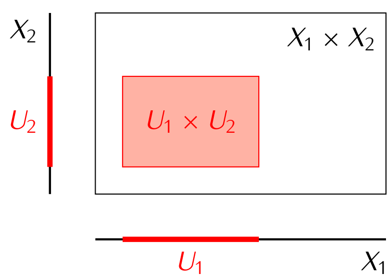
Proof. Let μ be a metric on Y. Define a metric ϱ on X by $\varrho(x_1, x_2) = \mu(j(x_1), j(x_2))$. It is easy to check that the topology on X is induced by the metric ϱ (exercise). $\square$

Let now X be a space as in Theorem 12.1. In order to show that X is metrizable it will be enough to construct an embedding $j\colon X \to Y$ where Y is metrizable. The space Y will be obtained as a product of topological spaces:

12.6 Definition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The *product topology* on $\prod_{i\in I} X_i$ is the topology generated by the basis

$$\mathcal{B} = \left\{\prod_{i\in I} U_i \mid U_i \text{ is open in } X_i \text{ and } U_i \neq X_i \text{ for finitely many indices } i \text{ only}\right\}$$

12.7 Note. 1) If X_1, X_2 are topological spaces then the product topology on $X_1 \times X_2$ is the topology induced by the basis $\mathcal{B} = \{U_1 \times U_2 \mid U_1 \text{ is open in } X_1,\ U_2 \text{ is open in } X_2\}$.



2) In general if $X_1, \dots, X_n$ are topological spaces then the product topology on $X_1 \times \dots \times X_n$ is the topology generated by the basis $\mathcal{B} = \{U_1 \times \dots \times U_n \mid U_i \text{ is open in } X_i\}$.

3) If $\{X_i\}_{i=1}^{\infty}$ is an infinitely countable family of topological spaces then the basis of the product topology on $\prod_{i=1}^{\infty} X_i$ consists of all sets of the form

$$U_1 \times \dots \times U_n \times X_{n+1} \times X_{n+2} \times X_{n+3} \times \dots$$

where $n \geq 0$ and $U_i \subseteq X_i$ is an open set for $i = 1, \dots, n$.

12.8 Proposition. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces and for $j \in I$ let*

$$p_j\colon \prod_{i\in I} X_i \to X_j$$

be the projection onto the j-th factor: $p_j((x_i)_{i\in I}) = x_j$. Then:

1) *for any* $j \in I$ *the function* p_j *is continuous.*
2) *A function* $f\colon Y \to \prod_{i\in I} X_i$ *is continuous if and only if the composition* $p_j f\colon Y \to X_j$ *is continuous for all* $j \in I$

Proof. Exercise. □

12.9 Note. Notice that the basis $\mathcal{B}$ given in Definition 12.6 consists of all sets of the form

$$p_{i_1}^{-1}(U_1) \cap \cdots \cap p_{i_n}^{-1}(U_{i_n})$$

where $i_1, \dots, i_n \in I$ and $U_{i_1} \subseteq X_{i_1}, \dots, U_{i_n} \subseteq X_{i_n}$ are open sets.

12.10 Proposition. *If* $\{X_i\}_{i=1}^{\infty}$ *is a countable family of metrizable spaces then* $\prod_{i=1}^{\infty} X_i$ *is also a metrizable space.*

Proof. Let ϱ_i be a metric on X_i. We can assume that for any $x, x' \in X_i$ we have $\varrho_i(x, x') \leq 1$. Indeed, if ϱ_i does not have this property then we can replace it by the metric ϱ_i' given by:

$$\varrho_i'(x, x') = \begin{cases} \varrho_i(x, x') & \text{if } \varrho_i(x, x') \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

The metrics ϱ_i and ϱ_i' are equivalent (exercise), and so they define the same topology on the space X_i.

Given metrics ϱ_i on X_i satisfying the above condition define a metric ϱ_∞ on $\prod_{i=1}^{\infty} X_i$ by:

$$\varrho_\infty((x_i), (x_i')) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varrho_i(x_i, x_i')$$

The topology induced by the metric ϱ_∞ on $\prod_{i=1}^{\infty} X_i$ is the product topology (exercise). □

12.11 Example. The *Hilbert cube* is the topological space $[0,1]^{\aleph_0}$ obtained as the infinite countable product of the closed interval $[0,1]$:

$$[0,1]^{\aleph_0} = \prod_{i=1}^{\infty} [0,1]$$

Elements of $[0,1]^{\aleph_0}$ are infinite sequences $(t_i) = (t_1, t_2, \dots)$ where $t_i \in [0,1]$ for $i = 1, 2, \dots$ The Hilbert cube is a metric space with a metric ϱ given by

$$\varrho((t_i), (s_i)) = \sum_{i=1}^{\infty} \frac{1}{2^i} |t_i - s_i|$$

Theorem 12.1 is a consequence of the following fact:

12.12 Theorem. *If X is a second countable normal space then there exists an embedding $j\colon X \to [0,1]^{\aleph_0}$.*

Theorem 12.12 will follow in turn from a more general result on embeddings of topological spaces:

12.13 Definition. Let X be a topological space and let $\{f_i\}_{i\in I}$ be a family of continuous functions $f_i\colon X \to [0,1]$. We say that the family $\{f_i\}_{i\in I}$ *separates points from closed sets* if for any point $x_0 \in X$ and any closed set $A \subseteq X$ such that $x_0 \notin A$ there is a function $f_j \in \{f_i\}_{i\in I}$ such that $f_j(x_0) > 0$ and $f_j|_A = 0$.

12.14 Embedding Lemma. *Let X be a T_1-space. If $\{f_i\colon X \to [0,1]\}_{i\in I}$ is a family that separates points from closed sets then the map*

$$f_\infty\colon X \to \prod_{i\in I}[0,1]$$

given by $f_\infty(x) = (f_i(x))_{i\in I}$ is an embedding.

12.15 Note. If the family $\{f_i\}_{i\in I}$ in Lemma 12.14 is infinitely countable then f_∞ is an embedding of X into the Hilbert cube $[0,1]^{\aleph_0}$.

We will show first that Theorem 12.12 follows from Lemma 12.14, and then we will prove the lemma.

Proof of Theorem 12.12. Let $\mathcal{B} = \{V_i\}_{i=1}^\infty$ be a countable basis of X, and let S the set given by

$$S := \{(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid \ \overline{V}_i \subseteq V_j\}$$

If $(i,j) \in S$ then the sets $\overline{V}_i$ and $X \smallsetminus V_j$ are closed and disjoint, so by the Urysohn Lemma 10.1 there is a continuous function $f_{ij}\colon X \to [0,1]$ such that

$$f_{ij}(x) = \begin{cases} 1 & \text{if } x \in \overline{V_i} \\ 0 & \text{if } x \in X \smallsetminus V_j \end{cases}$$

We will show that the family $\{f_{ij}\}_{(i,j)\in S}$ separates points from closed sets. Take $x_0 \in X$ and let $A \subseteq X$ be an closed set such that $x_0 \notin A$. Since $\mathcal{B} = \{V_i\}_{i=1}^\infty$ is a basis of X there is $V_j \in \mathcal{B}$ such that $x_0 \in V_j$ and $V_j \subseteq X \smallsetminus A$. Using Lemma 10.3 we also obtain that there exists $V_i \in \mathcal{B}$ such that $x_0 \in V_i$ and $\overline{V}_i \subseteq V_j$. We have $f_{ij}(x_0) = 1$. Also, since $A \subseteq X \smallsetminus V_j$ we have $f_{ij}|_A = 0$.

By the Embedding Lemma 12.14 the family $\{f_{ij}\}_{(i,j)\in S}$ defines an embedding

$$f_\infty\colon X \to \prod_{(i,j)\in S}[0,1]$$

The set S is countable. If it is infinite then $\prod_{(i,j)\in S}[0,1] \cong [0,1]^{\aleph_0}$. If S is finite then $\prod_{(i,j)\in S}[0,1] \cong [0,1]^N$ for some $N \geq 0$ and $[0,1]^N$ can be identified with a subspace of $[0,1]^{\aleph_0}$. □

Proof of Theorem 12.1. Follows from Theorem 12.12, Lemma 12.5, and the fact that the Hilbert cube is a metric space (12.11). ☐

It remains to prove Lemma 12.14:

Proof of Lemma 12.14. We need to show that the function f_∞ satisfies the following conditions:

1) f_∞ is continuous;
2) f_∞ is 1-1;
3) $f_\infty\colon X \to f_\infty(X)$ is a homeomorphism.

1) Let $p_j\colon \prod_{i\in I}[0,1] \to [0,1]$ be the projection onto the j-th coordinate. Since $p_j f_\infty = f_j$, thus $p_j f_\infty$ is a continuous function for all $j \in I$. Therefore by Proposition 12.8 the function f_∞ is continuous.

2) Let $x, y \in X$, $x \neq y$. Since X is a T_1-space the set $\{y\}$ is closed in X. Therefore there is a function $f_j \in \{f_i\}_{i\in I}$ such that $f_j(x) > 0$ and $f_j(y) = 0$. In particular $f_j(x) \neq f_j(y)$. Since $f_j = p_j f_\infty$ this gives $p_j f_\infty(x) \neq p_j f_\infty(y)$. Therefore $f_\infty(x) \neq f_\infty(y)$.

3) Let $U \subseteq X$ be an open set. We need to prove that the set $f_\infty(U)$ is open in $f(X)$. It will suffice to show that for any $x_0 \in U$ there is a set V open in $f_\infty(X)$ such that $f_\infty(x_0) \in V$ and $V \subseteq f_\infty(U)$.

Given $x_0 \in U$ let $f_j \in \{f_i\}_{i\in I}$ be a function such that $f_j(x_0) > 0$ and $f_j|_{X\smallsetminus U} = 0$. Let $p_j\colon \prod_{i\in I}[0,1] \to [0,1]$ be the projection onto the j-th coordinate. Define

$$V := f_\infty(X) \cap p_j^{-1}((0,1])$$

The set V is open in $f_\infty(X)$ since $p_j^{-1}((0,1])$ is open in $\prod_{i\in I}[0,1]$. Notice that

$$V = \{f_\infty(x) \mid \ x \in X \text{ and } p_j f_\infty(x) > 0\}$$

Since $p_j f_\infty(x_0) = f_j(x_0) > 0$ we have $f(x_0) \in V$. Finally, if $f_\infty(x) \in V$ then $f_j(x) > 0$ which means that $x \in U$, and so $f_\infty(x) \in f_\infty(U)$. This gives $V \subseteq f_\infty(U)$. ☐

One can show that the following holds:

12.16 Proposition. *Every second countable regular space is normal.*

Proof. Exercise. ☐

As a consequence Theorem 12.1 can be reformulated as follows:

12.17 Urysohn Metrization Theorem (v.2). *Every second countable regular space is metrizable.*

While every metrizable space is normal (and regular) such spaces do not need to be second countable. For example, any discrete space X is metrizable, but if X consists of uncountably many points it does not have a countable basis (Exercise 4.10). This means that the converse of the Urysohn Metrization Theorem does not hold. However, this theorem can be generalized to give conditions that are both sufficient and necessary for metrizability of a space. We finish this chapter by giving the statement of such result without proof.

12.18 Definition. Let X be a topological space. A collection $\mathcal{U} = \{U_i\}_{i\in I}$ of open sets in X is *locally finite* if each point $x \in X$ has an open neighborhood V_x such that $V_x \cap U_i \neq \varnothing$ for finitely many $i \in I$ only.

A collection $\mathcal{U}$ is *countably locally finite* if it can be decomposed into a countable union $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ where each collection $\mathcal{U}_n$ is locally finite.

12.19 Nagata-Smirnov Metrization Theorem. *Let X be a topological space. The following conditions are equivalent:*

1) *X is metrizable.*
2) *X is regular and it has a basis which is countably locally finite.*

Exercises to Chapter 12

E12.1 Exercise. Show that the product topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ is the same as the topology induced by the Euclidean metric.

E12.2 Exercise. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The *box topology* on $\prod_{i\in I} X_i$ is the topology generated by the basis

$$\mathcal{B} = \left\{\prod_{i\in I} U_i \mid U_i \text{ is open in } X_i\right\}$$

Notice that for products of finitely many spaces the box topology is the same as the product topology, but that it differs if we take infinite products.

Let $X = \prod_{n=1}^{\infty}[0,1]$ be the product of countably many copies of the interval $[0,1]$. Consider X as a topological space with the box topology. Show that the map $f\colon [0,1] \to X$ given by $f(t) = (t,t,t,\dots)$ is not continuous.

E12.3 Exercise. Prove Proposition 12.8

E12.4 Exercise. Let $\{X_i\}_{i\in I}$ be a family of topological spaces and for $i \in I$ let A_i be a closed set in X_i. Show that the set $\prod_{i\in I} A_i$ is closed in the product topology on $\prod_{i\in I} X_i$.

E12.5 Exercise. Let X and Y be non-empty topological spaces. Show that the space $X \times Y$ is connected if and only if X and Y are connected.

E12.6 Exercise. Assume that X, Y are spaces such that $\mathbb{R} \cong X \times Y$. Show that either X or Y is consists of only one point.

E12.7 Exercise. Let X, Y be topological spaces. For a (not necessarily continuous) function $f\colon X \to Y$ the *graph* of f is the subspace $\Gamma(f)$ of $X \times Y$ given by

$$\Gamma(f) = \{(x, f(x)) \in X \times Y \mid x \in X\}$$

Show that if Y is a Hausdorff space and $f\colon X \to Y$ is a continuous function then $\Gamma(f)$ is closed in $X \times Y$.

E12.8 Exercise. Let X_1, X_2 be topological spaces, and for $i = 1, 2$ let $p_i\colon X_1 \times X_2 \to X_i$ be the projection map.

a) Show that if a set $U \subseteq X_1 \times X_2$ is open in $X_1 \times X_2$ then $p_i(U)$ is open in X_i.

b) Is it true that if $A \subseteq X_1 \times X_2$ is a closed set then $p_i(A)$ must be closed is X_i ? Justify your answer.

E12.9 Exercise. The goal of this exercise is to complete the proof of Proposition 12.10. For $i = 1, 2, \dots$ let (X_i, ϱ_i) be a metric space such that $\varrho_i(x, x') \leq 1$ for all $x, x' \in X_i$. Let ϱ_∞ be a metric the Cartesian product $\prod_{i=1}^{\infty} X_i$ given by

$$\varrho_\infty((x_i), (x'_i)) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varrho_i(x_i, x'_i)$$

Show that the topology defined by ϱ_∞ is the same as the product topology.

E12.10 Exercise. The goal of this exercise is to give a proof of Proposition 12.16. Let X be a second countable regular space and let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$.

a) Show that there exist countable families of open sets $\{U_1, U_2, \dots\}$ and $\{V_1, V_2, \dots\}$ such that

(i) $A \subseteq \bigcup_{i=1}^{\infty} U_i$ and $B \subseteq \bigcup_{i=1}^{\infty} V_i$
(ii) for all $i \geq 1$ we have $\overline{U}_i \cap B = \varnothing$ and $\overline{V}_i \cap A = \varnothing$

b) For $n \geq 1$ define

$$U'_n := U_n \smallsetminus \bigcup_{i=1}^{n} \overline{V_i} \quad \text{and} \quad V'_n := V_n \smallsetminus \bigcup_{i=1}^{n} \overline{U}_i$$

Let $U' = \bigcup_{n=1}^{\infty} U'_n$ and $V' = \bigcup_{n=1}^{\infty} V'_n$. Show that U' and V' are open sets, that $A \subseteq U'$ and $B \subseteq V'$, and that $U' \cap V' = \varnothing$.

E12.11 Exercise. Let $\mathbb{R}_{disc}$ denote the real line with the discrete topology and let $X = \prod_{n=1}^{\infty} \mathbb{R}_{disc}$.

a) Show that X is not second countable.

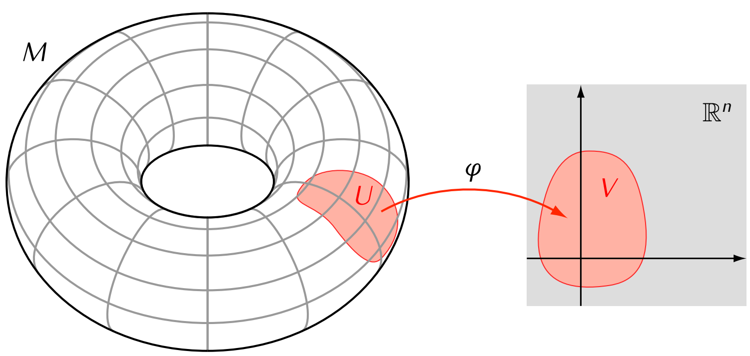
b) By Proposition 12.10 we know that X is a metrizable space. Verify this fact without using Proposition 12.10, but using instead only topological properties of X and the Nagata-Smirnov Metrization Theorem 12.19.

13 | Metrization of Manifolds

Manifolds are among the most important objects in geometry in topology. In this chapter we introduce manifolds and look at some of their basic examples and properties. In particular, as an application of the Urysohn Metrization Theorem, we show that every manifold is a metrizable space.

13.1 Definition. A *topological manifold of dimension n* is a topological space M which is a Hausdorff, second countable, and such that every point of M has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ (we say that M is *locally homeomorphic* to $\mathbb{R}^n$).

13.2 Note. Let M be a manifold of dimension n. If $U \subseteq M$ is an open set and $\varphi\colon U \to V$ is a homeomorphism of U with some open set $V \subseteq \mathbb{R}^n$ then we say that U is a *coordinate neighborhood* and φ is a *coordinate chart* on M.



13.3 Lemma. *If M is an n-dimensional manifold then:*

1) *for any point $x \in M$ there exists a coordinate chart $\varphi\colon U \to V$ such that $x \in U$, V is an open ball $V = B(y, r)$, and $\varphi(x) = y$;*

2) for any point $x \in M$ there exists a coordinate chart $\psi\colon U \to V$ such that $x \in U$, $V = \mathbb{R}^n$, and $\psi(x) = 0$.

Proof. Exercise. □

13.4 Example. A space M is a manifold of dimension 0 if and only if M is a countable (finite or infinite) discrete space.

13.5 Example. If U is an open set in $\mathbb{R}^n$ then U is an n-dimensional manifold. The identity map $\mathrm{id}\colon U \to U$ is then a coordinate chart defined on the whole manifold U. In particular $\mathbb{R}^n$ is an n-dimensional manifold.

13.6 Example. The n-dimensional sphere

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

is an n-dimensional manifold. Indeed, let $x = (x_1, \dots, x_{n+1}) \in S^n$. We need to show that there exists an open neighborhood of x which is homeomorphic to an open subset of $\mathbb{R}^n$. Choose $i \in \{1, 2, \dots, n+1\}$ such that $x_i \neq 0$. Assume that $x_i > 0$. Take $U_i^+ = \{(y_1, \dots, y_{n+1}) \in S^n \mid y_i > 0\}$. The set U_i^+ is open in S^n and $x \in U_i^+$. We have a continuous map

$$h_i^+\colon U_i^+ \to B(0, 1) \subseteq \mathbb{R}^n$$

given by $h(y_1, \dots, y_{n+1}) = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{n+1})$. This map is a homeomorphism with the inverse $(h_i^+)^{-1}\colon B(0, 1) \to U_i^+$ given by

$$(h_i^+)^{-1}(t_1, \dots, t_n) = \left(t_1, \dots, t_{i-1}, \sqrt{1 - (t_1^2 + \dots + t_n^2)}, t_i, \dots t_n, \right)$$

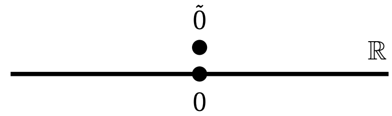
If $x_i < 0$ then we can construct in a similar way a coordinate chart $h_i^-\colon U_i^- \to B(0, 1)$ where $U_i^- = \{(y_1, \dots, y_{n+1}) \in S^n \mid y_i < 0\}$.

13.7 Proposition. *If M is an m-dimensional manifold and N is an n-dimensional manifold then $M \times N$ is an $m + n$-dimensional manifold.*

Proof. Exercise. □

13.8 Example. The *torus* is the space $T^2 := S^1 \times S^1$. Since S^1 is a manifold of dimension 1, thus by Proposition 13.7 T^2 is a manifold of dimension 2. Similarly, for any $n \geq 2$ the *n-dimensional torus* $T^n := \prod_{i=1}^n S^1$ is a manifold of dimension n.

13.9 Note. There exist topological spaces that are locally homeomorphic to $\mathbb{R}^n$, but do not satisfy the the other conditions of the definition of a manifold (13.1). For example, the line with double origin is a topological space L defined as follows. As a set L consist of all points of the real line $\mathbb{R}$ and one additional point that we will denote by $\tilde{0}$:



A basis $\mathcal{B}$ of the topology on L consists of the following sets:

1) any open set in $\mathbb{R}$ is in $\mathcal{B}$;
2) for any $a < 0$ and $b > 0$ the set $(a, 0) \cup \{\tilde{0}\} \cup (0, b)$ is in $\mathcal{B}$.

Notice that L is locally homeomorphic to $\mathbb{R}$. Indeed, since $\mathbb{R}$ is an open set in L thus any point of $L \smallsetminus \{\tilde{0}\}$ has an open neighborhood homeomorphic to $\mathbb{R}$. Also, for any $a < 0 < b$ the set $(a, 0) \cup \{\tilde{0}\} \cup (0, b)$ is an open neighborhood of $\tilde{0}$ which is homeomorphic to the open interval (a, b). On the other hand L is not a Hausdorff space since the point $\tilde{0}$ cannot be separated by open sets from $0 \in \mathbb{R}$. Therefore L is not a manifold. There exist also spaces (e.g. Alexandroff long line) that are locally homeomorphic to $\mathbb{R}^n$ and are Hausdorff, but are not second countable.

The following theorem says that the dimension of a manifold is well defined:

13.10 Invariance of Dimension Theorem. *If M is a non-empty topological space such that M is a manifold of dimension m and M is also a manifold of dimension n then $m = n$.*

In other words if a space is locally homeomorphic to $\mathbb{R}^m$ then if cannot be locally homeomorphic to $\mathbb{R}^n$ for $n \neq m$. While this sounds obvious the proof for arbitrary m and n is actually quite involved and goes beyond the scope of this course. The proof is much simpler for $m = 0$ and $m = 1$ (exercise).

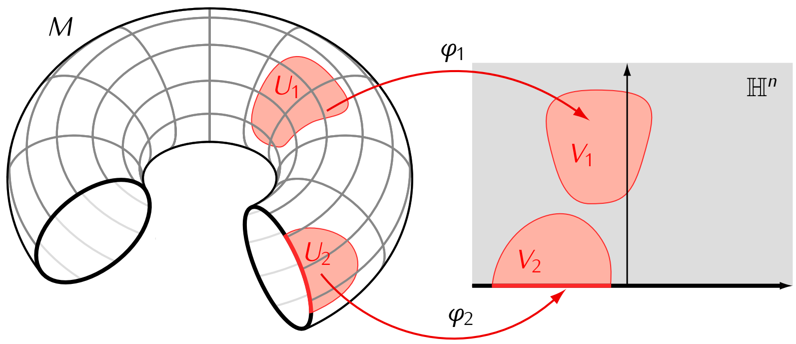
An slight generalization of the notion of a manifold is a manifold with boundary. Let $\mathbb{H}^n$ denote the subspace of $\mathbb{R}^n$ given by $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

13.11 Definition. A *topological n-dimensional manifold with boundary* is a topological space M which is a Hausdorff, second countable, and such that every point of M has an open neighborhood homeomorphic to an open subset of $\mathbb{H}^n$.

As before, if M is a manifold with boundary, U is an open set in M, V is an open set in $\mathbb{H}^n$ and $\varphi\colon U \to V$ is a homeomorphism then we say that φ is a coordinate chart on M.

13.12 Let $\partial\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{H}^n \mid x_n = 0\}$. If M is an n-dimensional manifold with boundary, $\varphi\colon U \to V$ is a coordinate chart, and $x \in U$ then there are two possibilities:

1) $\varphi(x) \in \partial\mathbb{H}^n$
2) $\varphi(x) \notin \partial\mathbb{H}^n$



In the first case we say that the point x is a *boundary point* of M, and in the the second case that x is an *interior point* of M. The next theorem says that a point cannot be a boundary point and an interior point of M at the same time:

13.13 Theorem. *Let M be an n-dimensional manifold with boundary, let $x_0 \in M$ and let $\varphi\colon U \to V$ be a local coordinate chart such that $x_0 \in U$. If $\varphi(x_0) \in \partial\mathbb{H}^n$ then for any other local coordinate chart $\psi\colon U' \to V'$ such that $x_0 \in U'$ we have $\psi(x_0) \in \partial\mathbb{H}^n$.*

The proof in the general case requires similar machinery as the proof of Theorem 13.10, and so we will omit it here. The case when $n = 1$ is much simpler (exercise).

13.14 Definition. Let M be a manifold with boundary. The subspace of M consisting of all boundary points of M is called *the boundary of M* and it is denoted by ∂M.

13.15 Example. The space $\mathbb{H}^n$ is trivially an n-dimensional manifold with boundary.

13.16 Example. For any n the closed n-dimensional ball

$$\overline{B}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$$

is an n-dimensional manifold with boundary (exercise). In this case we have $\partial\overline{B}^n = S^{n-1}$.

13.17 Example. If M is a manifold (without boundary) then we can consider it as a manifold with boundary. where $\partial M = \varnothing$.

13.18 Example. If M is an m-dimensional manifold with boundary and N is an n-dimensional manifold without boundary then $M \times N$ is an $(m+n)$-dimensional manifold with boundary (exercise). In such case we have: $\partial(M \times N) = \partial M \times N$. For example the *solid torus* $\overline{B}^2 \times S^1$ is a 3-dimensional manifold with boundary, and $\partial(\overline{B}^2 \times S^1) = S^1 \times S^1 = T^2$.

Even more generally, if M is an m-dimensional manifold with boundary and N is an n-dimensional manifold with boundary then $M \times N$ is an $(m+n)$-dimensional manifold with boundary and $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$ (exercise).

13.19 Proposition. *If M is an n-dimensional manifold with boundary then:*

1) *$M \smallsetminus \partial M$ is an open subset of M and it is an n-dimensional manifold (without boundary);*
2) *∂M is a closed subset of M and it is an $(n-1)$-dimensional manifold (without boundary).*

Proof. Exercise. □

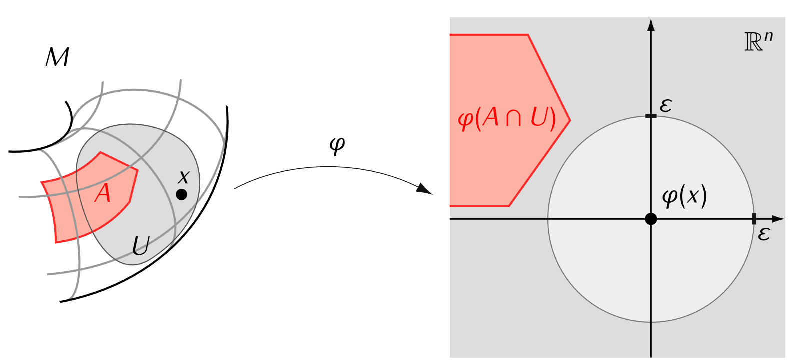
13.20 Theorem. *Every topological manifold (with or without boundary) is metrizable.*

Our argument will use the following fact, the proof of which will be postponed until later (see Exercise 15.5).

13.21 Lemma. *Let M be an n-dimensional topological manifold, and let $\varphi\colon U \to V$ be a coordinate chart on M. If $\bar{B}(x, r)$ is a closed ball in $\mathbb{R}^n$ such that $\bar{B}(x, r) \subseteq V$ then the set $\varphi^{-1}(\bar{B}(x, r))$ is closed in M.*

Proof of Theorem 13.20. We will use Urysohn Metrization Theorem 12.17. Since by definition every manifold is second countable it will be enough to prove that manifolds are regular topological spaces.

Let M be an n-dimensional manifold, let $A \subseteq M$ be a closed set, and let $x \in M$ be a point such that $x \notin A$. We need to show that there exists open sets $W, W' \subseteq M$ such that $A \subseteq W$, $x \in W'$ and $W \cap W' = \varnothing$. Assume first that x does not belong to the boundary of M. We can find an open neighborhood U of x and homeomorphism $\varphi\colon U \to \mathbb{R}^n$ such that $\varphi(x) = 0$. Since A is closed in M the set $A \cap U$ is closed in U, and so $\varphi(A \cap U)$ is closed in $\mathbb{R}^n$. Therefore the set $\mathbb{R}^n \smallsetminus \varphi(A \cap U)$ is open in $\mathbb{R}^n$. Since $0 = \varphi(x) \in \mathbb{R}^n \smallsetminus \varphi(A \cap U)$ we can find an open ball $B(0, \varepsilon)$ such that $B(0, \varepsilon) \subseteq \mathbb{R}^n \smallsetminus \varphi(A \cap U)$:



Take $W = M \smallsetminus \varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2}))$ and $W' = \varphi^{-1}(B(0, \frac{\varepsilon}{2}))$. Notice that $x \in W'$. Also, since W' is open in

U and U is open in M we obtain that W' is open in M. Next, by Lemma 13.21 the set $\varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2}))$ is closed in M, so W is open in M. Moreover, since $W' \subseteq \varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2}))$ we obtain that $W \cap W' = \varnothing$. It remains to show that $A \subseteq W$, or equivalently that $A \cap \varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2})) = \varnothing$. If $y \in A$ and $y \notin U$ then $y \notin \varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2}))$ since $\varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2})) \subseteq U$. Also, if $y \in A \cap U$ then $y \notin \varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2}))$ by the choice of ε, and so we are done. In case when $x \in \partial M$ we can use a similar argument.

$\square$

Exercises to Chapter 13

E13.1 Exercise. Prove Lemma 13.3.

E13.2 Exercise. Let M be an n-dimensional manifold, let $x_0 \in M$ and let $W \subseteq M$ be an open set such that $x_0 \in W$. Show that there exists a coordinate neighborhood $U \subseteq M$ such that $x_0 \in U$ and $U \subseteq W$.

E13.3 Exercise. The goal of this exercise is to prove the Invariance of Dimension Theorem 13.10 in small dimensions.

a) Let M be a manifold of dimension 0. Show that M is not locally homeomorphic to $\mathbb{R}^n$ for any $n \neq 0$.

b) Let M be a manifold of dimension 1. Show that M is not locally homeomorphic to $\mathbb{R}^n$ for any $n \neq 1$.

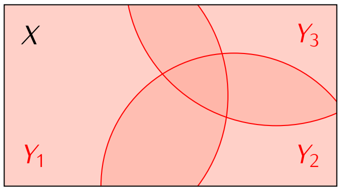
E13.4 Exercise. Prove Theorem 13.13 in the case when M is a 1-dimensional manifold with boundary.

E13.5 Exercise. Let M be an m-dimensional manifold with boundary and N an n-dimensional manifold with boundary Show that $M \times N$ is an $(m+n)$-dimensional manifold with boundary and $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$

E13.6 Exercise. Prove Proposition 13.19.

14 | Compact Spaces

14.1 Definition. Let X be a topological space. A *cover* of X is a collection $\mathcal{Y} = \{Y_i\}_{i\in I}$ of subsets of X such that $\bigcup_{i\in I} Y_i = X$.



If the sets Y_i are open in X for all $i \in I$ then $\mathcal{Y}$ is an *open cover* of X. If $\mathcal{Y}$ consists of finitely many sets then $\mathcal{Y}$ is a *finite cover* of X.

14.2 Definition. Let $\mathcal{Y} = \{Y_i\}_{i\in I}$ be a cover of X. A *subcover* of $\mathcal{Y}$ is cover $\mathcal{Y}'$ of X such that every element of $\mathcal{Y}'$ is in $\mathcal{Y}$.

14.3 Example. Let $X = \mathbb{R}$. The collection

$$\mathcal{Y} = \{(m, n) \subseteq \mathbb{R} \mid m, n \in \mathbb{Z},\ m < n\}$$

is an open cover of $\mathbb{R}$, and the collection

$$\mathcal{Y}' = \{(-n, n) \subseteq \mathbb{R} \mid n = 1, 2, \dots\}$$

is a subcover of $\mathcal{Y}$.

14.4 Definition. A space X is *compact* if every open cover of X contains a finite subcover.

14.5 Example. A discrete topological space X is compact if and only if X consists of finitely many points.

14.6 Example. Let X be a subspace of $\mathbb{R}$ given by

$$X = \{0\} \cup \{ \tfrac{1}{n} \mid n = 1, 2, \dots \}$$

The space X is compact. Indeed, let $\mathcal{U} = \{U_i\}_{i \in I}$ be any open cover of X and let $0 \in U_0$. Then there exists $N > 0$ such that $\frac{1}{n} \in U_{i_0}$ for all $n > N$. For $n = 1, \dots, N$ let $U_{i_n} \in \mathcal{U}$ be a set such that $\frac{1}{n} \in U_{i_n}$. We have:

$$X = U_{i_0} \cup U_{i_1} \cup \dots \cup U_{i_N}$$

so $\{U_{i_0}, U_{i_1}, \dots, U_{i_N}\}$ is a finite subcover of $\mathcal{U}$.

14.7 Example. The real line $\mathbb{R}$ is not compact since the open cover

$$\mathcal{Y} = \{(n-1, n+1) \subseteq \mathbb{R} \mid n \in \mathbb{Z}\}$$

does not have any finite subcover.

14.8 Proposition. *Let $f\colon X \to Y$ be a continuous function. If X is compact and f is onto then Y is compact.*

Proof. Exercise. □

14.9 Corollary. *Let $f\colon X \to Y$ be a continuous function. If $A \subseteq X$ is compact then $f(A) \subseteq Y$ is compact.*

Proof. The function $f|_A\colon A \to f(A)$ is onto, so this follows from Proposition 14.8. □

14.10 Corollary. *Let X, Y be topological spaces. If X is compact and $Y \cong X$ then Y is compact.*

Proof. Follows from Proposition 14.8. □

14.11 Example. For any $a < b$ the open interval $(a, b) \subseteq \mathbb{R}$ is not compact since $(a, b) \cong \mathbb{R}$.

14.12 Proposition. *For any $a < b$ the closed interval $[a, b] \subseteq \mathbb{R}$ is compact.*

Proof. Let $\mathcal{U}$ be an open cover of $[a, b]$ and let

$$A = \{x \in [a, b] \mid \text{the interval } [a, x] \text{ can be covered by a finite number of elements of } \mathcal{U}\}$$

Let $x_0 := \sup A$.

Step 1. We will show that $x_0 > a$. Indeed, let $U \in \mathcal{U}$ be a set such that $a \in U$. Since U is open we have $[a, a + \varepsilon) \subseteq U$ for some $\varepsilon > 0$. It follows that $x \in A$ for all $x \in [a, a + \varepsilon)$. Therefore $x_0 \geq a + \varepsilon$.

Step 2. Next, we will show that $x_0 \in A$. Let $U_0 \in \mathcal{U}$ be a set such that $x_0 \in U_0$. Since U_0 is open and $x_0 > a$ there exists $\varepsilon_1 > 0$ such that $(x_0 - \varepsilon_1, x_0] \subseteq U_0$. Also, since $x_0 = \sup A$ there is $x \in A$ such that $x \in (x_0 - \varepsilon_1, x_0]$. Notice that

$$[a, x_0] = [a, x] \cup (x_0 - \varepsilon_1, x_0]$$

By assumption the interval $[a, x]$ can be covered by a finite number of sets from $\mathcal{U}$ and $(x_0 - \varepsilon_1, x_0]$ is covered by $U_0 \in \mathcal{U}$. As a consequence $[a, x_0]$ can be covered by a finite number of elements of $\mathcal{U}$, and so $x_0 \in A$.

Step 3. In view of Step 2 it suffices to show that $x_0 = b$. To see this take again $U_0 \in \mathcal{U}$ to be a set such that $x_0 \in \mathcal{U}$. If $x_0 < b$ then there exists $\varepsilon_2 > 0$ such that $[x_0, x_0 + \varepsilon_2) \subseteq U_0$. Notice that for any $x \in (x_0, x_0 + \varepsilon_2)$ the interval $[a, x]$ can be covered by a finite number of elements of $\mathcal{U}$, and thus $x \in A$. Since $x > x_0$ this contradicts the assumption that $x_0 = \sup A$.

□

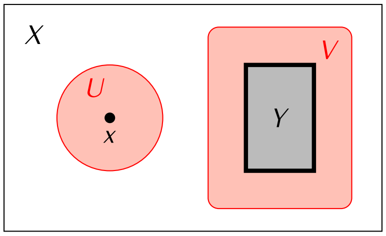
14.13 Proposition. *Let X be a compact space. If Y is a closed subspace of X then Y is compact.*

Proof. Exercise. □

14.14 Proposition. *Let X be a Hausdorff space and let $Y \subseteq X$. If Y is compact then it is closed in X.*

Proposition 14.14 is a direct consequence of the following fact:

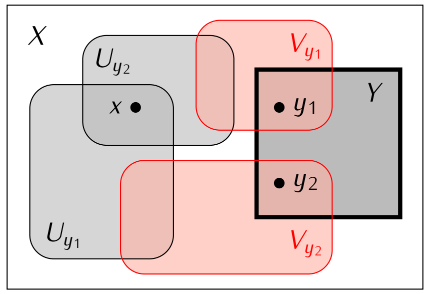
14.15 Lemma. *Let X be a Hausdorff space, let $Y \subseteq X$ be a compact subspace, and let $x \in X \smallsetminus Y$. There exists open sets $U, V \subseteq X$ such that $x \in U$, $Y \subseteq V$ and $U \cap V = \varnothing$.*



Proof. Since X is a Hausdorff space for any point $y \in Y$ there exist open sets U_y and V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \varnothing$. Notice that $Y \subseteq \bigcup_{y \in Y} V_y$. Since Y is compact we can find a finite number of points $y_1, \dots, y_n \in Y$ such that

$$Y \subseteq V_{y_1} \cup \cdots \cup V_{y_n}$$

Take $V = V_{y_1} \cup \cdots \cup V_{y_n}$ and $U := U_{y_1} \cap \cdots \cap U_{y_n}$.



□

Proof of Proposition 14.14. By Lemma 14.15 for each point $x \in X \smallsetminus Y$ we can find an open set $U_x \subseteq X$ such that $x \in U_x$ and $U_x \subseteq X \smallsetminus Y$. Therefore $X \smallsetminus Y$ is open and so Y is closed. □

14.16 Corollary. *Let X be a compact Hausdorff space. A subspace $Y \subseteq X$ is compact if and only if Y is closed in X.*

Proof. Follows from Proposition 14.13 and Proposition 14.14. □

14.17 Proposition. *Let $f\colon X \to Y$ be a continuous bijection. If X is a compact space and Y is a Hausdorff space then f is a homeomorphism.*

Proof. By Proposition 6.13 it suffices to show that for any closed set $A \subseteq X$ the set $f(A) \subseteq Y$ is closed. Let $A \subseteq X$ be a closed set. By Proposition 14.13 A is a compact space and thus by Corollary 14.9 $f(A)$ is a compact subspace of Y. Since Y is a Hausdorff space using Proposition 14.14 we obtain that $f(A)$ is closed in Y.

□

14.18 Theorem. *If X is a compact Hausdorff space then X is normal.*

Proof. Step 1. We will show first that X is a regular space (9.9). Let $A \subseteq X$ be a closed set and let $x \in X \smallsetminus A$. We need to show that there exists open sets $U, V \subseteq X$ such that $x \in U$, $A \subseteq V$ and $U \cap V = \varnothing$. Notice that by Proposition 14.13 the set A is compact. Since X is Hausdorff existence of the sets U and V follows from Lemma 14.15.

Step 2. Next, we show that X is normal. Let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$. By Step 1 for every $x \in A$ we can find open sets U_x and V_x such that $x \in U_x$, $B \subseteq V_x$ and $U_x \cap V_x = \varnothing$. The collection $\mathcal{U} = \{U_x\}_{x\in A}$ is an open cover of A. Since A is compact there is a finite number of points $x_1, \dots, x_m \in A$ such that $\{U_{x_1}, \dots, U_{x_m}\}$ is a cover of A. Take $U := \bigcup_{i=1}^m U_{x_i}$ and $V := \bigcap_{i=1}^m V_{x_i}$. Then U and V are open sets, $A \subseteq U$, $B \subseteq V$ and $U \cap V = \varnothing$. □

Exercises to Chapter 14

E14.1 Exercise. Prove Proposition 14.8.

E14.2 Exercise. Prove Proposition 14.13

E14.3 Exercise. Let X be a Hausdorff space and let $A \subseteq X$. Show that the following conditions are equivalent:

(i)) A is compact

(ii)) A is closed in X and in any open cover $\{U_i\}_{i\in I}$ of X there exists a finite number of sets $U_{i_1}, \dots, U_{i_n}$ such that $A \subseteq \bigcup_{k=1}^{n} U_{i_k}$.

E14.4 Exercise. a) Let X be a compact space and for $i = 1, 2, \dots$ let $A_i \subseteq X$ be a non-empty closed set. Show that if $A_{i+1} \subseteq A_i$ for all i then $\bigcap_{i=1}^{\infty} A_i \neq \varnothing$.

b) Give an example of a (non-compact) space X and closed non-empty sets $A_i \subseteq X$ satisfying $A_{i+1} \subseteq A_i$ for $i = 1, 2, \dots$ such that $\bigcap_{i=1}^{\infty} A_i = \varnothing$.

E14.5 Exercise. a) Let X be a compact Hausdorff space and for $i = 1, 2, \dots$ let $A_i \subseteq X$ be a closed, connected set. Show that if $A_{i+1} \subseteq A_i$ for all i then $\bigcap_{i=1}^{\infty} A_i$ is connected.

b) Give an example of a space X and subspaces $A_1 \subseteq A_2 \subseteq \dots \subseteq X$ such that A_i is connected for each i but $\bigcap_{i=1}^{\infty} A_i$ is not connected.

E14.6 Exercise. The goal of this exercise is to show that if $f\colon X \to \mathbb{R}$ is a continuous function and X is a compact space then there exist points $x_1, x_2 \in X$ such that $f(x_1)$ is the minimum value of f and $f(x_2)$ is the maximum value.

Let X be a compact space and let $f\colon X \to \mathbb{R}$ be a continuous function.

a) Show that there exists $C > 0$ such that $|f(x)| < C$ for all $x \in X$.

b) By part a) there exists $C > 0$ such that $f(X) \subseteq [-C, C]$. This implies that $\inf f(X) \neq -\infty$ and $\sup f(X) \neq +\infty$. Show that there are points $x_1, x_2 \in X$ such that $f(x_1) = \inf f(X)$ and that $f(x_2) = \sup f(X)$.

E14.7 Exercise. Let (X, ϱ) be a compact metric space, and let $f\colon X \to X$ be a function such that $\varrho(f(x), f(y)) < \varrho(x, y)$ for all $x, y \in X$, $x \neq y$.

a) Show that the function $\varphi\colon X \to \mathbb{R}$ given by $\varphi(x) = \varrho(x, f(x))$ is continuous.

b) Show that there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

E14.8 Exercise. Let $f\colon X \to Y$ be a continuous map such for any closed set $A \subseteq X$ the set $f(A)$ is closed in Y.

a) Let $y \in Y$. Show that if $U \subseteq X$ is an open set and $f^{-1}(y) \subseteq U$ then there exists an open set $V \subseteq Y$ such that $y \in V$ and $f^{-1}(V) \subseteq U$.

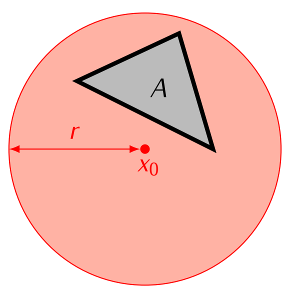
b) Show that if Y is compact and $f^{-1}(y)$ is compact for each $y \in Y$ then X is compact.

E14.9 Exercise. Let X, Y be topological spaces, and let $p_1\colon X \times Y \to X$ be the projection map: $p_1(x, y) = x$. Show that if Y is compact then for any closed set $A \subseteq X \times Y$ the set $p_1(A) \subseteq X$ is closed in X.

15 | Heine-Borel Theorem

We have seen already that a closed interval $[a, b] \subseteq \mathbb{R}$ is a compact space (14.12). Our next goal is to prove Heine-Borel Theorem 15.3 which gives a simple description of compact subspaces of $\mathbb{R}^n$.

15.1 Definition. Let (X, ϱ) be a metric space. A set $A \subseteq X$ is *bounded* if there exists an open ball $B(x_0, r) \subseteq X$ such that $A \subseteq B(x_0, r)$.



15.2 Proposition. *Let (X, ϱ) be a metric space and let $A \subseteq X$. The following conditions are equivalent:*

1) *A is bounded.*
2) *For each $x \in X$ there exists $r_x > 0$ such that $A \subseteq B(x, r_x)$.*
3) *There exists $R > 0$ such that $\varrho(x_1, x_2) < R$ for all $x_1, x_2 \in A$.*

Proof. Exercise. $\square$

15.3 Heine-Borel Theorem. *A set $A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded.*

15.4 Note. The statement of Heine-Borel Theorem is not true if we replace $\mathbb{R}^n$ by an arbitrary metric space. Take e.g. $X = (0, 1)$ with the usual metric $d(x, y) = |x - y|$. Let $A = X$. The set A is closed in

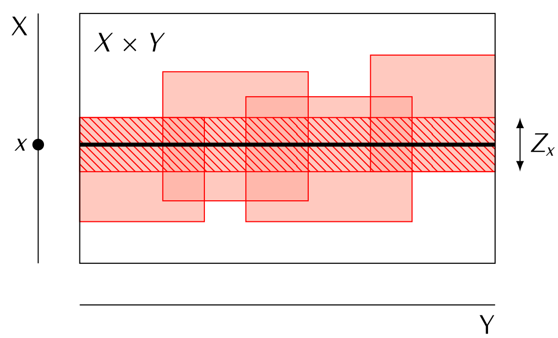
X. Also, A is bounded since $d(x, y) < 1$ for all $x, y \in A$. However A is not compact.

The proof of Heine-Borel Theorem will make use of the following fact:

15.5 Theorem. *If X, Y are compact spaces then the space $X \times Y$ is also compact.*

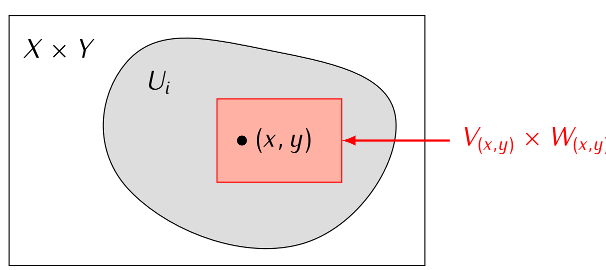
Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of $X \times Y$. Assume first that each set U_i is of the form $U_i = V_i \times W_i$ with V_i open in X, and W_i is open in Y. We will show that $\mathcal{U}$ has a finite subcover,

Step 1. We will show first that for every point $x \in X$ there is an open set $Z_x \subseteq X$ such that $Z_x \times Y$ can be covered by a finite number of elements of $\mathcal{U}$. Consider the subspace $\{x\} \times Y \subseteq X \times Y$. Since $\{x\} \times Y \cong Y$ is compact there is a finite number of sets $V_{i_1} \times W_{i_1}, \dots, V_{i_n} \times W_{i_n} \in \mathcal{U}$ such that $\{x\} \times Y \subseteq \bigcup_{j=1}^{n} V_{i_j} \times W_{i_j}$. We can take $Z_x = \bigcap_{j=1}^{n} V_{i_j}$.



Step 2. The family $\{Z_x\}_{x\in X}$ is a on open cover of X. Since X is compact we have $X = \bigcup_{k=1}^{m} Z_{x_k}$ for some $x_1, \dots, x_m \in X$. It follows that $X \times Y = \bigcup_{k=1}^{m}(Z_{x_k} \times Y)$. Since each set $Z_{x_k} \times Y$ is covered by a finite number of elements of $\mathcal{U}$ it follows that $X \times Y$ is also covered by a finite number of elements of $\mathcal{U}$.

Assume now that $\mathcal{U} = \{U_i\}_{i\in I}$ is an arbitrary open cover of $X \times Y$. For every point $(x, y) \in X \times Y$ let $V_{(x,y)} \times W_{(x,y)}$ be a set such that $V_{(x,y)}$ is open in X, $W_{(x,y)}$ is open in Y, $(x, y) \in V_{(x,y)} \times W_{(x,y)}$ and $V_{(x,y)} \times W_{(x,y)} \subseteq U_i$ for some $i \in I$:



The family $\{V_{(x,y)} \times W_{(x,y)}\}_{(x,y)\in X\times Y}$ is an open cover of $X \times Y$. By the argument above we can find points $(x_1, y_1), \dots, (x_n, y_n) \in X \times Y$ such that $X \times Y = \bigcup_{j=1}^{n} V_{(x_j,y_j)} \times W_{(x_j,y_j)}$. For $j = 1, \dots, n$ let $U_{i_j} \in \mathcal{U}$ be a set such that $V_{(x_j,y_j)} \times W_{(x_j,y_j)} \subseteq U_{i_j}$. We have

$$X \times Y = \bigcup_{j=1}^{n} V_{(x_j,y_j)} \times W_{(x_j,y_j)} \subseteq \bigcup_{j=1}^{n} U_{i_j}$$

which means that $\{U_{j_1}, \dots, U_{j_n}\}$ is a finite subcover of $\mathcal{U}$. □

15.6 Corollary. *If $X_1, \dots, X_n$ are compact spaces spaces then the space $X_1 \times \cdots \times X_n$ is compact.*

Proof. Follows from Theorem 15.5 by induction with respect to n. □

15.7 Corollary. *For $i = 1, \dots, n$ let $[a_i, b_i] \subseteq \mathbb{R}$ be a closed interval. The closed box*

$$[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$$

is compact.

Proof. This follows from Proposition 14.12 and Corollary 15.6. □

Proof of Theorem 15.3. (⇒) Exercise.

(⇐) If $A \subseteq \mathbb{R}^n$ is a closed and bounded set then $A \subseteq B(0, r)$ for some $r > 0$. Notice that $B(0, r) \subseteq J^n$ where $J = [-r, r] \subseteq \mathbb{R}$. As a consequence A is a closed subspace of J^n. By Corollary 15.7 the space J^n is a compact. Since closed subspaces of compact spaces are compact (Proposition 14.13) we obtain that A is compact.

□

Exercises to Chapter 15

E15.1 Exercise. Prove the implication (⇒) of Theorem 15.3.

E15.2 Exercise. Let X, Y be topological spaces. Show that the converse of Theorem 15.5 holds. That is, show that if $X \times Y$ is a compact space then X and Y are compact spaces.

E15.3 Exercise. Let $f\colon X \times [0, 1] \to Y$ be a continuous function, and let $U \subseteq Y$ be an open set. Show that the set

$$V = \{x \in X \mid f(\{x\} \times [0, 1]) \subseteq U\}$$

is open in X.

E15.4 Exercise. Let A, B be compact subspaces of $\mathbb{R}^n$. Show that the set

$$A + B = \{x + y \in \mathbb{R}^n \mid x \in A,\ y \in B\}$$

is also compact.

E15.5 Exercise. In Chapter 13 while proving that topological manifolds are metrizable we omitted the proof of Lemma 13.21. We are now in position to fill this gap. Prove Lemma 13.21.

16 | Compact Metric Spaces

We have seen previously that many questions related to metric spaces (e.g. whether a subset of a metric space is closed or whether a function between metric spaces is continuous) can be resolved by looking at convergence of sequences. Our main goal in this chapter the proof of Theorem 16.2 which says that also compactness of metric spaces can be characterized in terms convergence of sequences.

16.1 Definition. A topological space X is *sequentially compact* if every sequence $\{x_n\} \subseteq X$ contains a convergent subsequence.

16.2 Theorem. *A metric space (X, ϱ) is compact if and only if it is sequentially compact.*

16.3 Note. The statement of Theorem 16.2 is not true for general topological spaces: there exist spaces that are compact but not sequentially compact, and there exist spaces that are sequentially compact but not compact.

16.4 Lemma. *Let (X, ϱ) be a metric space. If a sequence $\{x_n\} \subseteq X$ does not contain any convergent subsequence then $\{x_n\}$ is a closed set in X.*

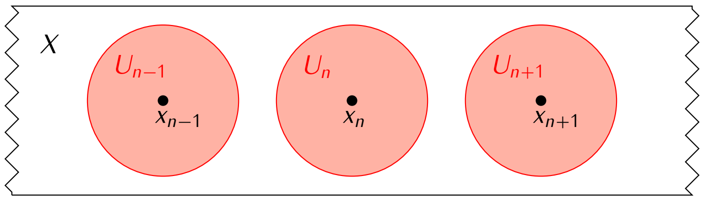
Proof. Exercise. $\square$

16.5 Lemma. *Let (X, ϱ) be a metric space. If a sequence $\{x_n\} \subseteq X$ does not contain any convergent subsequence then for each $k = 1, 2, \dots$ there exists $\varepsilon_k > 0$ such that $B(x_k, \varepsilon_k) \cap \{x_n\} = x_k$.*

Proof. Exercise. $\square$

Proof of Theorem 16.2 ($\Rightarrow$). Assume that (X, ϱ) is a metric space and that $\{x_n\} \subseteq X$ is a sequence without a convergent subsequence. By Lemma 16.4 the set $U_0 = X \smallsetminus \{x_n\}$ is open. For $k = 1, 2, \dots$

denote $U_k := B(x_k, \varepsilon_k)$ where $B(x_k, \varepsilon_k)$ is the open ball given by Lemma 16.5. The the family of sets $\{U_0, U_1, U_2, \dots\}$ is an open cover of X that has no finite subcover. Therefore X is not compact.



□

16.6 Definition. Let (X, ϱ) be a metric space, and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of X. A *Lebesgue number* for $\mathfrak{U}$ is a number $\lambda_{\mathfrak{U}} > 0$ such that for every $x \in X$ we have $B(x, \lambda_{\mathfrak{U}}) \subseteq U_i$ for some $U_i \in \mathfrak{U}$.

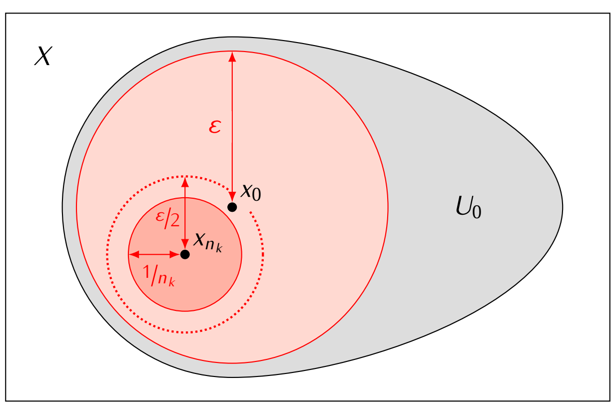
16.7 Note. For a general metric space (X, ϱ) and an open cover $\mathfrak{U}$ of X a Lebesgue number for $\mathfrak{U}$ may not exist (exercise).

16.8 Lemma. *If (X, ϱ) is a sequentially compact metric space then for any open cover $\mathfrak{U}$ of X there exists a Lebesgue number for $\mathfrak{U}$.*

Proof. We argue by contradiction. Assume that $\mathfrak{U}$ is an open cover of X without a Lebesgue number. This implies that for any $n \geq 1$ there is $x_n \in X$ such that $B(x_n, \frac{1}{n})$ is not contained in any element of $\mathfrak{U}$. Since X is sequentially compact the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$. Let $x_{n_k} \to x_0$ and let $U_0 \in \mathfrak{U}$ be a set such that $x_0 \in U_0$. We can find $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq U_0$ and $k > 0$ such that $\frac{1}{n_k} < \frac{\varepsilon}{2}$ and $\varrho(x_0, x_{n_k}) < \frac{\varepsilon}{2}$. This gives:

$$B(x_{n_k}, \tfrac{1}{n_k}) \subseteq B(x_{n_k}, \tfrac{\varepsilon}{2}) \subseteq B(x_0, \varepsilon) \subseteq U_0$$

which is impossible by the choice of x_{n_k}.



□

16.9 Definition. Let (X, ϱ) be a metric space. For $\varepsilon > 0$ an *ε-net* in X is a set of points $\{x_i\}_{i\in I} \subseteq X$ such that $X = \bigcup_{i\in I} B(x_i, \varepsilon)$.

16.10 Note. A set $\{x_i\}_{i\in I}$ is an ε-net in X if and only if for every $x \in X$ there is $i \in I$ such that $\varrho(x, x_i) < \varepsilon$.

16.11 Lemma. *Let (X, ϱ) be a sequentially compact metric space. For every $\varepsilon > 0$ there exists a finite ε-net in X.*

Proof. Assume that for some $\varepsilon > 0$ the space X does not have a finite ε-net. Choose any point $x_1 \in X$. We have $B(x_1, \varepsilon) \neq X$ (since otherwise the set $\{x_1\}$ would be an ε-net in X), so we can find $x_2 \in X$ such that $x_2 \notin B(x_1, \varepsilon)$. Next, since $\{x_1, x_2\}$ is not an ε-net there exists $x_3 \in X$ such that $x_3 \notin \bigcup_{i=1}^{2} B(x_i, \varepsilon)$. Arguing by induction we get an infinite sequence $\{x_n\} \subseteq X$ such that

$$x_n \notin \bigcup_{i=1}^{n-1} B(x_i, \varepsilon)$$

for $n = 1, 2, \dots$ This means that for any $n \neq m$ we have $\varrho(x_n, x_m) > \varepsilon$. As a consequence $\{x_n\}$ does not contain any convergent subsequence (exercise), and so the space X is not sequentially compact. $\square$

Proof of Theorem 16.2 ($\Leftarrow$) . Assume that the space (X, ϱ) is sequentially compact and let $\mathcal{U}$ be an open cover of X. We need to show that $\mathcal{U}$ contains a finite subcover. By Lemma 16.8 there exists a Lebesgue number $\lambda_{\mathcal{U}}$ for $\mathcal{U}$. Also, by Lemma 16.11, we can find in X a finite $\lambda_{\mathcal{U}}$-net $\{x_1, \dots, x_n\}$. For $i = 1, \dots, n$ let $U_i \in \mathcal{U}$ be a set such that $B(x_i, \lambda_{\mathcal{U}}) \subseteq U_i$. We have:

$$X = \bigcup_{i=1}^{n} B(x_i, \lambda_{\mathcal{U}}) \subseteq \bigcup_{i=1}^{n} U_i$$

Therefore $\{U_1, \dots, U_n\}$ is a finite subcover of $\mathcal{U}$. $\square$

16.12 Corollary. *If (X, ϱ) is a compact metric space then for any open cover $\mathcal{U}$ of X there exists a Lebesgue number for $\mathcal{U}$.*

Proof. Follows from Theorem 16.2 and Lemma 16.8. $\square$

Exercises to Chapter 16

E16.1 Exercise. Prove Lemma 16.4.

E16.2 Exercise. Prove Lemma 16.5.

E16.3 Exercise. Give an example of an open covering $\mathcal{U}$ of the open interval $(0, 1)$ (with the usual metric) such that there does not exist a Lebesgue number for $\mathcal{U}$.

E16.4 Exercise. The goal of this exercise is to fill one missing detail in the proof of Lemma 16.11. Let (X, ϱ) be a metric space and let $\{x_n\}$ be a sequence in X. Assume that for some $\varepsilon > 0$ we have $\varrho(x_n, x_m) > \varepsilon$ for all $m \neq n$. Show that $\{x_n\}$ does not contain any convergent subsequence.

E16.5 Exercise. Let (X, ϱ) be a metric space and let $A \subseteq X$ be a set such that $A \cap K$ is compact for every compact set $K \subseteq X$. Show that A is closed in X.

E16.6 Exercise. Recall that if (X, ϱ) is a metric space then a sequence $\{x_n\}$ in X is a Cauchy sequence if for each $\varepsilon > 0$ there exists $N > 0$ such that $\varrho(x_n, x_m) < \varepsilon$ for all $n, m > N$. The space (X, ϱ) is a *complete metric space* if each Cauchy sequence in X is convergent.

Let (X, ϱ) be a metric space. Show that the following conditions are equivalent.

(i) X is compact
(ii) The space X is a complete metric space and for any $\varepsilon > 0$ there exists a finite ε-net in X.

E16.7 Exercise. Let (X, ϱ) be a metric space. We will say that a function $f : X \to \mathbb{R}$ is *bounded* if there is $K > 0$ such that $|f(x)| < K$ for all $x \in X$. Show that the following conditions are equivalent:

(i) X is compact
(ii) every continuous function $f : X \to \mathbb{R}$ is bounded.

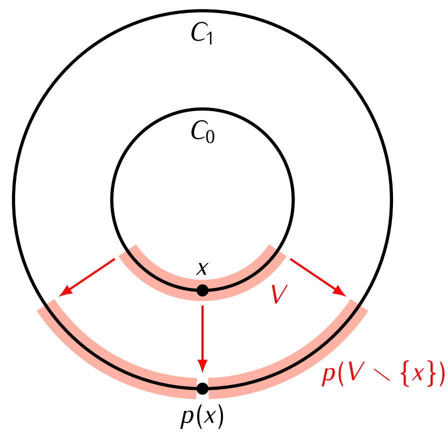
(Hint: Show that if X is non-compact then it contains a sequence $\{x_n\}$ with no convergent subsequence and such that $x_n \neq x_m$ for all $n \neq m$. Let A be the subspace of X consisting of all points of this sequence. Show the function $f : A \to \mathbb{R}$ given by $f(x_n) = n$ is continuous).

E16.8 Exercise. Theorem 16.2 characterizes compactness in metric spaces. One can ask if every compact Hausdorff space is metrizable. The goal of this exercise is to show that this is not true in general.

a) Recall that a space X is separable if it contains a countable dense subset. Show that any compact metric space is separable.

b) The *Alexandroff double circle* is a topological space X defined as follows. The points of X are the points of two concentric circles: C_0 (the inner circle) and C_1 (the outer circle). Let $p : C_0 \to C_1$ denote the radial projection map. A basis $\mathcal{B}$ of the topology on X consists of two types of sets:

(i) If $y \in C_1$ then $\{y\} \in \mathcal{B}$.
(ii) If $V \subseteq C_0$ is an open arch with center at the point x then then $V \cup p(V \smallsetminus \{x\}) \in \mathcal{B}$.



Show that X is a compact Hausdorff space, but that it is not separable. By part a) this will imply that X is not metrizable.

E16.9 Exercise. Let X be the Alexandroff double circle defined in Exercise 16.8. Is X sequentially compact? Justify your answer.

E16.10 Exercise. Let (X, ϱ) be a compact metric space and let $f\colon X \to X$ be a continuous function such that $\varrho(f(x), f(y)) \geq \varrho(x, y)$ for all $x, y \in X$. Show that f is a homeomorphism.

E16.11 Exercise. Let (X, ϱ) be a compact metric space, and let $f\colon X \to X$ be a function such that $\varrho(f(x), f(y)) < \varrho(x, y)$ for all $x, y \in X$, $x \neq y$. By Exercise 14.7 there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$. Let x be an arbitrary point in X and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and $x_n = f(x_{n-1})$ for $n > 1$. Show that the sequence $\{x_n\}$ converges to the point x_0.

E16.12 Exercise. Let (X, ϱ), (Y, μ) be metric spaces. We say that a function $f\colon X \to Y$ is *uniformly continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x_1, x_2 \in X$ and $\varrho(x_1, x_2) < \delta$ then $\mu(f(x_1), f(x_2)) < \varepsilon$.

a) Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ which is not uniformly continuous. Justify your answer.

b) Show that if $f\colon X \to Y$ is continuous function and X is a compact space then f is uniformly continuous.

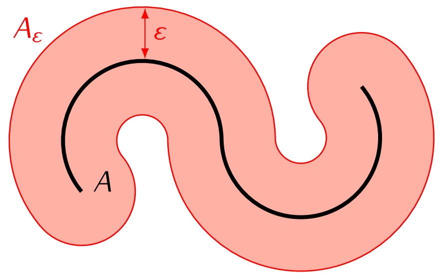
E16.13 Exercise. Let $U \subseteq \mathbb{R}^n$ be an open set and let $D \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be the set consisting of all pairs $(x, y) \in U \times U$ for which the whole line segment joining x and y is contained in U:

$$D = \{(x, y) \in U \times U \mid tx + (1 - t)y \in U \text{ for all } t \in [0, 1]\}$$

Show that D is open in $\mathbb{R}^n \times \mathbb{R}^n$.

E16.14 Exercise. For $A \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ define

$$A_\varepsilon := \{x \in \mathbb{R}^n \mid d(x, y) < \varepsilon \text{ for some } y \in A\}$$



Let $A \subseteq U \subseteq \mathbb{R}^n$ where A is compact and U is open in $\mathbb{R}^n$. Show that there exists $\varepsilon > 0$ such that $A_\varepsilon \subseteq U$.

E16.15 Exercise. Let (X, ϱ) be a metric space, and let $a, b \in X$. For $\varepsilon > 0$ we will say that a sequence of points $(x_1, \dots, x_n)$ is an *ε-chain connecting a and b* if $x_1 = a$, $x_n = b$, and $\varrho(x_i, x_{i+1}) < \varepsilon$ for $i = 1, \dots, n-1$.

Let (X, ϱ) be a compact metric space. Show that the following conditions are equivalent:

1) the space X is connected;
2) for any points $a, b \in X$ and any $\varepsilon > 0$ there exists *ε-chain* connecting a and b.

17 | Tychonoff Theorem

We have seen already that a product of finitely many compact spaces is compact (15.6) . The main goal here is to show that the same is true for arbitrary products of compact spaces:

17.1 Tychonoff Theorem. *If $\{X_s\}_{s\in S}$ is a family of topological spaces and X_s is compact for each $s \in S$ then the product space $\prod_{s\in S} X_s$ is compact.*

The proof of Theorem 17.1 involves two main ideas. The first is reformulation of compactness in terms of closed sets.

17.2 Definition. Let $\mathcal{A}$ be a family of subsets of a space X. The family $\mathcal{A}$ is *centered* if for any finite number of sets $A_1, \dots, A_n \in \mathcal{A}$ we have $A_1 \cap \dots \cap A_n \neq \varnothing$

17.3 Example. If $\mathcal{A} = \{A_i\}_{i\in I}$ is a family of subsets of X such that $\bigcap_{i\in I} A_i = \varnothing$ then $\mathcal{A}$ is centered.

17.4 Example. Let $X = \mathbb{R}$. For $n \in \mathbb{Z}$ define $A_n = (n, +\infty)$. Then the family $\{A_n\}_{n\in\mathbb{Z}}$ is centered even though $\bigcap_{n\in\mathbb{Z}} A_n = \varnothing$.

17.5 Lemma. *Let X be a topological space. The following conditions are equivalent:*

1) *The space X is compact.*
2) *For any centered family $\mathcal{A}$ of closed subsets of X we have $\bigcap_{A\in\mathcal{A}} A \neq \varnothing$.*

Proof. 2) $\Rightarrow$ 1) Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. We need to show that $\mathcal{U}$ has a finite subcover. For $i \in I$ define $A_i := X \smallsetminus U_i$. This gives a family $\mathcal{A} = \{A_i\}_{i\in I}$ of closed sets in X. We have:

$$\bigcap_{i\in I} A_i = \bigcap_{i\in I} (X \smallsetminus U_i) = X \smallsetminus \bigcup_{i\in I} U_i = X \smallsetminus X = \varnothing$$

This implies that $\mathcal{A}$ is not a centered family, so there exist sets $A_{i_1}, \dots, A_{i_n} \in \mathcal{A}$ such that $A_{i_1} \cap \dots \cap A_{i_n} =$

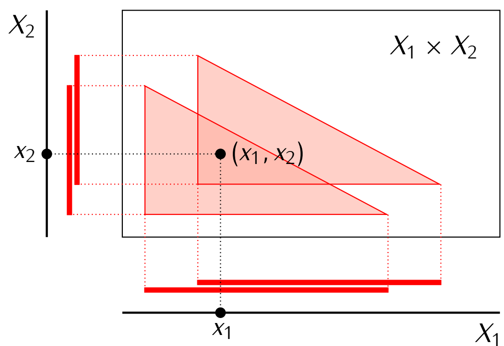
$\varnothing$. This gives:

$$\varnothing = A_{i_1} \cap \cdots \cap A_{i_n} = (X \smallsetminus U_{i_1}) \cap \cdots \cap (X \smallsetminus U_{i_n}) = X \smallsetminus (U_{i_1} \cup \cdots \cup U_{i_n})$$

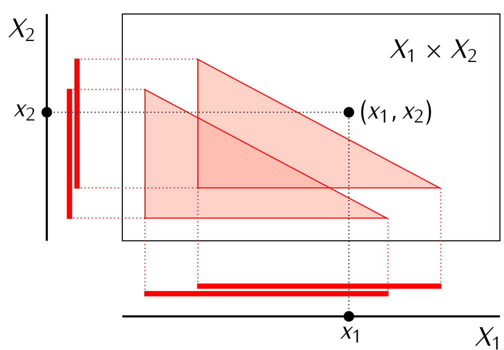
Therefore $X = U_{i_1} \cup \cdots \cup U_{i_n}$, and so $\{U_{i_1}, \ldots, U_{i_n}\}$ is a finite subcover of $\mathcal{U}$.

1) $\Rightarrow$ 2) Follows from a similar argument. $\square$

Having Lemma 17.5 at our disposal we can try to prove the Theorem 17.1 in the following way. Given a centered family $\mathcal{A}$ of subsets of $\prod_{s\in S} X_s$ we need to show that $\bigcap_{A\in\mathcal{A}} A \neq \varnothing$. Let $p_{s_0}\colon \prod_{s\in S} X_s \to X_{s_0}$ be the projection onto the s_0-th factor. For each $s \in S$ the family $\{\,\overline{p_s(A)}\,\}_{A\in\mathcal{A}}$ is a centered family of closed subsets of X_s. Since X_s is compact we can find $x_s \in X_s$ such that $x_s \in \bigcap_{A\in\mathcal{A}} \overline{p_s(A)}$. If we can show that the point $(x_s)_{s\in S} \in \prod_{s\in S} X_s$ is in $\bigcap_{A\in\mathcal{A}} A$ then we are done.



The problem with this approach is that in general not every choice of points $x_s \in \bigcap_{A\in\mathcal{A}} \overline{p_s(A)}$ will give a point $(x_s)_{s\in S}$ that belongs to $\bigcap_{A\in\mathcal{A}} A$:



This brings in the second main idea of the proof of Tychonoff Theorem, which (modulo a few technical details) can be outlined as follows. We will start with an arbitrary centered family $\mathcal{A}$ of closed subsets of $\prod_{s\in S} X_s$, but then we will replace it by a certain family $\mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{M}$. This inclusion will give $\bigcap_{M\in\mathcal{M}} M \subseteq \bigcap_{A\in\mathcal{A}} A$, so it will be enough to show that $\bigcap_{M\in\mathcal{M}} M \neq \varnothing$. The advantage of working with the family $\mathcal{M}$ will be that for any choice of points $x_s \in \bigcap_{M\in\mathcal{M}} \overline{p_s(M)}$ the point $(x_s)_{s\in S}$ will belong to $\bigcap_{M\in\mathcal{M}} M$, which will let us avoid the issues indicated above.

The main difficulty is to show that for a given centered family $\mathcal{A}$ we can find a family $\mathcal{M}$ that has the above propreties. This will be accomplished using Zorn's Lemma. This lemma is a very useful result in set theory that appears in proofs of many theorems in various areas of mathematics. Here is a concise introduction to Zorn's Lemma:

17.6 Definition. A *partially ordered set* (or *poset*) is a set S equipped with a binary relation $\leq$ satisfying

(i) $x \leq x$ for all $x \in S$ (reflexivity)
(ii) if $x \leq y$ and $y \leq x$ then $y = x$ (antisymmetry)
(iii) if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity).

17.7 Definition. A *linearly ordered set* is a poset $(S, \leq)$ such that for any $x, y \in S$ we have either $x \leq y$ or $y \leq x$.

17.8 Example. If A is a set and S is the set of all subsets of A then S is a poset with ordering given by inclusion of subsets.

17.9 Definition. If $(S, \leq)$ is a poset then an element $x \in S$ is a *maximal element* of S if we have $x \leq y$ only for $y = x$.

17.10 Example. If S is the set of all subsets of a set A ordered by inclusion then S has only one maximal element: the whole set A.

If we take S' to be the set of all *proper* subsets of a A then S' has many maximal elements: for every $a \in A$ the set $A - \{a\}$ is a maximal element of S'.

17.11 Example. In general a poset does not need to have any maximal elements. For example, take the set of integers $\mathbb{Z}$ with the usual ordering $\leq$. The set $\mathbb{Z}$ does not have any maximal elements since for every number $n \in \mathbb{Z}$ we can find a larger number (e.g. $n + 1$).

17.12 Note. If $(S, \leq)$ is a poset and $T \subseteq S$ then T is also a poset with ordering inherited from S.

17.13 Definition. Let $(S, \leq)$ is a poset and let $T \subseteq S$. An *upper bound of* T is an element $x \in S$ such that $y \leq x$ for all $y \in T$.

17.14 Definition. If $(S, \leq)$ is a poset. A *chain* in S is a subset $T \subseteq S$ such that T is linearly ordered.

17.15 Zorn's Lemma. *If $(S, \leq)$ is a non-empty poset such that every chain in S has an upper bound* *in S then S contains a maximal element.*

Proof. See any book on set theory. □

We are finally ready for the proof of the Tychonoff Theorem:

Proof of Theorem 17.1. Let $X = \prod_{s \in S} X_s$ where X_s is a compact space for each $s \in S$. Let $\mathcal{A}$ be a centered family of closed subsets of X. We will show that there exists $x = (x_s)_{s \in S} \in X$ such that $x \in \bigcap_{A \in \mathcal{A}} A$. Let T denote the set consisting of all centered families $\mathcal{F}$ of (not necessarily closed) subsets of X such that $\mathcal{A} \subseteq \mathcal{F}$. The set T is partially ordered by the inclusion.

Claim. Every chain in T has an upper bound.

Indeed, if $\{\mathcal{F}_j\}_{j \in J}$ is a chain in T then take $\mathcal{F} = \bigcup_{j \in J} \mathcal{F}_j$. Since $\mathcal{F}$ is a centered family (exercise) and $\mathcal{F}_j \subseteq \mathcal{F}$ for all $j \in J$ thus $\mathcal{F}$ is an upper bound of $\{\mathcal{F}_j\}_{j \in J}$.

By Zorn's Lemma 17.15 we obtain that the set T contains a maximal element $\mathcal{M}$. We will show that there exists $x \in X$ such that

$$x \in \bigcap_{M \in \mathcal{M}} \overline{M}$$

Since $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{A}$ consists of closed sets we have $\bigcap_{M \in \mathcal{M}} \overline{M} \subseteq \bigcap_{A \in \mathcal{A}} A$. Therefore it will follow that $x \in \bigcap_{A \in \mathcal{A}} A$, and thus $\bigcap_{A \in \mathcal{A}} A \neq \varnothing$.

Construction of the element x proceeds as follows. For $s \in S$ let $p_s \colon X \to X_s$ by the projection onto the s-th coordinate. For each $s \in S$ the family $\{\overline{p_s(M)}\}_{M \in \mathcal{M}}$ is a centered family of closed subsets of X_s, so by compactness of X_s there is $x_s \in X_s$ such that $x_s \in \bigcap_{M \in \mathcal{M}} \overline{p_s(M)}$. We set $x = (x_s)_{s \in S}$.

In order to see that $x \in \bigcap_{M \in \mathcal{M}} \overline{M}$ notice that $\mathcal{M}$ has the following property:

$$\text{if } B \subseteq X \text{ and } B \cap M \neq \varnothing \text{ for all } M \in \mathcal{M} \text{ then } B \in \mathcal{M} \tag{$*$}$$

Indeed, if $\mathcal{M}' = \mathcal{M} \cup \{B\}$ then $\mathcal{M}' \in T$ (exercise) and $\mathcal{M} \subseteq \mathcal{M}'$, so by the maximality of $\mathcal{M}$ we must have $\mathcal{M} = \mathcal{M}'$.

For $s \in S$ let $U_s \subseteq X_s$ be an open neighborhood of x_s. Since $x_s \in \overline{p_s(M)}$ for all $M \in \mathcal{M}$, thus $U_s \cap p_s(M) \neq \varnothing$ for all $M \in \mathcal{M}$. Equivalently, $p_s^{-1}(U_s) \cap M \neq \varnothing$ for all $M \in \mathcal{M}$. By property ($*$) we obtain that $p^{-1}(U_s) \in \mathcal{M}$ for all $s \in S$. Since $\mathcal{M}$ is a centered family we obtain

$$p^{-1}(U_{s_1}) \cap \cdots \cap p^{-1}(U_{s_n}) \cap M \neq \varnothing \quad \text{for all } M \in \mathcal{M} \tag{$**$}$$

Recall that by (12.9) the sets of the form $p^{-1}(U_{s_1}) \cap \cdots \cap p^{-1}(U_{s_n})$. are precisely the open neighborhoods of x that belong to the basis of the product topology on X, and thus any open neighborhood of X contains a neighborhood of this type. Therefore using ($**$) we obtain that if $M \in \mathcal{M}$ then for any open neighborhood U of x we have $M \cap U \neq \varnothing$. This means that for every $M \in \mathcal{M}$ we have $x \in \overline{M}$, and thus $x \in \bigcap_{M \in \mathcal{M}} \overline{M}$.

□

17.16 Proposition. *If X_i is a Hausdorff space for each $i \in I$ then the product space $\prod_{i\in I} X_i$ is also Hausdorff.*

Proof. Exercise. □

17.17 Corollary. *If X_i is a compact Hausdorff space space for each $i \in I$ then the product space $\prod_{i\in I} X_i$ is also compact Hausdorff.*

Proof. Follows from Tychonoff Theorem 17.1 and Proposition 17.16. □

Exercises to Chapter 17

E17.1 Exercise. This problem does not involve topology, it is an exercise in using Zorn's Lemma 17.15. A subset $H \subseteq \mathbb{R}$ is a *subgroup* of $\mathbb{R}$ if it satisfies three conditions:

1) $0 \in H$
2) if $x \in H$ then $-x \in H$
3) if $x, y \in H$ then $x + y \in H$

For example, the set of integers $\mathbb{Z}$ and the set of rational numbers $\mathbb{Q}$ are subgroups of $\mathbb{R}$. Show that for any real number $r \neq 0$ there exists a subgroup $H \subseteq \mathbb{R}$ such that $r \notin H$, but $r \in H'$ for any subgroup H' such that $H \subseteq H'$ and $H \neq H'$.

E17.2 Exercise. This is another exercise on Zorn's Lemma. Recall (1.24) that any binary relation on a set S is formally defined as a subset $R \subseteq S \times S$. We say that R is a *partial order relation* if S equipped with this relation is a partially ordered set (17.6). In the subset notation this mean that R satisfies the following conditions:

(i) $(x, x) \in R$ for all $x \in S$
(ii) if $(x, y) \in R$ and $(y, x) \in S$ then $x = y$
(iii) if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

A partial order relation R is a *linear order relation* if X equipped with this relation becomes a linearly ordered set (17.7) . Explicitly, this mean that R satisfies conditions (i) - (iii), and that for any $x, y \in S$ either $(x, y) \in R$ or $(y, x) \in R$.

If R, R' are binary relations on S then we will say that R' *extends* R if $R \subseteq R'$.

a) Show that if R is a partial order relation on S and $x_0, y_0 \in S$ are elements such that $(x_0, y_0) \notin R$ and $(y_0, x_0) \notin R$ then R can be extended to a partial order relation R' such that $(x_0, y_0) \in R'$.

b) Show that if R is a partial order relation on a set S then R can be extended to a linear order relation $\overline{R}$ on S.

E17.3 Exercise. The goal of this exercise is to complete two details in the proof of the Tychonoff Theorem 17.1.

a) For $j \in J$ let $\mathcal{F}_j$ be a centered family of subsets of a space X. Show that if the set $\{\mathcal{F}_j\}_{j\in J}$ is linearly ordered with respect to inclusion then $\mathcal{F} = \bigcup_{j\in J} \mathcal{F}_j$ is a centered family.

b) Let T denote the collection of all centered families of subsets of X. Consider T with ordering given by inclusion. Let $\mathcal{M}$ be a maximal element in T, and let $A \subseteq X$ be s set such that $A \cap M \neq \varnothing$ for all $M \in \mathcal{M}$. Show that the family $\mathcal{M}' = \mathcal{M} \cup \{A\}$ is centered.

E17.4 Exercise. Prove Proposition 17.16.

E17.5 Exercise. The *Cantor set* is the subspace C of the real line defined as follows. Take $A_0 = [0, 1]$. The set A_1 is then obtained by removing the open middle third subinterval of A_0:

$$A_1 = [0, 1] \smallsetminus (\tfrac{1}{3}, \tfrac{2}{3}) = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$

Next, A_2 is obtained from A_2 by removing open middle third subinterval out of each connected component of A_2. Explicitly:

$$A_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$$

Inductively we construct A_{n+1} from A_n by removing the middle third open subintervals from all connected components of A_n. Then we define $C = \bigcap_{n=0}^{\infty} A_n$.

Show that the Cantor set is homeomorphic to the space $\prod_{n=1}^{\infty} D$ where D is the discrete space with two elements $D = \{0, 1\}$.

18 | Compactification

We have seen that compact Hausdorff spaces have several interesting properties that make this class of spaces especially important in topology. If we are working with a space X which is not compact we can ask if X can be embedded into some compact Hausdorff space Y. If such embedding exists we can identify X with a subspace of Y, and some arguments that work for compact Hausdorff spaces will still apply to X. This approach leads to the notion of a *compactification* of a space. Our goal in this chapter is to determine which spaces have compactifications. We will also show that compactifications of a given space X can be ordered, and we will look for the largest and smallest compactifications of X.

18.1 Proposition. *Let X be a topological space. If there exists an embedding $j\colon X \to Y$ such that Y is a compact Hausdorff space then there exists an embedding $j_1\colon X \to Z$ such that Z is compact Hausdorff and $\overline{j_1(X)} = Z$.*

Proof. Assume that we have an embedding $j\colon X \to Y$ where Y is a compact Hausdorff space. Let $\overline{j(X)}$ be the closure of $j(X)$ in Y. The space $\overline{j(X)}$ is compact (by Proposition 14.13) and Hausdorff, so we can take $Z = \overline{j(X)}$ and define $j_1\colon X \to Z$ by $j_1(x) = j(x)$ for all $x \in X$. ☐

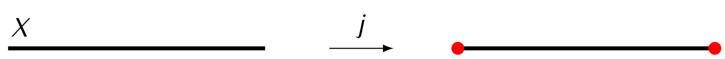
18.2 Definition. A space Z is a *compactification* of X if Z is compact Hausdorff and there exists an embedding $j\colon X \to Z$ such that $\overline{j(X)} = Z$.

18.3 Corollary. *Let X be a topological space. The following conditions are equivalent:*

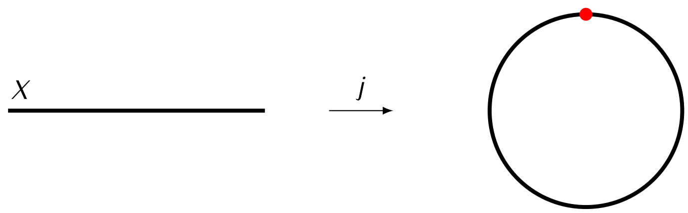
1) *There exists a compactification of X.*
2) *There exists an embedding $j\colon X \to Y$ where Y is a compact Hausdorff space.*

Proof. Follows from Proposition 18.1. ☐

18.4 Example. The closed interval $[-1, 1]$ is a compactification of the open interval $(-1, 1)$. with the embedding $j\colon (-1, 1) \to [-1, 1]$ is given by $j(t) = t$ for $t \in (-1, 1)$.



18.5 Example. The unit circle $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ is another compactification of the interval $(-1, 1)$. The embedding $j\colon (-1, 1) \to S^1$ is given by $j(t) = (\sin \pi t, -\cos \pi t)$.

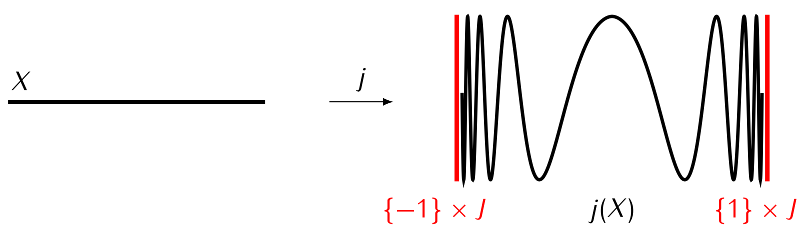


18.6 Example. A more complex compactification of the space $X = (-1, 1)$ can be obtained as follows. Let $J = [-1, 1]$. Consider the function $j\colon X \to J \times J$ given by

$$j(t) = \left(t,\, \cos\left(\tfrac{|t|}{1-|t|}\right)\right)$$

The map j is an embedding, and so $\overline{j(X)} \subseteq J \times J$ is a compactification of X. We have:

$$\overline{j(X)} = \{-1\} \times J \cup j(X) \cup \{1\} \times J$$



18.7 Theorem. *A space X has a compactification if and only if X is completely regular (i.e. it is a $T_{3\,1/2}$-space).*

Proof. ($\Rightarrow$) Assume that X has a compactification. Let $j\colon X \to Y$ be an embedding where Y is a compact Hausdorff space. By Theorem 14.18 the space Y is normal, so it is completely regular. Since subspaces of completely regular spaces are completely regular (exercise) we obtain that $j(X) \subseteq Y$ is completely regular. Finally, since $j(X) \cong X$ we get that X is completely regular.

($\Leftarrow$) Assume that X is completely regular . We need to show that there exists an embedding $j\colon X \to Y$ where Y is a compact Hausdorff space. Let $C(X)$ denote the set of all continuous functions $f\colon X \to [0, 1]$. Complete regularity of X implies that $C(X)$ is a family of functions that separates points from closed sets in X (12.13). Consider the map

$$j_X \colon : X \to \prod_{f \in C(X)} [0, 1]$$

given by $j_X(x) = (f(x))_{f \in C(X)}$. By the Embedding Lemma 12.14 we obtain that this map is an embedding. It remains to notice that by Corollary 17.17 the space $\prod_{f \in C(X)}[0, 1]$ is compact Hausdorff. $\square$

18.8 Note. In the part ($\Rightarrow$) of the proof of Theorem 18.7 we used the fact that subspaces of completely regular spaces are completely regular. An analogous property does not hold for normal spaces: a subspace of a normal space need not be normal. For this reason it is not true that a space that has a compactification must be a normal space.

18.9 Definition. Let X be a completely regular space and let $j_X \colon X \to \prod_{f \in C(X)}[0, 1]$ be the embedding defined in the proof of Theorem 18.7 and let $\beta(X)$ be the closure of $j_X(X)$ in $\prod_{f \in C(X)}[0, 1]$. The compactification $j_X \colon X \to \beta(X)$ is called the *Čech-Stone compactification* of X.

The Čech-Stone compactification is the largest compactification of a space X in the following sense:

18.10 Definition. Let X be a space and let $i_1 \colon X \to Y_1$, $i_1 \colon X \to Y_1$ be compactifications of X. We will write $Y_1 \geq Y_2$ if there exists a continuous function $g \colon Y_1 \to Y_2$ such that $i_2 = g i_1$:

$$\begin{tikzcd} & X \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ Y_1 \arrow[rr, "g"'] & & Y_2 \end{tikzcd}$$

18.11 Proposition. *Let $i_1 \colon X \to Y_1$, $i_1 \colon X \to Y_1$ be compactifications of a space X.*

1) If $Y_1 \geq Y_2$ then there exists only one map $g \colon Y_1 \to Y_2$ satisfying $i_2 = g i_1$. Moreover g is onto.

2) $Y_1 \geq Y_2$ and $Y_2 \geq Y_1$ if and only if the map $g \colon Y_1 \to Y_2$ is a homeomorphism.

Proof. Exercise. $\square$

18.12 Theorem. *Let X be a completely regular space and let $j_X \colon X \to \beta(X)$ be the Čech-Stone compactification of X. For any compactification $i \colon X \to Y$ of X we have $\beta(X) \geq Y$.*

The proof Theorem 18.12 will use the following fact:

18.13 Lemma. *If $f \colon X_1 \to X_2$ is a continuous map of compact Hausdorff spaces then $f(\overline{A}) = \overline{f(A)}$ for any $A \subseteq X_1$.*

Proof. Exercise. $\square$

Proof of Theorem 18.12. Let $i\colon X \to Y$ be a compactification of X. We need to show that there exists a map $g\colon \beta(X) \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "j_X"'] \arrow[dr, "i"] & \\ \beta(X) \arrow[rr, "g"'] & & Y \end{tikzcd}$$

Let $C(X)$, $C(Y)$ denote the sets of all continuous functions $X \to [0,1]$ and $Y \to [0,1]$ respectively. Consider the continuous functions $j_X\colon X \to \prod_{f\in C(X)}[0,1]$ and $j_Y\colon Y \to \prod_{f'\in C(Y)}[0,1]$ defined as in the proof of Theorem 18.7. Notice that we have a continuous function

$$i_*\colon \prod_{f\in C(X)}[0,1] \to \prod_{f'\in C(Y)}[0,1]$$

given by $i_*((t_f)_{f\in C(X)}) = (s_{f'})_{f'\in C(Y)}$ where $s_{f'} = t_{if'}$. Moreover, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j_X"'] & Y \arrow[d, "j_Y"] \\ \prod_{f\in C(X)}[0,1] \arrow[r, "i_*"'] & \prod_{f'\in C(Y)}[0,1] \end{tikzcd}$$

We have:

$$i_*(\beta(X)) = i_*(\overline{j_X(X)}) = \overline{i_*j_X(X)} = \overline{j_Y i(X)} = j_Y(\overline{i(X)}) = j_Y(Y)$$

Here the first equality comes from the definition of $\beta(X)$, the second from Lemma 18.13, the third from commutativity of the diagram above, the fourth again from Lemma 18.13, and the last from the assumption that $i\colon X \to Y$ is a compactification. Since the map $j_Y\colon Y \to \prod_{f'\in C(Y)}[0,1]$ is embedding the map $j_Y\colon Y \to j_Y(Y)$ is a homeomorphism. We can take $g = j_Y^{-1} i_*\colon \beta(X) \to Y$.

$\square$

Motivated by the fact that Čech-Stone compactification is the largest compactification of a space X one can ask if the smallest compactification of X also exists. If X is a non-compact space then we need to add at least one point to X to compactify it. If adding only one point suffices then it gives an obvious candidate for the smallest compactification:

18.14 Definition. A space Z is a *one-point compactification* of a space X if Z is a compactification of X with embedding $j\colon X \to Z$ such that the set $Z \smallsetminus j(X)$ consists of only one point.

18.15 Example. The unit circle S^1 is a one-point compactification of the open interval $(0,1)$.

18.16 Proposition. *If a space X has a one-point compactification $j\colon X \to Z$ then this compactification is unique up to homeomorphism. That is, if $j'\colon X \to Z'$ is another one-point compactification of X then there exists a homeomorphism $h\colon Z \to Z'$ such that $j' = jh$.*

Proof. Exercise. □

Our next goal is to determine which spaces admit a one-point compactification.

18.17 Definition. A topological space X is *locally compact* if every point $x \in X$ has an open neighborhood $U_x \subseteq X$ such that the the closure $\overline{U}_x$ is compact.

18.18 Note. 1) If X is a compact space then X is locally compact since for any $x \in X$ we can take $U_x = X$.

2) The real line $\mathbb{R}$ is not compact but it is locally compact. For $x \in \mathbb{R}$ we can take $U_x = (x-1, x+1)$, and then $\overline{U}_x = [x-1, x+1]$ is compact. Similarly, for each $n \geq 0$ the space $\mathbb{R}^n$ is a non-compact but locally compact.

3) The set $\mathbb{Q}$ of rational numbers, considered as a subspace of the real line, is not locally compact (exercise).

18.19 Theorem. *Let X be a non-compact topological space. The following conditions are equivalent:*

1) *The space X is locally compact and Hausdorff.*
2) *There exists a one-point compactification of X.*

Proof. 1) ⇒ 2) Assume that X locally compact and Hausdorff. We define a space X^+ as follows. Points of X^+ are points of X and one extra point that we will denote by ∞:

$$X^+ := X \cup \{\infty\}$$

A set $U \subseteq X^+$ is open if either of the following conditions holds:

(i) $U \subseteq X$ and U is open in X
(ii) $U = \{\infty\} \cup (X \smallsetminus K)$ where $K \subseteq X$ is a compact set.

The collection of subsets of X^+ defined in this way is a topology on X^+ (exercise). One can check that the function $j\colon X \to X^+$ given by $j(x) = x$ is continuous and that it is an embedding (exercise). Moreover, since X is not compact for every open neighborhood U of ∞ we have $U \cap X \neq \varnothing$, so $\overline{j(X)} = X^+$.

To see that X^+ is a compact space assume that $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of X^+. Let $U_{i_0} \in \mathcal{U}$ be a set such that $\infty \in U_{i_0}$. By the definition of the topology on X^+ we have $X^+ \smallsetminus U_{i_0} = K$ where $K \subseteq X$ is a compact set. Compactness of K gives that

$$K \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$$

for some $U_1, \dots, U_{i_n} \in \mathcal{U}$. It follows that $\{U_{i_0}, U_{i_1}, \dots, U_{i_n}\}$ is a finite cover of X^+.

It remains to check that X^+ is a Hausdorff space (exercise).

2) $\Rightarrow$ 1) Let $j\colon X \to Z$ be a one-point compactification of X. Since $X \cong j(X)$ it suffices to show that the space $j(X)$ is locally compact and Hausdorff. We will denote by ∞ the unique point in $Z \smallsetminus j(X)$.

Since Z is a Hausdorff space and subspaces of a Hausdorff space are Hausdorff we get that $j(X)$ is a Hausdorff space.

Next, we will show that $j(X)$ is locally compact. Let $x \in j(X)$. Since Z is Hausdorff there are sets $U, V \subseteq Z$ open in Z such that $x \in U$, $\infty \in V$, and $U \cap V = \varnothing$. Since $\infty \notin U$ the set U is open in X. Let $\overline{U}$ denote the closure of U in X. We will show that $\overline{U}$ is a compact set. Notice that we have

$$\overline{U} \subseteq Z \smallsetminus V \subseteq Z$$

Since $Z \smallsetminus V$ is closed in the compact space Z thus it is compact by Proposition 14.13. Also, since $\overline{U}$ is a closed subset of $Z \smallsetminus V$, thus $\overline{U}$ is compact by the same result. □

18.20 Corollary. *If X is a locally compact Hausdorff space then X is completely regular.*

Proof. Follows from Theorem 18.7 and Theorem 18.19. □

18.21 Corollary. *Let X be a topological space. The following conditions are equivalent:*

1) *The space X is locally compact and Hausdorff .*
2) *There exists an embedding $i\colon X \to Y$ where Y is compact Hausdorff space and $i(X)$ is an open set in Y.*

Proof. 1) $\Rightarrow$ 2) If X is compact then we can take i to be the identity map $\mathrm{id}_X\colon X \to X$. If X is not compact take the one-point compactification $j\colon X \to X^+$. By the definition of topology on X^+ the set $j(X)$ is open in X^+.

2) $\Rightarrow$ 1) exercise. □

The next proposition says that one-point compactification, when it exists, is the smallest compactification of a space in the sense of Definition 18.10:

18.22 Proposition. *Let X be a non-compact, locally compact space and let $j\colon X \to X^+$ be the one-point compactification of X. For every compactification $i\colon X \to Y$ of X we have $Y \geq X^+$.*

Proof. Exercise. □

One can also show that if a space X is not locally compact (and so it does not have a one-point compactification) then no compactification of X has the property of being the smallest (see Exercise 18.13).

Exercises to Chapter 18

E18.1 Exercise. Show that a subspace of a completely regular space is completely regular (this will complete the proof of Theorem 18.7).

E18.2 Exercise. Prove Proposition 18.11.

E18.3 Exercise. Prove Lemma 18.13.

E18.4 Exercise. Consider the set $\mathbb{Q}$ of rational numbers with the subspace topology of the real line. Show that $\mathbb{Q}$ is not locally compact.

E18.5 Exercise. Prove Proposition 18.16.

E18.6 Exercise. The goal of this exercise is to fill one of the gaps left in the proof of Theorem 18.19. Let X be a locally compact Hausdorff space and let $X^+ = X \cup \{\infty\}$ be the space defined in part 1) $\Rightarrow$ 2) of the proof of (18.19). Show that X^+ is a Hausdorff space.

E18.7 Exercise. Prove the implication 2) $\Rightarrow$ 1) of Corollary 18.21.

E18.8 Exercise. A continuous function $f\colon X \to Y$ is *proper* if for every compact set $A \subseteq Y$ the set $f^{-1}(A) \subseteq X$ is compact. Let X, Y be locally compact, Hausdorff spaces and let X^+, Y^+ be their one-point compactifications. Let $f\colon X \to Y$ be a continuous function. Show that the following conditions are equivalent:

1) The function f is proper.
2) The function $f^+\colon X^+ \to Y^+$ given by $f^+(x) = f(x)$ for $x \in X$ and $f^+(\infty) = \infty$ is continuous.

E18.9 Exercise. Let (X, ϱ), (Y, μ) be metric spaces and let $f\colon X \to Y$ be a continuous function. Show that the following conditions are equivalent:

1) f is proper (Exercise 18.8)
2) If $\{x_n\} \subseteq X$ is a sequence such that $\{f(x_n)\} \subseteq Y$ converges then $\{x_n\} \subseteq X$ has a convergent subsequence.

E18.10 Exercise. Let X, Y be locally compact Hausdorff spaces, and let $i\colon X \to Y$ be an embedding such that $i(X)$ is an open in Y. Define $j^\sharp : Y^+ \to X^+$ as follows:

$$j^\sharp(y) = \begin{cases} j^{-1}(y) & \text{if } y \in j(X) \\ \infty & \text{otherwise} \end{cases}$$

Show that $j^\sharp$ is a continuous function.

E18.11 Exercise. Let X be topological space and let $j\colon X \to Y$ be a compactification of X. Show that if X is locally compact the set $j(X)$ is open in Y.

E18.12 Exercise. Prove Proposition 18.22.

E18.13 Exercise. The goal of this exercise is to show that the smallest compactification of a non-compact space X exists only if X has a one-point compactification (i.e. if X is a locally compact space).

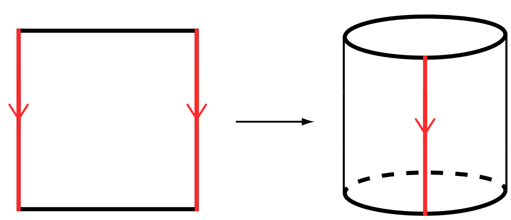
Let X be a completely regular non-compact space. Assume that there exists a compactification $j\colon X \to Y$ of X such that for any other compactification $i\colon X \to Z$ we have $Z \geq Y$. Show that Y is a one-point compactification of X. As a consequence X must be locally compact. (Hint: Assume that Y is not a one-point compactification of X and let $y_1, y_2 \in Y \smallsetminus j(X)$. Show that the space $W = Y \smallsetminus \{y_1, y_2\}$ has a one-point compactification $k\colon W \to W^+$ and that $kj\colon : X \to W^+$ is a compactification of X. Show that it is not true that $W^+ \geq Y$).

19 | Quotient Spaces

So far we have encountered two methods of constructing new topological spaces from old ones:

- given a space X we can obtain new spaces by considering subspaces of X;
- given two (or more) spaces X_1, X_2 we can obtain a new space by taking their product $X_1 \times X_2$.

Here we will consider another, very useful construction of a *quotient space* of a given topological space. This construction will let us produce, in particular, interesting examples of manifolds. Intuitively, a quotient space of a space X is a space Y which is obtained by identifying some points of X. For example, if we take the square $X = [0, 1] \times [0, 1]$ and identify each point $(0, t)$ with the point $(1, t)$ for $t \in [0, 1]$ we obtain a space Y that looks like a cylinder:



In order to make this precise we need to specify the following:

1) what are the points of Y;
2) what is the topology on Y.

The first part is done by considering Y as the set of *equivalence classes* of some *equivalence relation* on X. The second part is done by defining the *quotient topology*. We explain these notions below.

19.1 Definition. Let X be a set. An *equivalence relation on* X is a binary relation $\sim$ satisfying three properties:

1) $x \sim x$ for all $x \in X$ (reflexivity)
2) if $x \sim y$ then $y \sim x$ (symmetry)
3) if $x \sim y$ and $y \sim z$ then $x \sim z$ (transitivity)

19.2 Example. Let $X = [0,1] \times [0,1]$. Define a relation on X as follows. For any $(s,t) \in X$ we set $(s,t) \sim (s,t)$. Also, for any $t \in [0,1]$ we set $(0,t) \sim (1,t)$ and $(1,t) \sim (0,t)$. This relation is an equivalence relation that identifies corresponding points of the vertical edges of the square $[0,1] \times [0,1]$.

19.3 Example. Define a relation $\sim$ on the set of real numbers $\mathbb{R}$ as follows: $r \sim s$ if $s = r + n$ for some $n \in \mathbb{Z}$. One can check that this is an equivalence relation (exercise).

19.4 Definition. Let X we a set with an equivalence relation $\sim$ and let $x \in X$. The *equivalence class* of x is the subset $[x] \subseteq X$ consisting of all elements that are in the relation with x:

$$[x] = \{y \in X \mid x \sim y\}$$

19.5 Example. Take $X = [0,1] \times [0,1]$ with the equivalence relation defined as in Example 19.2. If $(s,t) \in X$ and $s \neq 0,1$ then $[(s,t)]$ consists of a single point: $[(s,t)] = \{(s,t)\}$. If $s = 0,1$ then $[(s,0)]$ consists of two points: $[(0,t)] = [(1,t)] = \{(0,t),(1,t)\}$.

19.6 Example. Take $\mathbb{R}$ with the equivalence relation defined as in Example 19.3. For $r \in \mathbb{R}$ we have:

$$[r] = \{r + n \mid n \in \mathbb{Z}\}$$

For example: $[1] = \{1 + n \mid n \in \mathbb{Z}\} = \mathbb{Z}$. Notice that $[1] = [2]$ and $[\sqrt{2}] = [\sqrt{2} + 1]$.

19.7 Proposition. *Let X be a set with an equivalence relation $\sim$. For $x, y \in X$ we have $[x] = [y]$ if and only if $x \sim y$.*

Proof. ($\Rightarrow$) Since $x \sim x$ we have $x \in [x]$. Therefore if $[x] = [y]$ then $x \in [y]$ and so $x \sim y$.

($\Leftarrow$) Assume that $x \sim y$ and that $z \in [x]$. This gives $z \sim x$ and by transitivity $z \sim y$. Therefore $z \in [y]$. This shows that $[x] \subseteq [y]$. In the same way we can show that $[y] \subseteq [x]$. Thus we get $[x] = [y]$. $\square$

19.8 Corollary. *Let X be a set with an equivalence relation $\sim$ and let $x, y \in X$. If $[x] \cap [y] \neq \varnothing$ then $[x] = [y]$.*

Proof. Assume that $[x] \cap [y] \neq \varnothing$ and let $z \in [x] \cap [y]$. This means that $z \sim x$ and $z \sim y$. Using transitivity we get that $x \sim y$, and so by Proposition 19.7 $[x] = [y]$. $\square$

19.9 Note. Corollary 19.8 shows that an equivalence relation $\sim$ on a set X splits X into a disjoint union of distinct equivalence classes of $\sim$. The opposite is also true. Namely, assume that we have a family $\{A_i\}_{i \in I}$ of subsets of X such that $A_i \cap A_j = \varnothing$ for $i \neq j$ and $\bigcup_{i \in I} A_i = X$. We can define a relation $\sim$ on X such that $x \sim y$ if and only if both x and y are elements of the same subset A_i. This relation is an equivalence relation and its equivalence classes are the sets A_i.

19.10 Definition. Let X be a set with an equivalence relation $\sim$. The *quotient set* of X is the set $X/{\sim}$ whose elements are all distinct equivalence classes of $\sim$. The function

$$\pi\colon X \to X/{\sim}$$

given by $\pi(x) = [x]$ is called the *quotient map*.

19.11 Note. Let X be a set with an equivalence relation $\sim$, and let $f\colon X \to Y$ be a function. Assume that for each $x, x' \in X$ such that $x \sim x'$ we have $f(x) = f(x')$. Then we can define a function $\bar{f}\colon X/{\sim} \to Y$ by $\bar{f}([x]) = f(x)$. We have $f = \bar{f}\pi$, i.e. the following diagram commutes:

$$\begin{tikzcd} & X/{\sim} \arrow[d, "\bar{f}"] \\ X \arrow[ur, "\pi"] \arrow[r, "f"'] & Y \end{tikzcd}$$

19.12 Definition. Let X be a topological space and let $\sim$ be an equivalence relation on X. The *quotient topology* on the set $X/{\sim}$ is the topology where a set $U \subseteq X/{\sim}$ is open if and only if the set $\pi^{-1}(U)$ is open in X. The set $X/{\sim}$ with this topology is called the *quotient space* of X taken with respect to the relation $\sim$.

19.13 Proposition. *Let X, Y be a topological spaces and let $\sim$ be an equivalence relation on X. A function $f\colon X/{\sim} \to Y$ is continuous if and only if the function $f\pi\colon X \to Y$ is continuous.*

Proof. Exercise. □

19.14 Note. Let X be a space with an equivalence relation $\sim$ and let $f\colon X \to Y$ be a continuous function. If for each $x, x' \in X$ such that $x \sim x'$ we have $f(x) = f(x')$ then as in (19.11) we obtain a function $\bar{f}\colon X/{\sim} \to Y$, $\bar{f}([x]) = f(x)$. Since the function $\bar{f}\pi = f$ is continuous thus by Proposition 19.13 $\bar{f}$ is a continuous function.

19.15 Example. Take the closed interval $[-1, 1]$ with the equivalence relation $\sim$ such that $(-1) \sim 1$ (and $t \sim t$ for all $t \in [-1, 1]$). We will show that the quotient space $[-1.1]/{\sim}$ is homeomorphic to the circle S^1. Consider the function $f\colon [-1, 1] \to S^1$ given by $f(x) = (\sin \pi x,\ -\cos \pi x)$:

Since $f(1) = f(-1)$ by (19.14) we get the induced continuous function $\bar{f}\colon [-1, 1]/{\sim} \to S^1$. We will prove that $\bar{f}$ is a homeomorphism. First, notice that $\bar{f}$ is a bijection. Next, since $[-1, 1]$ is a compact space and the quotient map $\pi\colon [-1, 1] \to [-1, 1]/{\sim}$ is onto by Proposition 14.8 we obtain that the space $[-1, 1]/{\sim}$ is compact. Therefore we can use Proposition 14.17 which says that any continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

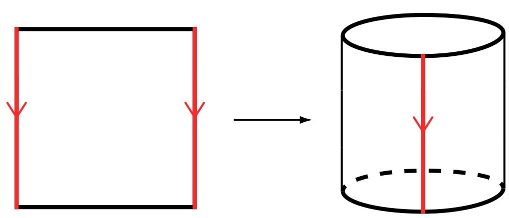
This example can be generalized as follows. Take the closed unit ball

$$\bar{B}^n = \{x \in \mathbb{R}^n \mid d(0, x) \leq 1\}$$

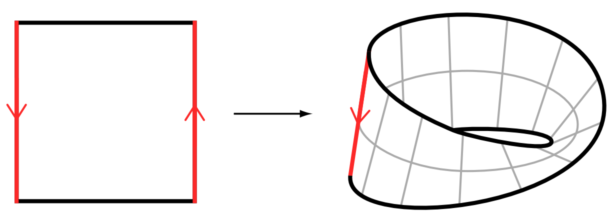
The unit sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid d(0, x) = 1\}$ is a subspace of $\bar{B}^n$. Consider the equivalence relation $\sim$ on $\bar{B}^n$ that identifies all points of S^{n-1}: $x \sim x'$ for all $x, x' \in S^{n-1}$. Using similar arguments as above one can show that $\bar{B}^n/\sim$ is homeomorphic to the sphere S^n (exercise). Notice that for $n = 1$ we have $\bar{B}^1 = [-1, 1]$ and $S^0 = \{-1, 1\}$ so in this case we recover the homeomorphism $[-1, 1]/\sim \cong S^1$.

19.16 Note. Let X be a space and let $A \subseteq X$. Consider the equivalence relation on X that identifies all points of A: $x \sim x'$ for all $x, x' \in A$. The quotient space $X/\sim$ is usually denoted by X/A. Using this notation the homeomorphism given in Example 19.15 can be written as $\bar{B}^n/S^{n-1} \cong S^n$.

19.17 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation defined as in Example 19.2: $(0, t) \sim (1, t)$ for all $t \in [0, 1]$. Using arguments similar as in Example 19.15 we can show that the quotient space is homeomorphic to the cylinder $S^1 \times [0, 1]$:

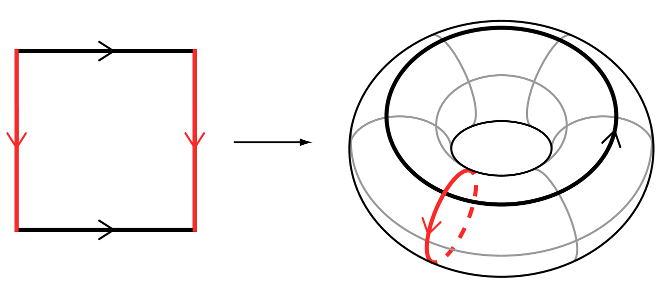


19.18 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, 1 - t)$ for all $t \in [0, 1]$. The space obtained as a quotient space is called the *Möbius band*:

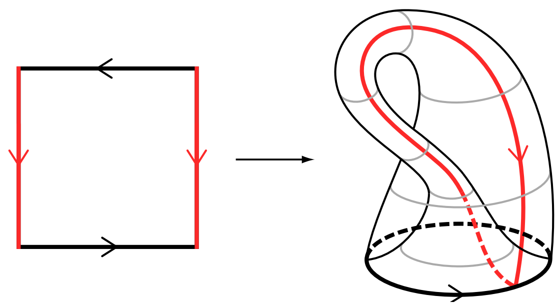


The Möbius band is a 2-dimensional manifold with boundary, and its boundary is homeomorphic to S^1.

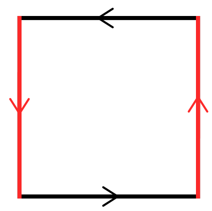
19.19 Example. Take the square $[0, 1] \times [0, 1]$ with the equivalence relation given by $(0, t) \sim (1, t)$ for all $t \in [0, 1]$ and $(s, 0) \sim (s, 1)$ for all $s \in [0, 1]$. Using arguments similar to these given in Example 19.15 one can show that the quotient space in this case is homeomorphic to the torus:



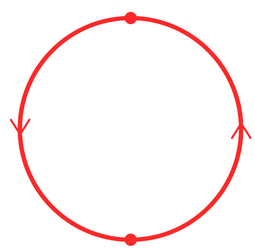
19.20 Example. Take the square $[0,1] \times [0,1]$ with the equivalence relation given by $(0,t) \sim (1,t)$ for all $t \in [0,1]$ and $(s,0) \sim (1-s,1)$ for all $s \in [0,1]$. The resulting quotient space is called the *Klein bottle*. One can show that the Klein bottle is a two dimensional manifold.



19.21 Example. Following the scheme of the last two examples we can consider the square $[0,1]\times[0,1]$ with the equivalence relation given by $(0,t) \sim (1,1-t)$ and $(s,0) \sim (1-s,1)$ for all $s,t \in [0,1]$:



The resulting quotient space is homeomorphic to the space $\mathbb{RP}^2$ which is defined as follows. Take the the 2-dimensional closed unit ball $\overline{B}^2$. The boundary of $\overline{B}^2$ is the circle S^1. Consider the equivalence relation $\sim$ on $\overline{B}^2$ that identifies each point $(x_1, x_2) \in S^1$ with its antipodal point $(-x_1, -x_2)$:



We define $\mathbb{RP}^2 = \overline{B}^2/\sim$. This space is called the *2-dimensional real projective space* and it is a 2-dimensional manifold. One can show that $\mathbb{RP}^2$ (and also the Klein bottle) cannot be embedded into $\mathbb{R}^3$. For this reason it is harder to visualize it.

19.22 Example. The construction of $\mathbb{RP}^2$ given in Example 19.21 can be generalized to higher dimensions. Consider the n-dimensional closed unit ball $\overline{B}^n$. The boundary $\overline{B}^n$ is the sphere S^{n-1}. Similarly as before we can consider the equivalence relation $\sim$ on $\overline{B}^n$ that identifies antipodal points of S^{n-1}:

$$(x_1, \dots, x_n) \sim (-x_1, \dots, -x_n)$$

for all $(x_1, \dots, x_n) \in S^{n-1}$. The quotient space $\overline{B}^n/\sim$ is denoted by $\mathbb{RP}^n$ and is called the *n-dimensional real projective space*. The space $\mathbb{RP}^n$ is an n-dimensional manifold. For another perspective on projective spaces see Exercise 19.5.

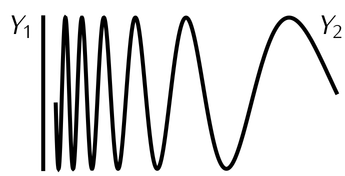
Exercises to Chapter 19

E19.1 Exercise. Consider the real line $\mathbb{R}$ with the equivalence relation defined as in Example 19.3. Show that the quotient space $\mathbb{R}/\sim$ is homeomorphic with S^1.

E19.2 Exercise. Take the closed interval $[0, 1]$ with the equivalence relation $\sim$ defined as in Example 19.15. Let $\pi\colon [0, 1] \to [0, 1]/\sim$ be the quotient map. The set $U = [0, \frac{1}{2})$ which is open subset of $[0, 1]$. Show that $\pi(U)$ is not open in $[0, 1]/\sim$.

E19.3 Exercise. Let $\overline{B}^n \subseteq \mathbb{R}^n$ be the closed unit ball (see Example 19.15). Show that $\overline{B}^n/S^{n-1}$ is homeomorphic to S^n.

E19.4 Exercise. Recall that the topologists sine curve Y is the subspace of $\mathbb{R}^2$ consisting of the vertical line segment $Y_1 = \{(0, y) \mid -1 \leq y \leq 1\}$ and the curve $Y_2 = \{(x, \sin(\frac{1}{x})) \mid x > 0\}$:



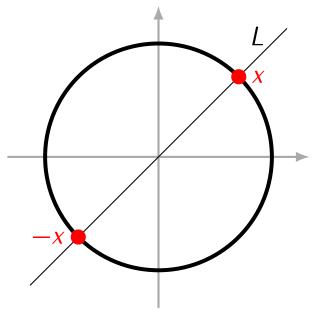
Show that the space Y/Y_1 is homeomorphic to the half line $[0, +\infty)$.

E19.5 Exercise. Consider the unit sphere S^n with the equivalence relation that identifies antipodal points of S^n:

$$(x_1, \dots, x_{n+1}) \sim (-x_1, \dots, -x_{n+1})$$

for all $(x_1, \dots, x_{n+1})$. Show that the quotient space $S^n/\sim$ is homeomorphic to the projective space $\mathbb{RP}^n$ (19.22).

Note: This construction lets us interpret $\mathbb{RP}^n$ as the space of straight lines in $\mathbb{R}^{n+1}$ that pass through the origin. Indeed, any such line L intersects the sphere S^n at two points: some point x and its antipodal point $-x$:



Since $\mathbb{RP}^n$ is obtained by identifying antipodal points we get a bijective correspondence between elements of $\mathbb{RP}^n$ and lines in $\mathbb{R}^{n+1}$ passing through the origin.

E19.6 Exercise. A *pointed topological space* is a pair (X, x_0) where X is a topological space and $x_0 \in X$. The *smash product* of pointed spaces (X, x_0) and (Y, y_0) is the quotient space

$$X \wedge Y = (X \times Y)/A$$

where $A = (X \times \{y_0\}) \cup (\{x_0\} \times Y)$

a) Let X, Y be a locally compact spaces (18.17). Show that the space $X \times Y$ is locally compact.

b) By part a) and Corrollary 17.17 if X, Y are locally compact Hausdorff spaces then the space $X \times Y$ is also locally compact and Hausdorff. By Theorem 18.19 we have in such case one-point compactifications X^+, Y^+, and $(X \times Y)^+$ of the spaces X, Y, and $X \times Y$ respectively. Recall that $X^+ = X \cup \{\infty\}$ and $Y^+ = Y \cup \{\infty\}$. Consider (X^+, ∞) and (Y^+, ∞) as pointed spaces. Show that there is a homeomorphism:

$$X^+ \wedge Y^+ \cong (X \times Y)^+$$

20 | Embeddings of Manifolds

We have seen so far several examples of manifolds. Some of them (e.g. S^n) are defined as subspaces of a Euclidean space $\mathbb{R}^m$ for some m, but some other (e.g. the Klein bottle (19.20), or the projective spaces (19.22)) are defined more abstractly. A natural question is if every manifold is homeomorphic to a subspace of some Euclidean space $\mathbb{R}^m$, or equivalently if it can be embedded into $\mathbb{R}^m$. Our next goal is to show that this is in fact true, at least in the case of compact manifolds.

We begin with some technical preparation.

20.1 Definition. Let X be a topological space and let $f\colon X \to \mathbb{R}$ be a continuous function. The *support* of f is the closure of the subset of X consisting of points with non-zero values:

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$$

20.2 Definition. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. A *partition of unity subordinate to* $\mathcal{U}$ is a family of continuous functions $\{\lambda_i\colon X \to [0,1]\}_{i\in I}$ such that

(i) $\operatorname{supp}(\lambda_i) \subseteq U_i$ for each $i \in I$;
(ii) each point $x \in X$ has an open neighborhood U_x such that $U_x \cap \operatorname{supp}(\lambda_i) \neq \varnothing$ for finitely many $i \in I$ only;
(iii) for each $x \in X$ we have $\sum_{i\in I} \lambda_i(x) = 1$.

20.3 Note. Condition (iii) makes sense since by (ii) we have $\lambda_i(x) \neq 0$ for finitely many $i \in I$ only.

Partitions of unity are a very useful tool for gluing together functions defined on subsets of X to obtain a function defined on the whole space X:

20.4 Lemma. *Let X be a topological space, let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X and let $\{\lambda_i\}_{i\in I}$ be* *a partition of unity subordinate to $\mathcal{U}$.*

1) *Let $i \in I$ and let $f_i\colon U_i \to \mathbb{R}^n$ be a continuous function. Then the function $\tilde{f}_i\colon X \to \mathbb{R}^n$ given by*

$$\tilde{f}_i(x) = \begin{cases} \lambda_i(x) f_i(x) & \text{for } x \in U_i \\ 0 & \text{for } x \in X \smallsetminus U_i \end{cases}$$

   is continuous.

2) *Assume that for each $i \in I$ we have a continuous function $f_i\colon U_i \to \mathbb{R}^n$, and let $\tilde{f}_i\colon X \to \mathbb{R}^n$ be the function defined as above. Then the function $\tilde{f}\colon X \to \mathbb{R}^n$ given by*

$$\tilde{f}(x) = \sum_{i\in I} \tilde{f}_i(x)$$

   is continuous.

Proof. Exercise. □

20.5 Proposition. *Let X be a normal space. For any finite open cover $\{U_1, \dots, U_n\}$ of X there exists a partition of unity subordinate to this cover.*

The proof of Proposition 20.5 will use the following fact:

20.6 Finite Shrinking Lemma. *Let X be a normal space and let $\{U_1, \dots, U_n\}$ be a finite open cover of X. There exists an open cover $\{V_1, \dots, V_n\}$ of X such that $\overline{V}_i \subseteq U_i$ for each $i \geq 1$.*

Proof. We will argue by induction. Assume that for some $k < n$ we already have open sets $V_1, \dots, V_k$ such that $\overline{V}_i \subseteq U_i$ for all $1 \leq i \leq k$ and that $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ is a cover of X (at the start of induction we set $k = 0$). We will show that there exists an open set V_{k+1} such that $\overline{V}_{k+1} \subseteq U_{k+1}$ and that $\{V_1, \dots, V_{k+1}, U_{k+2} \dots, U_n\}$ still covers X. Take the set

$$W = V_1 \cup \dots \cup V_k \cup U_{k+2} \cup \dots \cup U_n$$

Notice that $W \cup U_{k+1} = X$. Therefore $X \smallsetminus W \subseteq U_{k+1}$. Since $X \smallsetminus W$ is a closed set by Lemma 10.3 there exists an open set V such that $X \smallsetminus W \subseteq V$ and $\overline{V} \subseteq U_{k+1}$. The first of these properties gives $W \cup V = X$, which means that $\{V_1, \dots, V_k, V, U_{k+2} \dots, U_n\}$ is an open cover of X. Therefore we can take $V_{k+1} = V$.

□

Lemma 20.6 can be generalized to infinite covers of normal spaces as follows:

20.7 Shrinking Lemma. *Let X be a normal space and let $\{U_i\}_{i\in I}$ be a open cover of X such that each point of X belongs to finitely many sets U_i only. There exists an open cover $\{V_i\}_{i\in I}$ of X such that $\overline{V}_i \subseteq U_i$ for all $i \in I$.*

Proof. Exercise. $\square$

Proof of Proposition 20.5. By Lemma 20.6 there exists an open cover $\{V_1, \dots, V_n\}$ of X such that $\overline{V}_i \subseteq U_i$ for all $i \geq 1$. Since X is a normal space by Lemma 10.3 for each $i \geq 1$ we can find an open set W_i such that $\overline{V}_i \subseteq W_i$ and $\overline{W}_i \subseteq U_i$. Using Urysohn Lemma 10.1 we get continuous functions $\mu_i \colon X \to [0, 1]$ such that $\mu_i(\overline{V}_i) \subseteq \{1\}$ and $\mu_i(X \smallsetminus W_i) \subseteq \{0\}$. Notice that $\operatorname{supp}(\mu_i) \subseteq \overline{W}_i \subseteq U_i$. Let $\mu = \sum_{i=1}^{n} \mu_i$. We claim that $\mu(x) > 0$ for all $x \in X$. Indeed, if $x \in X$ then $x \in V_j$ for some $j \geq 1$ and so $\mu_j(x) = 1$. For $i = 1, \dots, n$ let $\lambda_i \colon X \to [0, 1]$ be the function given by

$$\lambda_i(x) = \frac{\mu_i(x)}{\mu(x)}$$

The family $\{\lambda_1, \dots, \lambda_n\}$ is a partition of unity subordinate to the cover $\{U_1, \dots, U_n\}$ (exercise).

$\square$

20.8 Corollary. *If X is a compact Hausdorff space then for every open cover $\mathcal{U}$ of X there exists an partition of unity subordinate to $\mathcal{U}$.*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$. Since X is compact we can find a finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$ of $\mathcal{U}$. By Theorem 14.18 the space X is normal, so using Proposition 20.5 we obtain a partition of unity $\{\lambda_{i_1}, \dots, \lambda_{i_n}\}$ subordinate to the cover $\{U_{i_1}, \dots, U_{i_n}\}$. For $i \in I \smallsetminus \{i_1, \dots, i_n\}$ let $\lambda_i \colon X \to [0, 1]$ be the constant zero function. The family of functions $\{\lambda_i\}_{i \in I}$ is a partition of unity subordinate to the cover $\mathcal{U}$. $\square$

We are now ready to prove the embedding theorem for compact manifolds. We will consider first the case of manifolds without boundary:

20.9 Theorem. *If M is a compact manifold without boundary then for some $N \geq 0$ there exists an embedding $j \colon M \to \mathbb{R}^N$.*

20.10 Note. A compact manifold without boundary is called a *closed manifold*.

Proof of Theorem 20.9. Assume that M is an n-dimensional manifold. Since M is compact we can find a finite collection of coordinate charts $\{\varphi_i \colon U_i \to \mathbb{R}^n\}_{i=1}^{m}$ on M such that $\{U_i\}_{i=1}^{m}$ is an open cover of M. By Corollary 20.8 there exists a partition of unity $\{\lambda_i\}_{i=1}^{m}$ subordinate to this cover. For $i = 1, \dots, m$ let $\tilde{\varphi}_i \colon M \to \mathbb{R}^n$ be the function obtained from φ_i as in part 1) of Lemma 20.4. Consider the continuous function $j \colon M \to \mathbb{R}^{mn+m}$ defined as follows:

$$j(x) = (\tilde{\varphi}_1(x),\ \dots,\ \tilde{\varphi}_m(x),\ \lambda_1(x),\ \dots,\ \lambda_m(x))$$

We will show that j is a 1-1 function. Since M is a compact and $\mathbb{R}^{mn+m}$ is a Hausdorff space by Proposition 14.17 this will imply that j is a homeomorphism onto $j(M) \subseteq \mathbb{R}^{mn+m}$, and so it is an embedding. Assume then that $x, y \in M$ are points such that $j(x) = j(y)$. This means that $\tilde{\varphi}_i(x) = \tilde{\varphi}_i(y)$ and $\lambda_i(x) = \lambda_i(y)$ for all $i = 1, \ldots, m$. Since $\sum_{i=1}^{m} \lambda_i(x) = 1$ there exists $1 \leq i_0 \leq m$ such that $\lambda_{i_0}(x) \neq 0$, and so also $\lambda_{i_0}(y) \neq 0$. Since $\operatorname{supp}(\lambda_{i_0}) \subseteq U_{i_0}$ we obtain that $x, y \in U_{i_0}$. By definition of $\tilde{\varphi}_{i_0}$ we have $\tilde{\varphi}_{i_0}(z) = \lambda_{i_0}(z)\varphi_{i_0}(z)$ for all $z \in U_{i_0}$. Therefore we get

$$\lambda_{i_0}(x)\varphi_{i_0}(x) = \tilde{\varphi}_{i_0}(x) = \tilde{\varphi}_{i_0}(y) = \lambda_{i_0}(y)\varphi_{i_0}(y)$$

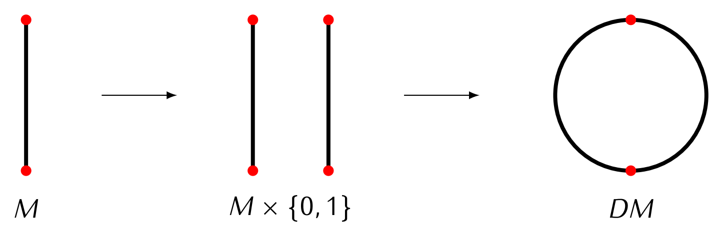
Dividing both sides by $\lambda_{i_0}(x) = \lambda_{i_0}(y)$ we obtain $\varphi_{i_0}(x) = \varphi_{i_0}(y)$. However, $\varphi_{i_0}\colon U_{i_0} \to \mathbb{R}^n$ is a homeomorphism, so in particular it is a 1-1 function. This shows that $x = y$. □

It is straightforward to generalize the proof of Theorem 20.9 to the case when M is a compact manifold with boundary. We will use however a slightly different argument to show that such manifolds can be embedded into Euclidean spaces.

20.11 Definition. Let M be a manifold with boundary ∂M. The *double* of M is the topological space

$$DM = M \times \{0, 1\}/\sim$$

where $\{0, 1\}$ is the discrete space with two points and $\sim$ is the equivalence relation on $M \times \{0, 1\}$ given by $(x, 0) \sim (x, 1)$ for all $x \in \partial M$.



20.12 Proposition. *If M is an n-dimensional manifold with boundary then DM is an n-dimensional manifold without boundary. Moreover, if M is compact then so is DM.*

Proof. Exercise. □

20.13 Corollary. *If M is a compact manifold with boundary then for some $N > 0$ there exists an embedding $M \to \mathbb{R}^N$.*

Proof. Take the double DM of M. By Proposition 20.12 DM is a closed manifold, so using Theorem 20.9 we obtain an embedding $j\colon DM \to \mathbb{R}^N$ for some $N \geq 0$. Notice that we also have an embedding $\pi i\colon M \to DM$ where $i\colon M \to M \times \{0, 1\}$ is the function given by $i(x) = (x, 0)$ and $\pi\colon M \times \{0, 1\} \to DM$ is the quotient map. Therefore we obtain an embedding

$$j\pi i\colon M \to \mathbb{R}^N$$

□

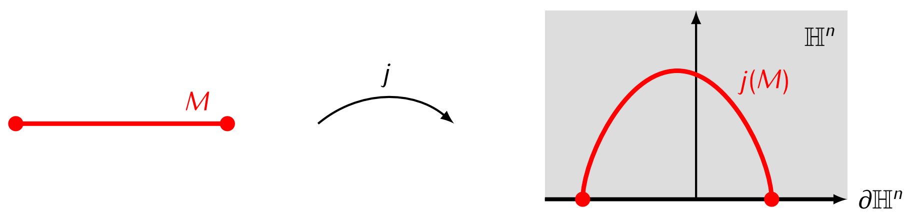
20.14 Note. Theorem 20.9 and Corollary 20.13 can be extended to non-compact manifolds: one can show that any manifold (compact or not, with or without boundary) can be embedded into the Euclidean space $\mathbb{R}^N$ for some $N \geq 0$. Moreover, it turns out that any n-dimensional manifold can be embedded into $\mathbb{R}^{2n+1}$. An interesting question is, given some specific manifold M (e.g. $M = \mathbb{RP}^n$) what is the smallest number N such that M can be embedded into $\mathbb{R}^N$.

Exercises to Chapter 20

E20.1 Exercise. Prove Lemma 20.4.

E20.2 Exercise. Prove Proposition 20.12.

E20.3 Exercise. Recall that $\mathbb{H}^n$ is the subspace of $\mathbb{R}^n$ given by $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and that $\partial\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{H}^n \mid x_n = 0\}$. Let M be a compact manifold with boundary ∂M. Show that for some $N \geq 0$ there exists an embedding $j\colon M \to \mathbb{H}^N$ such that $j(\partial M) \subseteq \partial\mathbb{H}^N$ and $j(M \smallsetminus \partial M) \subseteq \mathbb{H}^N \smallsetminus \partial\mathbb{H}^N$.



E20.4 Exercise. The goal of this exercise is to prove the general Shrinking Lemma 20.7. Let X be a normal space and let $\{U_i\}_{i\in I}$ be an open cover of X such that every point of X belongs to finitely many sets U_i only.

a) Let S be the set consisting of all pairs $(J, \{V_j\}_{j\in J})$ where J is a subset of I and $\{V_j\}_{j\in J}$ is a collection of open sets in X such that $\overline{V}_j \subseteq U_j$ for all $j \in J$, and $\{V_j\}_{j\in J} \cup \{U_i\}_{i\in I\smallsetminus J}$ is a cover of X. We define a partial order on S as follows. If $(J, \{V_j\}_{j\in J})$ and $(J', \{V'_j\}_{j\in J'})$ are elements of S then $(J, \{V_j\}_{j\in J}) \leq (J', \{V'_j\}_{j\in J'})$ if $J \subseteq J'$ and if $V_j = V'_j$ for all $j \in J$. Use Zorn's Lemma 17.15 to show that the set S has a maximal element.

b) Let S be the set defined above. Show that if $(J, \{V_j\}_{j\in J})$ is a maximal element of S then $J = I$. This gives that $\{V_j\}_{j\in J}$ is an open cover of X such that $\overline{V}_i \subseteq U_i$ for all $i \in I$.

21 | Mapping Spaces

21.1 Definition. Let X, Y be topological spaces. By $\mathrm{Map}(X, Y)$ we will denote the set of all continuous functions $f\colon X \to Y$.

Our main goal in this chapter is to show how the set $\mathrm{Map}(X, Y)$ can be given the structure of a topological space. Constructions of new topological spaces from existing topological spaces that we have already seen were motivated by the choice of continuous functions from or into the new space that we wanted to have. For example, the product topology was defined in such way, that a map $f\colon Y \to \prod_{i\in I} X_i$ is continuous if and only if its compositions with all projection maps $p_i f\colon Y \to X_i$ are continuous (12.8). Similarly, the quotient topology on a space $X/\sim$ was defined so that a function $f\colon X/\sim \to Y$ is continuous if and only if its composition with the quotient map $f\pi\colon X \to Y$ is continuous (19.13). The choice of topology on $\mathrm{Map}(X, Y)$ will be based on similar considerations.

Denote by $\mathrm{Func}(X, Y)$ the set of all functions (continuous or not) $X \to Y$. Any function $F\colon Z \times X \to Y$ defines a function $F_*\colon Z \to \mathrm{Func}(X, Y)$, where for $z \in Z$ the function $F_*(z)\colon X \to Y$ is given by $F_*(z)(x) = F(z, x)$. Conversely, any function $F_*\colon Z \to \mathrm{Func}(X, Y)$ defines a function $F\colon Z \times X \to Y$ given by $F(z, x) = F_*(z)(x)$. For any spaces X, Y, Z the assignment $F \mapsto F_*$ gives a bijective correspondence:

$$\left(\begin{array}{c}\text{functions}\\ Z \times X \to Y\end{array}\right) \cong \left(\begin{array}{c}\text{functions}\\ Z \to \mathrm{Func}(X, Y)\end{array}\right)$$

If $F\colon Z \times X \to Y$ is a continuous function, then for any $z \in Z$ the function $F_*(z)\colon X \to Y$ is continuous. This shows that in this case we get a well defined function

$$F_*\colon Z \to \mathrm{Map}(X, Y)$$

With this in mind, it is reasonable to attempt to define a topology on $\mathrm{Map}(X, Y)$ in such way, that for any function $F\colon Z \times X \to Y$ the induced function $F_*\colon Z \to \mathrm{Map}(X, Y)$ is continuous if any only if F is continuous. This motivates the following definition:

21.2 Definition. Let X, Y be a topological spaces, and let $\mathcal{T}$ be a topology on $\mathrm{Map}(X, Y)$.

1) We will say that the topology $\mathcal{T}$ is *lower admissible* if for any continuous function $F\colon Z \times X \to Y$ the function $F_*\colon Z \to \mathrm{Map}(X, Y)$ is continuous.
2) We will say that the topology $\mathcal{T}$ is *upper admissible* if for any function $F\colon Z \times X \to Y$ if the function $F_*\colon Z \to \mathrm{Map}(X, Y)$ is continuous then F is continuous.
3) We will say that the topology $\mathcal{T}$ is *admissible* if it is both lower and upper admissible.

The definition of upper admissible topology can be reformulated using the notion of the evaluation map:

21.3 Definition. Let X, Y be topological spaces. The *evaluation map* is the function

$$\mathrm{ev}\colon \mathrm{Map}(X, Y) \times X \to Y$$

given by $\mathrm{ev}((f, x)) = f(x)$.

Notice that $\mathrm{ev}_*\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Y)$ is the identity function. We have:

21.4 Lemma. *Let X, Y be topological spaces, and let $\mathcal{T}$ be a topology on* $\mathrm{Map}(X, Y)$*. The following conditions are equivalent:*

1) *The topology $\mathcal{T}$ is upper admissible.*
2) *The evaluation map* $\mathrm{ev}\colon \mathrm{Map}(X, Y) \times X \to Y$ *is continuous.*

Proof. 1) $\Rightarrow$ 2) For any choice of topology on $\mathrm{Map}(X, Y)$ the identity function $\mathrm{id}_{\mathrm{Map}(X,Y)}\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Y)$ is continuous. Since by assumption $\mathcal{T}$ is upper admissible and $\mathrm{ev}_* = \mathrm{id}_{\mathrm{Map}(X,Y)}$ this implies that ev is continuous.

2) $\Rightarrow$ 1) Assume that ev in continuos, and let $F\colon Z \times X \to Y$ be a function such that F_* is continuous. Then $F_* \times \mathrm{id}_X\colon Z \times X \to \mathrm{Map}(X, Y) \times X$ is a continuous function. Since $F = \mathrm{ev} \circ (F_* \times \mathrm{id}_X)$ it follows that F is continuous. $\square$

21.5 Example. Let X, Y be topological spaces. If we consider $\mathrm{Map}(X, Y)$ with the antidiscrete topology then every function $Z \to \mathrm{Map}(X, Y)$ is continuous. Therefore the antidiscrete topology on $\mathrm{Map}(X, Y)$ is lower admissible.

On the other hand, consider $\mathrm{Map}(X, Y)$ with the discrete topology. We will show that this topology is upper admissible. By Lemma 21.4 it suffices to verify that the evaluation map $\mathrm{ev}\colon \mathrm{Map}(X, Y) \times X \to Y$

is continuous, i.e. that for any open set $U \subseteq Y$ the set $\mathrm{ev}^{-1}(U)$ is open in $\mathrm{Map}(X, Y) \times X$. Notice that

$$\begin{aligned} \mathrm{ev}^{-1}(U) &= \{(f, x) \in \mathrm{Map}(X, Y) \times X \mid f(x) \in U\} \\ &= \{(f, x) \in \mathrm{Map}(X, Y) \times X \mid x \in f^{-1}(U)\} \\ &= \bigcup_{f \in \mathrm{Map}(X,Y)} \{f\} \times f^{-1}(U) \end{aligned}$$

For any $f \in \mathrm{Map}(X, Y)$ the set $f^{-1}(U)$ is open in X, and since the topology on $\mathrm{Map}(X, Y)$ is discrete the set $\{f\}$ is open in $\mathrm{Map}(X, Y)$. It follows that $\mathrm{ev}^{-1}(U)$ is open in $\mathrm{Map}(X, Y) \times X$.

21.6 Proposition. *Let X, Y be topological spaces.*

1) *If $\mathcal{U}, \mathcal{U}'$ are two topologies on $\mathrm{Map}(X, Y)$ such that $\mathcal{U} \subseteq \mathcal{U}'$ and $\mathcal{U}$ is upper admissible, then $\mathcal{U}'$ also is upper admissible.*
2) *If $\mathcal{L}, \mathcal{L}'$ are two topologies on $\mathrm{Map}(X, Y)$ such that $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathcal{L}$ is lower admissible, then $\mathcal{L}'$ also is lower admissible.*
3) *If $\mathcal{U}$, $\mathcal{L}$ are two topologies on $\mathrm{Map}(X, Y)$ such that $\mathcal{U}$ is upper admissible and $\mathcal{L}$ is lower admissible then $\mathcal{L} \subseteq \mathcal{U}$.*

Proof. Proofs of 1) and 2) are straightforward. To prove part 3), denote by $\mathrm{Map}(X, Y)_{\mathcal{U}}$ and $\mathrm{Map}(X, Y)_{\mathcal{L}}$ the set $\mathrm{Map}(X, Y)$ equipped with the topology, respectively, $\mathcal{U}$ and $\mathcal{L}$. Since $\mathcal{U}$ is upper admissible the evaluation map $\mathrm{ev}\colon \mathrm{Map}(X, Y)_{\mathcal{U}} \times X \to Y$ is continuous. Since $\mathcal{L}$ is lower admissible we get that $\mathrm{id}_{\mathrm{Map}(X,Y)} = \mathrm{ev}_*\colon \mathrm{Map}(X, Y)_{\mathcal{U}} \to \mathrm{Map}(X, Y)_{\mathcal{L}}$ is continuous. Therefore any set U open in $\mathrm{Map}(X, Y)_{\mathcal{L}}$ is also open in $\mathrm{Map}(X, Y)_{\mathcal{U}}$, and so $\mathcal{L} \subseteq \mathcal{U}$. ☐

21.7 Corollary. *Given spaces X and Y, if there exists an admissible topology on $\mathrm{Map}(X, Y)$ then such topology is unique.*

Proof. This follows directly from Proposition 21.6. ☐

The next proposition shows that in general an admissible topology on $\mathrm{Map}(X, Y)$ may not exist:

21.8 Proposition. *Let X be completely regular space. If there exist an admissible topology on $\mathrm{Map}(X, \mathbb{R})$ then X is locally compact.*

21.9 Example. Since the space $\mathbb{Q}$ of rational numbers is completely regular but not locally compact (Exercise 18.4), there is no admissible topology on $\mathrm{Map}(\mathbb{Q}, \mathbb{R})$.

The proof Proposition 21.8 will depend on the following fact:

21.10 Definition. Let X, Y be topological spaces. For sets $A \subseteq X$ and $B \subseteq Y$ denote

$$P(A, B) = \{f \in \mathrm{Map}(X, Y) \mid f(A) \subseteq B\}$$

21.11 Lemma. *Let X, Y topological spaces, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. Let $\mathcal{T}$ be a topology on* $\mathrm{Map}(X, Y)$ *with subbasis given by all sets $P(A, V)$ where $A \subseteq X$ is a closed set such that $A \subseteq U_i$ for some $i \in I$, and $V \subseteq Y$ is an open set. If X is a regular space then $\mathcal{T}$ upper admissible.*

Proof. Exercise. □

Proof of Proposition 21.8. Let $\mathcal{A}$ be an admissible topology on $\mathrm{Map}(X, \mathbb{R})$, and let $\mathrm{Map}(X, \mathbb{R})_{\mathcal{A}}$ denote $\mathrm{Map}(X, \mathbb{R})$ taken with this topology. Take $x_0 \in X$. We need to show that there exists an open set $V \subseteq X$ such that $x_0 \in V$ and $\overline{V}$ is compact.

Let $f \colon X \to \mathbb{R}$ be a constant function given by $f(x) = 0$ for all $x \in X$. Then $\mathrm{ev}((f, x_0)) \in (-1, 1)$. Since $(-1, 1)$ is open in $\mathbb{R}$ and the function $\mathrm{ev}\colon \mathrm{Map}(X, \mathbb{R})_{\mathcal{A}} \times X \to \mathbb{R}$ is continuous, there exist open sets $W \subseteq \mathrm{Map}(X, \mathbb{R})_{\mathcal{A}}$ and $V \subseteq X$ such that $f \in W$, $x_0 \in V$, and $ev(W \times V) \subseteq (-1, 1)$. We will prove that $\overline{V}$ is compact. It will suffice to show that if $\mathcal{U}$ is an open cover of X then $\overline{V} \subseteq U_{i_1} \cup \ldots U_{i_n}$ for some $i_1, \ldots, i_n \in I$ (Exercise 14.3). Let $\mathcal{U} = \{U_i\}_{\in I}$ be such open cover, and let $\mathcal{T}$ be a topology on $\mathrm{Map}(X, \mathbb{R})$ with subbasis consisting of all sets $P(A, Z)$ where $A \subseteq X$ is a closed, $A \subseteq U_i$ for some $i \in I$, and $Z \subseteq \mathbb{R}$ is an open set. By Lemma 21.11 $\mathcal{T}$ is upper admissible. Since $\mathcal{A}$ is lower admissible, by Proposition 21.6 we obtain that $\mathcal{A} \subseteq \mathcal{T}$. This implies that there exist elements $P(A_1, Z_1), \ldots P(A_n, Z_n)$ of the subbasis of $\mathcal{T}$ such that $f \in \bigcap_{k=1}^n P(A_k, Z_k) \subseteq W$. Notice that since $f(A_k) = 0$ for all k, we must have $0 \in \bigcap_{k=1}^n Z_k$. Assume that there exists a point $y \in V \smallsetminus \bigcup_{k=1}^n A_k$. Since the set $\bigcup_{k=1}^n A_k$ is closed in X and the space X is completely regular, this would give a continuous function $g\colon X \to \mathbb{R}$ such that $g(\bigcup_{k=1}^n A_k) = 0$ (and so $g \in \bigcap_{k=1}^n P(A_k, Z_k) \subseteq W$) and $g(y) = 1$. This is however impossible, since by the choice of W and V we have $h(v) \in (-1, 1)$ for every $h \in W$ and $v \in V$. Therefore $V \subseteq \bigcup_{k=1}^n A_k$, and since $\bigcup_{k=1}^n A_k$ is closed, also $\overline{V} \subseteq \bigcup_{k=1}^n A_k$. By assumption for each $k = 1, \ldots, n$ there exists $U_{i_k} \in \mathcal{U}$ such that $A_k \subseteq U_{i_k}$. This gives $\overline{V} \subseteq \bigcup_{k=1}^n U_{i_k}$.

□

In view of Proposition 21.8 a natural question is whether an admissible topology on $\mathrm{Map}(X, Y)$ exists when X is a completely regular, locally compact space. The condition that X is completely regular can replaced by the condition that X is Hausdorff, since every locally compact Hausdorff space is completely regular (18.20). Our next goal is to show that under these assumptions on X the set $\mathrm{Map}(X, Y)$ has an admissible topology, and that this topology can be described as follows:

21.12 Definition. Let X, Y be topological spaces. The *compact-open* topology on $\mathrm{Map}(X, Y)$ is the topology defined by the subbasis consisting of all sets $P(A, U)$ where $A \subseteq X$ is compact and $U \subseteq Y$ is an open set.

21.13 Theorem. *For any spaces X, Y the compact-open topology on* $\mathrm{Map}(X, Y)$ *is lower admissible.*

Proof. Consider $\mathrm{Map}(X, Y)$ as a space with compact-open topology, and let $F: Z \times X \to Y$ be a continuous function. We need to show that if $F_*: Z \to \mathrm{Map}(X, Y)$ is continuous. By Proposition 4.14 it is enough to show that for any compact set $A \subseteq Y$ and an open set $U \subseteq Y$ the set $F_*^{-1}(P(A, U))$ is open in Z. It will suffice to check that for any $z_0 \in F_*^{-1}(P(A, U))$ there exists an open neighborhood $V \subseteq Z$ such that $V \subseteq F_*^{-1}(P(A, U))$. Notice that

$$\begin{aligned} F_*^{-1}(P(A, U)) &= \{z \in Z \mid F(\{z\} \times A) \subseteq U\} \\ &= \{z \in Z \mid \{z\} \times A \subseteq F^{-1}(U)\} \end{aligned}$$

In particular, since $z_0 \in F_*^{-1}(P(A, U))$ we have $\{z_0\} \times A \subseteq F^{-1}(U)$. The set $F^{-1}(U)$ is open in $Z \times Y$, so $F^{-1}(U) = \bigcup_{i \in I}(V_i \times W_i)$ for some open sets $V_i \in Z$ and $W_i \in X$. Since $\{z_0\} \times A \cong A$ is compact, there exist $i_1, \dots i_n \in I$ such that $\{z_0\} \times A \subseteq \bigcup_{k=1}^n (V_{i_k} \times W_{i_k})$. Take $V = \bigcap_{k=1}^n V_{i_k}$. Then $V \times A \subseteq \bigcup_{k=1}^n V_{i_k} \times W_{i_k} \subseteq F^{-1}(U)$, and so $V \subseteq F_*^{-1}(P(A, U))$. □

21.14 Theorem. *Let X, Y be topological spaces. If X is locally compact Hausdorff space then the compact-open topology on* $\mathrm{Map}(X, Y)$ *is upper admisible.*

Proof. Let $\mathcal{C}$ denote the compact-open topology on $\mathrm{Map}(X, Y)$. Let $\mathcal{U} = \{U_i\}$ be an open cover of X such that $\overline{U}_i$ is compact for each $i \in I$. Such open cover exists by the assumption that X is locally compact. Let $\mathcal{T}$ be the topology on $\mathrm{Map}(X, Y)$ with subbasis consisting of all sets $P(A, V)$ where $A \subseteq X$ is a closed, $A \subseteq U_i$ for some $i \in I$, and $V \subseteq Y$ is an open. Notice that by Proposition 14.13 for any such $P(A, V)$ the set A is compact, since $A \subseteq \overline{U}_i$ for some $i \in I$, and $\overline{U}_i$ is compact. Therefore $P(A, V) \in \mathcal{C}$, and so $\mathcal{T} \subseteq \mathcal{C}$.

By Lemma 21.11 the topology $\mathcal{T}$ is upper admissible. Since by Theorem 21.13 $\mathcal{C}$ is lower admissible using Proposition 21.6 we obtain that $\mathcal{C} \subseteq \mathcal{T}$. This shows that $\mathcal{C} = \mathcal{T}$, and so $\mathcal{C}$ is upper admissible. □

21.15 Corollary. *If X is a locally compact Hausdorff space and Y is any space then the compact-open topology on* $\mathrm{Map}(X, Y)$ *is admissible.*

Proof. Follows from Theorem 21.13 and Theorem 21.14 . □

21.16 Note. Let X, Y, Z be topological spaces. By Corollary 21.15 if X is locally compact Hausdorff and $\mathrm{Map}(X, Y)$ is taken with the compact-open topology then the map

$$\Psi\colon \mathrm{Map}(Z \times X, Y) \to \mathrm{Map}(Z, \mathrm{Map}(X, Y))$$

given by $\Psi(F) = F_*$ is a well defined bijection. One can show that if in addition Z is a Hausdorff space, and both $\mathrm{Map}(Z \times X, Y)$ and $\mathrm{Map}(Z, \mathrm{Map}(X, Y))$ are considered as topological spaces with compact-open topology, then Ψ is a homeomorphism.

In some cases the compact open-topology on $\mathrm{Map}(X, Y)$ can be described more explicitly. Let X be a topological space and let S be a set. Recall (1.18) that the Cartesian product $\prod_{s \in S} X$ is formally defined as the set of all functions $S \to X$. We have:

21.17 Proposition. *Let X be a topological space, and let S be a set considered as a discrete topological space. There exists a homeomorphism*

$$\mathrm{Map}(S, X) \cong \prod_{s \in S} X$$

where $\mathrm{Map}(S, X)$ *is taken with the compact-open topology, and* $\prod_{s\in S} X$ *with the product topology.*

Proof. Exercise □

21.18 Note. In the special case where $S = \{*\}$ is a set consisting of a single point we obtain a homeomorphism $\mathrm{Map}(\{*\}, X) \cong X$.

Next, let X be a topological space and (Y, ϱ) be a metric space. If $f, g \colon X \to Y$ are continuous function then the function $\Phi_{f,g} \colon X \to \mathbb{R}$ given by $\Phi_{f,g}(x) = \varrho(f(x), g(x))$ is continuous (exercise). If X is a compact space then by Exercise 14.6 this function attains its maximum value at some point $x_0 \in X$. We have:

21.19 Proposition. *Let X be a compact Hausdorff space, and let (Y, ϱ) be a metric space. For $f, g \in \mathrm{Map}(X, Y)$ define*

$$d(f, g) = \max\{\varrho(f(x), g(x)) \mid x \in X\}$$

Then d is a metric on $\mathrm{Map}(X, Y)$. *Moreover, in the topology induced by this metric is the compact-open topology.*

Proof. Exercise. □

We conclude this chapter with a result that says that compact-open topology behaves well with respect to composition of functions:

21.20 Theorem. *Let X, Y, Z be topological spaces. Let*

$$\Phi \colon \mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

be a function given by $\Phi(f, g) = g \circ f$. If Y is a locally compact Hausdorff space, and all mapping spaces are equipped with the compact-open topology then Φ is continuous.

The proof will use the following fact:

21.21 Lemma. *Let X be a locally compact Hausdorff space, and let $A, W \subseteq X$ be sets such that A is compact, W is open, and $A \subseteq W$. Then there exists an open set $V \subseteq X$ such that $A \subseteq V$, $\overline{V} \subseteq W$, and $\overline{V}$ is compact.*

Proof. Exercise. □

Proof of Theorem 21.20. Let $A \subseteq X$ be a compact set, $U \subseteq Z$ be an open set, and let $(f, g) \in \Phi^{-1}(P(A, U))$. It will suffice to show that (f, g) has an open neighborhood contained in $\Phi^{-1}(P(A, U))$. Since $g \circ f(A) \subseteq U$, thus $f(A) \subseteq g^{-1}(U)$. By (14.9) the set $f(A)$ is compact, so using Lemma 21.21 we obtain that there exists an open set $V \subseteq Y$ such that $f(A) \subseteq V$, $\overline{V} \subseteq g^{-1}(U)$, and $\overline{V}$ is compact. It remains to notice that the set $P(A, V) \times P(\overline{V}, U)$ is an open neighborhood of (f, g) in $\mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z)$, and $P(A, V) \times P(\overline{V}, U) \subseteq \Phi^{-1}(P(A, U))$. □

Exercises to Chapter 21

E21.1 Exercise. Prove Proposition 21.8.

E21.2 Exercise. Prove Proposition 21.17.

E21.3 Exercise. Prove Proposition 21.19.

E21.4 Exercise. Prove Proposition 21.21.

E21.5 Exercise. Let X, Y be topological spaces, and let $A \subseteq X$, $B \subseteq Y$ be closed sets. Show that in the compact-open topology on $\mathrm{Map}(X, Y)$ the set $P(A, B)$ is closed.

E21.6 Exercise. Let X, Y, Z be topological spaces, and let $f \colon X \to Y$ be a continuous function.

a) Define a function $f_* \colon \mathrm{Map}(Z, X) \to \mathrm{Map}(Z, Y)$ by $f_*(g) = f \circ g$. Show that f_* is continuous.

b) Define a function $f^* \colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ by $f^*(g) = g \circ f$. Show that f^* is continuous.

All mapping spaces are considered with the compact-open topology.

E21.7 Exercise. Let X, Y_i, $i \in I$ be topological spaces. Show that there is a homeomorphism:

$$\mathrm{Map}(X, \textstyle\prod_{i\in I} Y_i) \simeq \prod_{i\in I} \mathrm{Map}(X, Y_i)$$

All mapping spaces are taken with the compact-open topology.