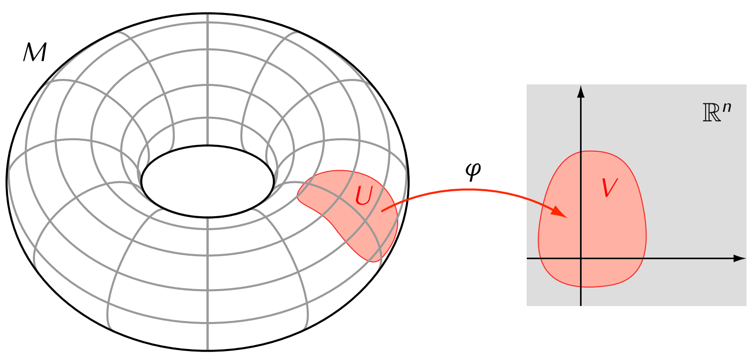
13 | Metrization of Manifolds

Manifolds are among the most important objects in geometry in topology. In this chapter we introduce manifolds and look at some of their basic examples and properties. In particular, as an application of the Urysohn Metrization Theorem, we show that every manifold is a metrizable space.

13.1 Definition. A *topological manifold of dimension n* is a topological space M which is a Hausdorff, second countable, and such that every point of M has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ (we say that M is *locally homeomorphic* to $\mathbb{R}^n$).

13.2 Note. Let M be a manifold of dimension n. If $U \subseteq M$ is an open set and $\varphi\colon U \to V$ is a homeomorphism of U with some open set $V \subseteq \mathbb{R}^n$ then we say that U is a *coordinate neighborhood* and φ is a *coordinate chart* on M.



13.3 Lemma. *If M is an n-dimensional manifold then:*

1) *for any point $x \in M$ there exists a coordinate chart $\varphi\colon U \to V$ such that $x \in U$, V is an open ball $V = B(y, r)$, and $\varphi(x) = y$;*

2) for any point $x \in M$ there exists a coordinate chart $\psi\colon U \to V$ such that $x \in U$, $V = \mathbb{R}^n$, and $\psi(x) = 0$.

Proof. Exercise. □

13.4 Example. A space M is a manifold of dimension 0 if and only if M is a countable (finite or infinite) discrete space.

13.5 Example. If U is an open set in $\mathbb{R}^n$ then U is an n-dimensional manifold. The identity map $\mathrm{id}\colon U \to U$ is then a coordinate chart defined on the whole manifold U. In particular $\mathbb{R}^n$ is an n-dimensional manifold.

13.6 Example. The n-dimensional sphere

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is an n-dimensional manifold. Indeed, let $x = (x_1, \dots, x_{n+1}) \in S^n$. We need to show that there exists an open neighborhood of x which is homeomorphic to an open subset of $\mathbb{R}^n$. Choose $i \in \{1, 2, \dots, n+1\}$ such that $x_i \neq 0$. Assume that $x_i > 0$. Take $U_i^+ = \{(y_1, \dots, y_{n+1}) \in S^n \mid y_i > 0\}$. The set U_i^+ is open in S^n and $x \in U_i^+$. We have a continuous map

$$h_i^+\colon U_i^+ \to B(0,1) \subseteq \mathbb{R}^n$$

given by $h(y_1, \dots, y_{n+1}) = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{n+1})$. This map is a homeomorphism with the inverse $(h_i^+)^{-1}\colon B(0,1) \to U_i^+$ given by

$$(h_i^+)^{-1}(t_1, \dots, t_n) = \left(t_1, \dots, t_{i-1}, \sqrt{1 - (t_1^2 + \cdots + t_n^2)}, t_i, \dots t_n,\right)$$

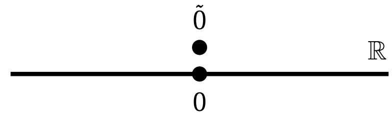
If $x_i < 0$ then we can construct in a similar way a coordinate chart $h_i^-\colon U_i^- \to B(0,1)$ where $U_i^- = \{(y_1, \dots, y_{n+1}) \in S^n \mid y_i < 0\}$.

13.7 Proposition. *If M is an m-dimensional manifold and N is an n-dimensional manifold then $M \times N$ is an $m+n$-dimensional manifold.*

Proof. Exercise. □

13.8 Example. The *torus* is the space $T^2 := S^1 \times S^1$. Since S^1 is a manifold of dimension 1, thus by Proposition 13.7 T^2 is a manifold of dimension 2. Similarly, for any $n \geq 2$ the *n-dimensional torus* $T^n := \prod_{i=1}^n S^1$ is a manifold of dimension n.

13.9 Note. There exist topological spaces that are locally homeomorphic to $\mathbb{R}^n$, but do not satisfy the the other conditions of the definition of a manifold (13.1). For example, the line with double origin is a topological space L defined as follows. As a set L consist of all points of the real line $\mathbb{R}$ and one additional point that we will denote by $\tilde{0}$:



A basis $\mathcal{B}$ of the topology on L consists of the following sets:

1) any open set in $\mathbb{R}$ is in $\mathcal{B}$;
2) for any $a < 0$ and $b > 0$ the set $(a, 0) \cup \{\tilde{0}\} \cup (0, b)$ is in $\mathcal{B}$.

Notice that L is locally homeomorphic to $\mathbb{R}$. Indeed, since $\mathbb{R}$ is an open set in L thus any point of $L \smallsetminus \{\tilde{0}\}$ has an open neighborhood homeomorphic to $\mathbb{R}$. Also, for any $a < 0 < b$ the set $(a, 0) \cup \{\tilde{0}\} \cup (0, b)$ is an open neighborhood of $\tilde{0}$ which is homeomorphic to the open interval (a, b). On the other hand L is not a Hausdorff space since the point $\tilde{0}$ cannot be separated by open sets from $0 \in \mathbb{R}$. Therefore L is not a manifold. There exist also spaces (e.g. Alexandroff long line) that are locally homeomorphic to $\mathbb{R}^n$ and are Hausdorff, but are not second countable.

The following theorem says that the dimension of a manifold is well defined:

13.10 Invariance of Dimension Theorem. *If M is a non-empty topological space such that M is a manifold of dimension m and M is also a manifold of dimension n then $m = n$.*

In other words if a space is locally homeomorphic to $\mathbb{R}^m$ then if cannot be locally homeomorphic to $\mathbb{R}^n$ for $n \neq m$. While this sounds obvious the proof for arbitrary m and n is actually quite involved and goes beyond the scope of this course. The proof is much simpler for $m = 0$ and $m = 1$ (exercise).

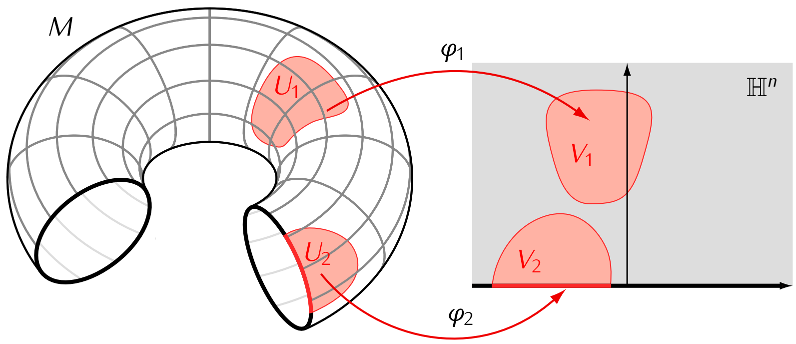
An slight generalization of the notion of a manifold is a manifold with boundary. Let $\mathbb{H}^n$ denote the subspace of $\mathbb{R}^n$ given by $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

13.11 Definition. A *topological n-dimensional manifold with boundary* is a topological space M which is a Hausdorff, second countable, and such that every point of M has an open neighborhood homeomorphic to an open subset of $\mathbb{H}^n$.

As before, if M is a manifold with boundary, U is an open set in M, V is an open set in $\mathbb{H}^n$ and $\varphi\colon U \to V$ is a homeomorphism then we say that φ is a coordinate chart on M.

13.12 Let $\partial\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{H}^n \mid x_n = 0\}$. If M is an n-dimensional manifold with boundary, $\varphi\colon U \to V$ is a coordinate chart, and $x \in U$ then there are two possibilities:

1) $\varphi(x) \in \partial\mathbb{H}^n$
2) $\varphi(x) \notin \partial\mathbb{H}^n$



In the first case we say that the point x is a *boundary point* of M, and in the the second case that x is an *interior point* of M. The next theorem says that a point cannot be a boundary point and an interior point of M at the same time:

13.13 Theorem. *Let M be an n-dimensional manifold with boundary, let $x_0 \in M$ and let $\varphi\colon U \to V$ be a local coordinate chart such that $x_0 \in U$. If $\varphi(x_0) \in \partial\mathbb{H}^n$ then for any other local coordinate chart $\psi\colon U' \to V'$ such that $x_0 \in U'$ we have $\psi(x_0) \in \partial\mathbb{H}^n$.*

The proof in the general case requires similar machinery as the proof of Theorem 13.10, and so we will omit it here. The case when $n = 1$ is much simpler (exercise).

13.14 Definition. Let M be a manifold with boundary. The subspace of M consisting of all boundary points of M is called *the boundary of M* and it is denoted by ∂M.

13.15 Example. The space $\mathbb{H}^n$ is trivially an n-dimensional manifold with boundary.

13.16 Example. For any n the closed n-dimensional ball

$$\overline{B}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

is an n-dimensional manifold with boundary (exercise). In this case we have $\partial\overline{B}^n = S^{n-1}$.

13.17 Example. If M is a manifold (without boundary) then we can consider it as a manifold with boundary. where $\partial M = \varnothing$.

13.18 Example. If M is an m-dimensional manifold with boundary and N is an n-dimensional manifold without boundary then $M \times N$ is an $(m + n)$-dimensional manifold with boundary (exercise). In such case we have: $\partial(M \times N) = \partial M \times N$. For example the *solid torus* $\overline{B}^2 \times S^1$ is a 3-dimensional manifold with boundary, and $\partial(\overline{B}^2 \times S^1) = S^1 \times S^1 = T^2$.

Even more generally, if M is an m-dimensional manifold with boundary and N is an n-dimensional manifold with boundary then $M \times N$ is an $(m+n)$-dimensional manifold with boundary and $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$ (exercise).

13.19 Proposition. *If M is an n-dimensional manifold with boundary then:*

1) *$M \smallsetminus \partial M$ is an open subset of M and it is an n-dimensional manifold (without boundary);*
2) *∂M is a closed subset of M and it is an $(n-1)$-dimensional manifold (without boundary).*

Proof. Exercise. □

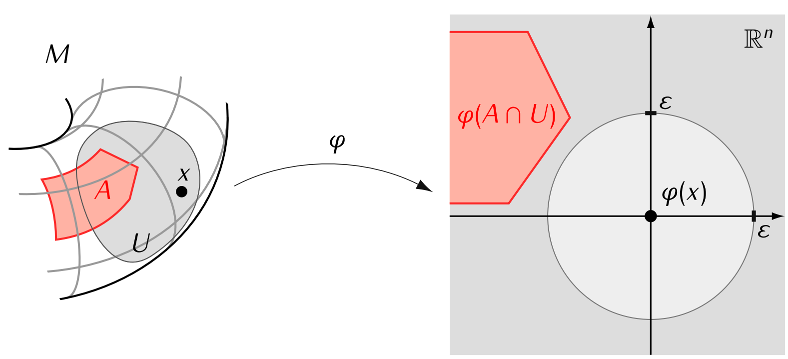
13.20 Theorem. *Every topological manifold (with or without boundary) is metrizable.*

Our argument will use the following fact, the proof of which will be postponed until later (see Exercise 15.5).

13.21 Lemma. *Let M be an n-dimensional topological manifold, and let $\varphi\colon U \to V$ be a coordinate chart on M. If $\bar{B}(x, r)$ is a closed ball in $\mathbb{R}^n$ such that $\bar{B}(x, r) \subseteq V$ then the set $\varphi^{-1}(\bar{B}(x, r))$ is closed in M.*

Proof of Theorem 13.20. We will use Urysohn Metrization Theorem 12.17. Since by definition every manifold is second countable it will be enough to prove that manifolds are regular topological spaces.

Let M be an n-dimensional manifold, let $A \subseteq M$ be a closed set, and let $x \in M$ be a point such that $x \notin A$. We need to show that there exists open sets $W, W' \subseteq M$ such that $A \subseteq W$, $x \in W'$ and $W \cap W' = \varnothing$. Assume first that x does not belong to the boundary of M. We can find an open neighborhood U of x and homeomorphism $\varphi\colon U \to \mathbb{R}^n$ such that $\varphi(x) = 0$. Since A is closed in M the set $A \cap U$ is closed in U, and so $\varphi(A \cap U)$ is closed in $\mathbb{R}^n$. Therefore the set $\mathbb{R}^n \smallsetminus \varphi(A \cap U)$ is open in $\mathbb{R}^n$. Since $0 = \varphi(x) \in \mathbb{R}^n \smallsetminus \varphi(A \cap U)$ we can find an open ball $B(0, \varepsilon)$ such that $B(0, \varepsilon) \subseteq \mathbb{R}^n \smallsetminus \varphi(A \cap U)$:



Take $W = M \smallsetminus \varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2}))$ and $W' = \varphi^{-1}(B(0, \frac{\varepsilon}{2}))$. Notice that $x \in W'$. Also, since W' is open in

U and U is open in M we obtain that W' is open in M. Next, by Lemma 13.21 the set $\varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2}))$ is closed in M, so W is open in M. Moreover, since $W' \subseteq \varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2}))$ we obtain that $W \cap W' = \varnothing$. It remains to show that $A \subseteq W$, or equivalently that $A \cap \varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2})) = \varnothing$. If $y \in A$ and $y \notin U$ then $y \notin \varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2}))$ since $\varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2})) \subseteq U$. Also, if $y \in A \cap U$ then $y \notin \varphi^{-1}(\bar{B}(0, \frac{\varepsilon}{2}))$ by the choice of ε, and so we are done. In case when $x \in \partial M$ we can use a similar argument.

□

Exercises to Chapter 13

E13.1 Exercise. Prove Lemma 13.3.

E13.2 Exercise. Let M be an n-dimensional manifold, let $x_0 \in M$ and let $W \subseteq M$ be an open set such that $x_0 \in W$. Show that there exists a coordinate neighborhood $U \subseteq M$ such that $x_0 \in U$ and $U \subseteq W$.

E13.3 Exercise. The goal of this exercise is to prove the Invariance of Dimension Theorem 13.10 in small dimensions.

a) Let M be a manifold of dimension 0. Show that M is not locally homeomorphic to $\mathbb{R}^n$ for any $n \neq 0$.

b) Let M be a manifold of dimension 1. Show that M is not locally homeomorphic to $\mathbb{R}^n$ for any $n \neq 1$.

E13.4 Exercise. Prove Theorem 13.13 in the case when M is a 1-dimensional manifold with boundary.

E13.5 Exercise. Let M be an m-dimensional manifold with boundary and N an n-dimensional manifold with boundary Show that $M \times N$ is an $(m+n)$-dimensional manifold with boundary and $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$

E13.6 Exercise. Prove Proposition 13.19.