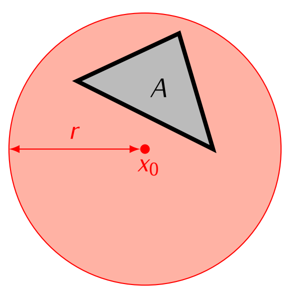
15 | Heine-Borel Theorem

We have seen already that a closed interval $[a, b] \subseteq \mathbb{R}$ is a compact space (14.12). Our next goal is to prove Heine-Borel Theorem 15.3 which gives a simple description of compact subspaces of $\mathbb{R}^n$.

15.1 Definition. Let (X, ϱ) be a metric space. A set $A \subseteq X$ is *bounded* if there exists an open ball $B(x_0, r) \subseteq X$ such that $A \subseteq B(x_0, r)$.



15.2 Proposition. *Let (X, ϱ) be a metric space and let $A \subseteq X$. The following conditions are equivalent:*

1) *A is bounded.*
2) *For each $x \in X$ there exists $r_x > 0$ such that $A \subseteq B(x, r_x)$.*
3) *There exists $R > 0$ such that $\varrho(x_1, x_2) < R$ for all $x_1, x_2 \in A$.*

Proof. Exercise. □

15.3 Heine-Borel Theorem. *A set $A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded.*

15.4 Note. The statement of Heine-Borel Theorem is not true if we replace $\mathbb{R}^n$ by an arbitrary metric space. Take e.g. $X = (0, 1)$ with the usual metric $d(x, y) = |x - y|$. Let $A = X$. The set A is closed in

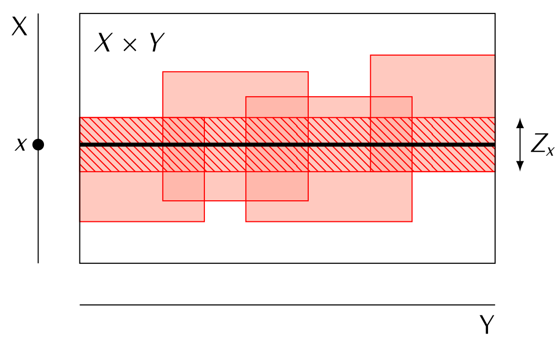
X. Also, A is bounded since $d(x, y) < 1$ for all $x, y \in A$. However A is not compact.

The proof of Heine-Borel Theorem will make use of the following fact:

15.5 Theorem. *If X, Y are compact spaces then the space $X \times Y$ is also compact.*

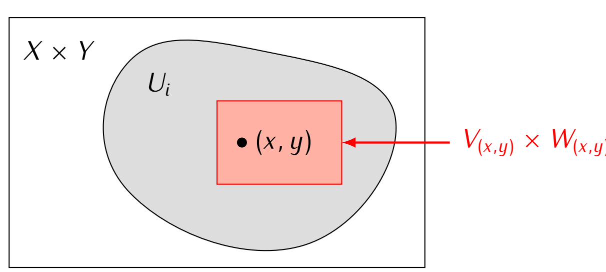
Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $X \times Y$. Assume first that each set U_i is of the form $U_i = V_i \times W_i$ with V_i open in X, and W_i is open in Y. We will show that $\mathcal{U}$ has a finite subcover,

Step 1. We will show first that for every point $x \in X$ there is an open set $Z_x \subseteq X$ such that $Z_x \times Y$ can be covered by a finite number of elements of $\mathcal{U}$. Consider the subspace $\{x\} \times Y \subseteq X \times Y$. Since $\{x\} \times Y \cong Y$ is compact there is a finite number of sets $V_{i_1} \times W_{i_1}, \dots, V_{i_n} \times W_{i_n} \in \mathcal{U}$ such that $\{x\} \times Y \subseteq \bigcup_{j=1}^{n} V_{i_j} \times W_{i_j}$. We can take $Z_x = \bigcap_{j=1}^{n} V_{i_j}$.



Step 2. The family $\{Z_x\}_{x \in X}$ is a on open cover of X. Since X is compact we have $X = \bigcup_{k=1}^{m} Z_{x_k}$ for some $x_1, \dots, x_m \in X$. It follows that $X \times Y = \bigcup_{k=1}^{m} (Z_{x_k} \times Y)$. Since each set $Z_{x_k} \times Y$ is covered by a finite number of elements of $\mathcal{U}$ it follows that $X \times Y$ is also covered by a finite number of elements of $\mathcal{U}$.

Assume now that $\mathcal{U} = \{U_i\}_{i \in I}$ is an arbitrary open cover of $X \times Y$. For every point $(x, y) \in X \times Y$ let $V_{(x,y)} \times W_{(x,y)}$ be a set such that $V_{(x,y)}$ is open in X, $W_{(x,y)}$ is open in Y, $(x, y) \in V_{(x,y)} \times W_{(x,y)}$ and $V_{(x,y)} \times W_{(x,y)} \subseteq U_i$ for some $i \in I$:



The family $\{V_{(x,y)} \times W_{(x,y)}\}_{(x,y) \in X \times Y}$ is an open cover of $X \times Y$. By the argument above we can find points $(x_1, y_1), \ldots, (x_n, y_n) \in X \times Y$ such that $X \times Y = \bigcup_{j=1}^{n} V_{(x_j,y_j)} \times W_{(x_j,y_j)}$. For $j = 1, \ldots, n$ let $U_{i_j} \in \mathcal{U}$ be a set such that $V_{(x_j,y_j)} \times W_{(x_j,y_j)} \subseteq U_{i_j}$. We have

$$X \times Y = \bigcup_{j=1}^{n} V_{(x_j,y_j)} \times W_{(x_j,y_j)} \subseteq \bigcup_{j=1}^{n} U_{i_j}$$

which means that $\{U_{j_1}, \ldots, U_{j_n}\}$ is a finite subcover of $\mathcal{U}$. □

15.6 Corollary. *If $X_1, \ldots, X_n$ are compact spaces spaces then the space $X_1 \times \cdots \times X_n$ is compact.*

Proof. Follows from Theorem 15.5 by induction with respect to n. □

15.7 Corollary. *For $i = 1, \ldots, n$ let $[a_i, b_i] \subseteq \mathbb{R}$ be a closed interval. The closed box*

$$[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$$

is compact.

Proof. This follows from Proposition 14.12 and Corollary 15.6. □

Proof of Theorem 15.3. (⇒) Exercise.

(⇐) If $A \subseteq \mathbb{R}^n$ is a closed and bounded set then $A \subseteq B(0, r)$ for some $r > 0$. Notice that $B(0, r) \subseteq J^n$ where $J = [-r, r] \subseteq \mathbb{R}$. As a consequence A is a closed subspace of J^n. By Corollary 15.7 the space J^n is a compact. Since closed subspaces of compact spaces are compact (Proposition 14.13) we obtain that A is compact.

□

Exercises to Chapter 15

E15.1 Exercise. Prove the implication (⇒) of Theorem 15.3.

E15.2 Exercise. Let X, Y be topological spaces. Show that the converse of Theorem 15.5 holds. That is, show that if $X \times Y$ is a compact space then X and Y are compact spaces.

E15.3 Exercise. Let $f \colon X \times [0, 1] \to Y$ be a continuous function, and let $U \subseteq Y$ be an open set. Show that the set

$$V = \{x \in X \mid f(\{x\} \times [0, 1]) \subseteq U\}$$

is open in X.

E15.4 Exercise. Let A, B be compact subspaces of $\mathbb{R}^n$. Show that the set

$$A + B = \{x + y \in \mathbb{R}^n \mid x \in A,\ y \in B\}$$

is also compact.

E15.5 Exercise. In Chapter 13 while proving that topological manifolds are metrizable we omitted the proof of Lemma 13.21. We are now in position to fill this gap. Prove Lemma 13.21.