21 | Embeddings of Manifolds

We have seen so far several examples of manifolds. Some of them (e.g. S^n) are defined as subspaces of a Euclidean space $\mathbb{R}^m$ for some m, but some other (e.g. the Klein bottle (19.20), or the projective spaces (19.22)) are defined more abstractly. A natural question is if every manifold is homeomorphic to a subspace of some Euclidean space $\mathbb{R}^m$, or equivalently if it can be embedded into $\mathbb{R}^m$. Our next goal is to show that this is in fact true, at least in the case of compact manifolds.

We begin with some technical preparation.

21.1 Definition. Let X be a topological space and let $f\colon X \to \mathbb{R}$ be a continuous function. The *support* of f is the closure of the subset of X consisting of points with non-zero values:

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$$

21.2 Definition. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. A *partition of unity subordinate to* $\mathcal{U}$ is a family of continuous functions $\{\lambda_i\colon X \to [0,1]\}_{i \in I}$ such that

(i) $\operatorname{supp}(\lambda_i) \subseteq U_i$ for each $i \in I$;
(ii) each point $x \in X$ has an open neighborhood U_x such that $U_x \cap \operatorname{supp}(\lambda_i) \neq \varnothing$ for finitely many $i \in I$ only;
(iii) for each $x \in X$ we have $\sum_{i \in I} \lambda_i(x) = 1$.

21.3 Note. Condition (iii) makes sense since by (ii) we have $\lambda_i(x) \neq 0$ for finitely many $i \in I$ only.

Partitions of unity are a very useful tool for gluing together functions defined on subsets of X to obtain a function defined on the whole space X:

21.4 Lemma. *Let X be a topological space, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and let $\{\lambda_i\}_{i \in I}$ be* *a partition of unity subordinate to $\mathcal{U}$.*

1) *Let $i \in I$ and let $f_i \colon U_i \to \mathbb{R}^n$ be a continuous function. Then the function $\tilde{f}_i \colon X \to \mathbb{R}^n$ given by*

$$\tilde{f}_i(x) = \begin{cases} \lambda_i(x) f_i(x) & \text{for } x \in U_i \\ 0 & \text{for } x \in X \smallsetminus U_i \end{cases}$$

   is continuous.

2) *Assume that for each $i \in I$ we have a continuous function $f_i \colon U_i \to \mathbb{R}^n$, and let $\tilde{f}_i \colon X \to \mathbb{R}^n$ be the function defined as above. Then the function $\tilde{f} \colon X \to \mathbb{R}^n$ given by*

$$\tilde{f}(x) = \sum_{i \in I} \tilde{f}_i(x)$$

   is continuous.

Proof. Exercise. □

21.5 Proposition. *Let X be a normal space. For any finite open cover $\{U_1, \dots, U_n\}$ of X there exists a partition of unity subordinate to this cover.*

The proof of Proposition 21.5 will use the following fact:

21.6 Finite Shrinking Lemma. *Let X be a normal space and let $\{U_1, \dots, U_n\}$ be a finite open cover of X. There exists an open cover $\{V_1, \dots, V_n\}$ of X such that $\overline{V}_i \subseteq U_i$ for each $i \geq 1$.*

Proof. We will argue by induction. Assume that for some $k < n$ we already have open sets $V_1, \dots, V_k$ such that $\overline{V}_i \subseteq U_i$ for all $1 \leq i \leq k$ and that $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ is a cover of X (at the start of induction we set $k = 0$). We will show that there exists an open set V_{k+1} such that $\overline{V}_{k+1} \subseteq U_{k+1}$ and that $\{V_1, \dots, V_{k+1}, U_{k+2} \dots, U_n\}$ still covers X. Take the set

$$W = V_1 \cup \dots \cup V_k \cup U_{k+2} \cup \dots \cup U_n$$

Notice that $W \cup U_{k+1} = X$. Therefore $X \smallsetminus W \subseteq U_{k+1}$. Since $X \smallsetminus W$ is a closed set by Lemma 10.3 there exists an open set V such that $X \smallsetminus W \subseteq V$ and $\overline{V} \subseteq U_{k+1}$. The first of these properties gives $W \cup V = X$, which means that $\{V_1, \dots, V_k, V, U_{k+2} \dots, U_n\}$ is an open cover of X. Therefore we can take $V_{k+1} = V$. □

Lemma 21.6 can be generalized to infinite covers of normal spaces as follows:

21.7 Shrinking Lemma. *Let X be a normal space and let $\{U_i\}_{i \in I}$ be a open cover of X such that each point of X belongs to finitely many sets U_i only. There exists an open cover $\{V_i\}_{i \in I}$ of X such that $\overline{V}_i \subseteq U_i$ for all $i \in I$.*

Proof. Exercise. ☐

Proof of Proposition 21.5. By Lemma 21.6 there exists an open cover $\{V_1, \dots, V_n\}$ of X such that $\overline{V}_i \subseteq U_i$ for all $i \geq 1$. Since X is a normal space by Lemma 10.3 for each $i \geq 1$ we can find an open set W_i such that $\overline{V}_i \subseteq W_i$ and $\overline{W}_i \subseteq U_i$. Using Urysohn Lemma 10.1 we get continuous functions $\mu_i \colon X \to [0, 1]$ such that $\mu_i(\overline{V}_i) \subseteq \{1\}$ and $\mu_i(X \smallsetminus W_i) \subseteq \{0\}$. Notice that $\mathrm{supp}(\mu_i) \subseteq \overline{W}_i \subseteq U_i$. Let $\mu = \sum_{i=1}^{n} \mu_i$. We claim that $\mu(x) > 0$ for all $x \in X$. Indeed, if $x \in X$ then $x \in V_j$ for some $j \geq 1$ and so $\mu_j(x) = 1$. For $i = 1, \dots, n$ let $\lambda_i \colon X \to [0, 1]$ be the function given by

$$\lambda_i(x) = \frac{\mu_i(x)}{\mu(x)}$$

The family $\{\lambda_1, \dots, \lambda_n\}$ is a partition of unity subordinate to the cover $\{U_1, \dots, U_n\}$ (exercise).

☐

21.8 Corollary. *If X is a compact Hausdorff space then for every open cover $\mathcal{U}$ of X there exists an partition of unity subordinate to $\mathcal{U}$.*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$. Since X is compact we can find a finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$ of $\mathcal{U}$. By Theorem 14.19 the space X is normal, so using Proposition 21.5 we obtain a partition of unity $\{\lambda_{i_1}, \dots, \lambda_{i_n}\}$ subordinate to the cover $\{U_{i_1}, \dots, U_{i_n}\}$. For $i \in I \smallsetminus \{i_1, \dots, i_n\}$ let $\lambda_i \colon X \to [0, 1]$ be the constant zero function. The family of functions $\{\lambda_i\}_{i \in I}$ is a partition of unity subordinate to the cover $\mathcal{U}$. ☐

We are now ready to prove the embedding theorem for compact manifolds. We will consider first the case of manifolds without boundary:

21.9 Theorem. *If M is a compact manifold without boundary then for some $N \geq 0$ there exists an embedding $j \colon M \to \mathbb{R}^N$.*

21.10 Note. A compact manifold without boundary is called a *closed manifold*.

Proof of Theorem 21.9. Assume that M is an n-dimensional manifold. Since M is compact we can find a finite collection of coordinate charts $\{\varphi_i \colon U_i \to \mathbb{R}^n\}_{i=1}^m$ on M such that $\{U_i\}_{i=1}^m$ is an open cover of M. By Corollary 21.8 there exists a partition of unity $\{\lambda_i\}_{i=1}^m$ subordinate to this cover. For $i = 1, \dots, m$ let $\tilde{\varphi}_i \colon M \to \mathbb{R}^n$ be the function obtained from φ_i as in part 1) of Lemma 21.4. Consider the continuous function $j \colon M \to \mathbb{R}^{mn+m}$ defined as follows:

$$j(x) = (\tilde{\varphi}_1(x),\ \dots,\ \tilde{\varphi}_m(x),\ \lambda_1(x),\ \dots,\ \lambda_m(x))$$

We will show that j is a 1-1 function. Since M is a compact and $\mathbb{R}^{mn+m}$ is a Hausdorff space by Proposition 14.18 this will imply that j is a homeomorphism onto $j(M) \subseteq \mathbb{R}^{mn+m}$, and so it is an embedding. Assume then that $x, y \in M$ are points such that $j(x) = j(y)$. This means that $\tilde{\varphi}_i(x) = \tilde{\varphi}_i(y)$ and $\lambda_i(x) = \lambda_i(y)$ for all $i = 1, \dots, m$. Since $\sum_{i=1}^{m} \lambda_i(x) = 1$ there exists $1 \leq i_0 \leq m$ such that $\lambda_{i_0}(x) \neq 0$, and so also $\lambda_{i_0}(y) \neq 0$. Since $\operatorname{supp}(\lambda_{i_0}) \subseteq U_{i_0}$ we obtain that $x, y \in U_{i_0}$. By definition of $\tilde{\varphi}_{i_0}$ we have $\tilde{\varphi}_{i_0}(z) = \lambda_{i_0}(z)\varphi_{i_0}(z)$ for all $z \in U_{i_0}$. Therefore we get

$$\lambda_{i_0}(x)\varphi_{i_0}(x) = \tilde{\varphi}_{i_0}(x) = \tilde{\varphi}_{i_0}(y) = \lambda_{i_0}(y)\varphi_{i_0}(y)$$

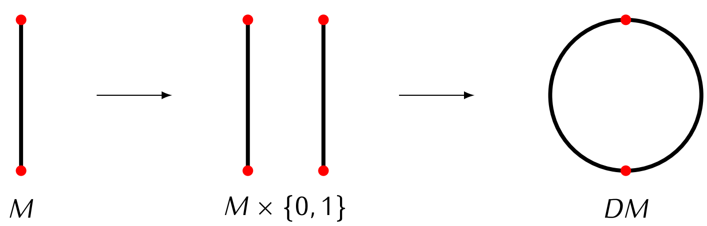
Dividing both sides by $\lambda_{i_0}(x) = \lambda_{i_0}(y)$ we obtain $\varphi_{i_0}(x) = \varphi_{i_0}(y)$. However, $\varphi_{i_0} \colon U_{i_0} \to \mathbb{R}^n$ is a homeomorphism, so in particular it is a 1-1 function. This shows that $x = y$. □

It is straightforward to generalize the proof of Theorem 21.9 to the case when M is a compact manifold with boundary. We will use however a slightly different argument to show that such manifolds can be embedded into Euclidean spaces.

21.11 Definition. Let M be a manifold with boundary ∂M. The *double* of M is the topological space

$$DM = M \times \{0, 1\}/\sim$$

where $\{0, 1\}$ is the discrete space with two points and $\sim$ is the equivalence relation on $M \times \{0, 1\}$ given by $(x, 0) \sim (x, 1)$ for all $x \in \partial M$.



21.12 Proposition. *If M is an n-dimensional manifold with boundary then DM is an n-dimensional manifold without boundary. Moreover, if M is compact then so is DM.*

Proof. Exercise. □

21.13 Corollary. *If M is a compact manifold with boundary then for some $N > 0$ there exists an embedding $M \to \mathbb{R}^N$.*

Proof. Take the double DM of M. By Proposition 21.12 DM is a closed manifold, so using Theorem 21.9 we obtain an embedding $j \colon DM \to \mathbb{R}^N$ for some $N \geq 0$. Notice that we also have an embedding $\pi i \colon M \to DM$ where $i \colon M \to M \times \{0, 1\}$ is the function given by $i(x) = (x, 0)$ and $\pi \colon M \times \{0, 1\} \to DM$ is the quotient map. Therefore we obtain an embedding

$$j\pi i \colon M \to \mathbb{R}^N$$

□

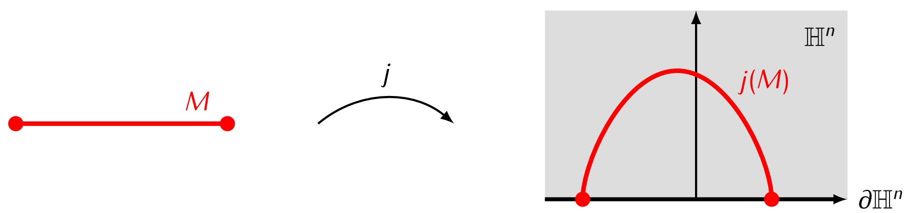
21.14 Note. Theorem 21.9 and Corollary 21.13 can be extended to non-compact manifolds: one can show that any manifold (compact or not, with or without boundary) can be embedded into the Euclidean space $\mathbb{R}^N$ for some $N \geq 0$. Moreover, it turns out that any n-dimensional manifold can be embedded into $\mathbb{R}^{2n+1}$. An interesting question is, given some specific manifold M (e.g. $M = \mathbb{RP}^n$) what is the smallest number N such that M can be embedded into $\mathbb{R}^N$.

Exercises to Chapter 21

E21.1 Exercise. Prove Lemma 21.4.

E21.2 Exercise. Prove Proposition 21.12.

E21.3 Exercise. Recall that $\mathbb{H}^n$ is the subspace of $\mathbb{R}^n$ given by $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and that $\partial\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{H}^n \mid x_n = 0\}$. Let M be a compact manifold with boundary ∂M. Show that for some $N \geq 0$ there exists an embedding $j\colon M \to \mathbb{H}^N$ such that $j(\partial M) \subseteq \partial\mathbb{H}^N$ and $j(M \smallsetminus \partial M) \subseteq \mathbb{H}^N \smallsetminus \partial\mathbb{H}^N$.



E21.4 Exercise. The goal of this exercise is to prove the general Shrinking Lemma 21.7. Let X be a normal space and let $\{U_i\}_{i\in I}$ be an open cover of X such that every point of X belongs to finitely many sets U_i only.

a) Let S be the set consisting of all pairs $(J, \{V_j\}_{j\in J})$ where J is a subset of I and $\{V_j\}_{j\in J}$ is a collection of open sets in X such that $\overline{V}_j \subseteq U_j$ for all $j \in J$, and $\{V_j\}_{j\in J} \cup \{U_i\}_{i\in I\smallsetminus J}$ is a cover of X. We define a partial order on S as follows. If $(J, \{V_j\}_{j\in J})$ and $(J', \{V'_j\}_{j\in J'})$ are elements of S then $(J, \{V_j\}_{j\in J}) \leq (J', \{V'_j\}_{j\in J'})$ if $J \subseteq J'$ and if $V_j = V'_j$ for all $j \in J$. Use Zorn's Lemma 17.15 to show that the set S has a maximal element.

b) Let S be the set defined above. Show that if $(J, \{V_j\}_{j\in J})$ is a maximal element of S then $J = I$. This gives that $\{V_j\}_{j\in J}$ is an open cover of X such that $\overline{V}_i \subseteq U_i$ for all $i \in I$.