7 | Connectedness

7.1 Let $[a, b] \subseteq \mathbb{R}$ be a closed interval and let $(a, b) \subseteq \mathbb{R}$ be an open interval. We would like to show that $[a, b]$ and (a, b) are non-homeomorphic topological spaces. The idea of a proof of this fact is as follows. Assume that there exists a homeomorphism

$$f \colon [a, b] \to (a, b)$$

Recall that by Proposition 6.11 for any $Y \subseteq [a, b]$ the function $f|_Y \colon Y \to f(Y)$ also would be a homeomorphism. If we take $Y = [a, b] \smallsetminus \{a\} = (a, b]$ then

$$f(Y) = f([a, b]) \smallsetminus \{f(a)\} = (a, b) \smallsetminus \{f(a)\}$$

Intuitively the spaces Y and $f(Y)$ are different in an essential way since Y comes in one piece while $f(Y)$ is split into two pieces by removal of the point $f(a)$:

$Y = [a, b] \smallsetminus \{a\}$ $\qquad$ $f(Y) = (a, b) \smallsetminus \{f(a)\}$

For this reason we can expect that the spaces are Y and $f(Y)$ are not homeomorphic, and that, as a consequence, $[a, b]$ and (a, b) are not homeomorphic as well.

In order to make this intuitive argument into a rigorous proof we need to define precisely what it means that a topological space is "in one piece" and then show that this feature is preserved by homeomorphisms. The property of being "in one piece" is captured by the definition of a connected space:

7.2 Definition. A topological space X is *connected* if for any two open sets $U, V \subseteq X$ such that $U \cup V = X$ and $U, V \neq \varnothing$ we have $U \cap V \neq \varnothing$.

7.3 Definition. If X is a topological space and $U, V \subseteq X$ are non-empty open sets such that $U \cap V = \varnothing$ and $U \cup V = X$ then we say that $\{U, V\}$ is a *separation* of X.

Thus, a space X is connected if there does not exists a separation of X.

7.4 Example. For $a < b$ take $(a, b) \subseteq \mathbb{R}$ and let $c \in (a, b)$. The space $X = (a, b) \smallsetminus \{c\}$ is not connected. Indeed, the sets $U = (a, c)$ and $V = (c, b)$ form a separation of X.

7.5 Proposition. *Let $a < b$. The intervals (a, b), $[a, b]$, $(a, b]$, and $[a, b)$ are connected topological spaces.*

Proof. Assume first that $[a, b]$ is a closed interval and that that $U, V \subseteq [a, b]$ are open sets such $a \in U$, $b \in V$, and $U \cup V = [a, b]$. We will show that $U \cap V \neq \varnothing$. Let $x_0 = \inf V$. There are two possibilities: either $x_0 \notin U$ or $x_0 \in U$. In the first case $x_0 \in V$ and $x_0 > a$. Since V is an open set there exists $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq V$. This implies that there is $x \in V$ such that $x < x_0$ which is impossible by the definition of x_0.

Thus the only possible option is $x_0 \in U$. Since U is an open set there exists $\varepsilon' > 0$ such that $[x_0, x_0 + \varepsilon') \subseteq U$. On the other hand, by the definition of x_0 we have $[x_0, x_0 + \varepsilon') \cap V \neq \varnothing$. Therefore $U \cap V \neq \varnothing$.

Assume now that I is an interval (either closed, open, or half-open) and that $U, V \subseteq I$ are non-empty open sets such that $U \cup V = I$. We will show that $U \cap V \neq \varnothing$. Let $c, d \in I$ be points such that $c \in U$ and $d \in V$. We can assume that $c < d$. Take $U' = U \cap [c, d]$ and $V' = V \cap [c, d]$. The sets U', V' are open in $[c, d]$, $c \in U'$, $d \in V'$, and $U' \cup V' = [c, d]$. By the observation above we have $U' \cap V' \neq \varnothing$, and so $U \cap V \neq \varnothing$.

□

One can show that intervals are in fact the only subspaces of $\mathbb{R}$ that are connected:

7.6 Proposition. *If X is a connected subspace of $\mathbb{R}$ then X is an interval (either open, closed, or half-closed, finite or infinite).*

Proof. Exercise. □

7.7 Going back to the argument outlined in 7.1, by Proposition 7.5 we get that the space $Y = (a, b]$ is connected, and the space $f(Y) = (a, b) \smallsetminus f(a)$ is not connected by Example 7.4. We still need to show however that a connected space cannot be homeomorphic to one that is not connected. In fact a stronger statement is true:

7.8 Proposition. *Let $f: X \to Y$ be a continuous function. If f is onto and the space X is connected then Y is also connected.*

Proof. Assume that Y is not connected and let $U, V \subseteq Y$ be a separation of X. Then the sets $f^{-1}(U), f^{-1}(V)$ form a separation of X which contradicts the assumption that X is connected. □

7.9 Corollary. *If $f: X \to Y$ is a continuous function and X is a connected space then $f(X)$ is connected.*

Proof. By restricting the range of f we obtain a function $f: X \to f(X)$ which is continuous and onto, and so it we can apply Proposition 7.8. □

A very useful consequence of Corollary 7.9 is the following fact:

7.10 Intermediate Value Theorem. *Let X be a connected topological space and let $f: X \to \mathbb{R}$ be a continuous function. If $a < b$ are points in $\mathbb{R}$ such that $a = f(x)$ and $b = f(y)$ for some $x, y \in X$ then for each $c \in [a, b]$ there exists $z \in X$ such that $c = f(z)$.*

Proof. By Corollary 7.9 the set $f(X)$ is connected, and so by Proposition 7.6 $f(X)$ is an interval. It follows that for any $a, b \in f(X)$ we have $[a, b] \subseteq f(X)$. □

Since every homeomorphism $f: X \to Y$ is onto directly from Corollary 7.9 we get:

7.11 Corollary. *If $X \cong Y$ and X is a connected space then Y is also connected.*

7.12 Corollary. *The space $\mathbb{R}$ is connected.*

Proof. This follows from Corollary 7.11 and Proposition 7.5 since $\mathbb{R} \cong (a, b)$ for any $a < b$. □

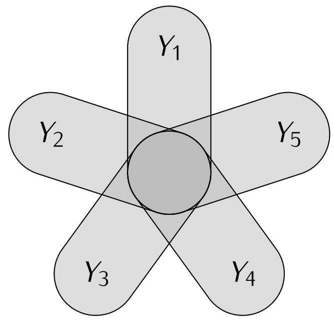
7.13 Note. A *topological invariant* is a property of topological spaces such that if a space X has this property and $X \cong Y$ then Y also has this property. By Corollary 7.11 connectedness is a topological invariant.

7.14 Proposition. *Let X be a topological space. The following conditions are equivalent :*

1) *X is connected*
2) *For any closed sets $A, B \subseteq X$ such that $A, B \neq X$ and $A \cap B = \varnothing$ we have $A \cup B \neq X$.*
3) *If $A \subseteq X$ is a set that is both open and closed then either $A = X$ or $A = \varnothing$.*
4) *If $D = \{0, 1\}$ is a space with the discrete topology then any continuous function $f: X \to D$ is a constant function.*

Proof. Exercise. □

7.15 Proposition. *Let X be a topological space and for $i \in I$ let Y_i be a subspace of X. Assume that $\bigcup_{i\in I} Y_i = X$ and $\bigcap_{i\in I} Y_i \neq \varnothing$. If Y_i is connected for each $i \in I$ then X is also connected.*



Proof. Let $D = \{0, 1\}$ be a space with the discrete topology and let $f\colon X \to D$ be a continuous function. By Proposition 7.14 it is enough to show that f is a constant function. Let $x_0 \in \bigcap_{i\in I} Y_i$. We can assume that $f(x_0) = 0$. For any $i \in I$ the function $f|_{Y_i}\colon Y_i \to D$ is constant since Y_i is connected. Since $x_0 \in Y_i$ and $f(x_0) = 0$ we get that $f(x) = 0$ for all $x \in Y_i$. Since this applies to all subspaces Y_i we obtain that $f(x) = 0$ for all $x \in \bigcup_{i\in I} Y_i = X$. □

7.16 Corollary. *The space $\mathbb{R}^n$ is connected for all $n \geq 1$.*

Proof. For $0 \neq x \in \mathbb{R}^n$ let $L_x \subseteq \mathbb{R}^n$ be the line passing through x and the origin:

$$L_x = \{tx \in \mathbb{R}^n \mid t \in \mathbb{R}\}$$

For every $x \in \mathbb{R}^n$ consider the continuous function $f_x\colon \mathbb{R} \to \mathbb{R}^n$ given by $f_x(t) = tx$. Since $\mathbb{R}$ is connected and $f_x(\mathbb{R}) = L_x$ if follows that L_x is connected. We have $\mathbb{R}^n = \bigcup_{x\in\mathbb{R}^n} L_x$ and $\bigcap_{x\in\mathbb{R}^n} L_x = \{0\}$. Therefore by Proposition 7.15 the space $\mathbb{R}^n$ is connected. □

7.17 Definition. Let X be a topological space. A *connected component* of X is a subspace $Y \subseteq X$ such that

1) Y is connected
2) if $Y \subseteq Z \subseteq X$ and Z is connected then $Y = Z$.

7.18 Proposition. *Let X be a topological space.*

1) For every point $x_0 \in X$ there exist a connected component $Y \subseteq X$ such that $x_0 \in Y$.
2) If Y, Y' are connected components of X then either $Y \cap Y' = \varnothing$ or $Y = Y'$.

Proof. 1) Given a point $x_0 \in X$ let $\{C_i\}_{i\in I}$ be the collection of all subspaces of X such that $x_0 \in C_i$ and C_i is connected. Define $Y := \bigcup_{i\in I} C_i$. We have $x_0 \in Y$. Also, since $x_0 \in \bigcap_{i\in I} C_i$ by Proposition

7.15 we obtain that Y is connected. If $Y \subseteq Z \subseteq X$ and Z is connected then $Z = C_{i_0}$ for some $i_0 \in I$, and so $Z = Y$. Therefore Y is a connected component of X.

2) Let Y, Y' be two connected components of X. Assume that $Y \cap Y' \neq \varnothing$. By Proposition 7.15 we get then that $Y \cup Y'$ is connected. Since $Y \subseteq Y \cup Y'$ we must have $Y = Y \cup Y'$. By the same argument we obtain that $Y' = Y \cup Y'$. Therefore $Y = Y'$

□

7.19 Corollary. *Let X be a topological space. If $Z \subseteq X$ is a connected subspace then there exists a connected component $Y \subseteq X$ such that $Z \subseteq Y$.*

Proof. Exercise. □

7.20 Corollary. *Let $f\colon X \to Y$ be a continuous function. If X is a connected space then there exists a connected component $Z \subseteq Y$ such that $f(X) \subseteq Z$.*

Proof. Exercise. □

Exercises to Chapter 7

E7.1 Exercise. Let X be a topological space and let $Y \subseteq X$ be a subspace. Show that if Y is a connected space and Y is dense in X then X is connected.

E7.2 Exercise. Prove Proposition 7.6.

E7.3 Exercise. Show that the sphere S^n is connected for all $n \geq 1$.

E7.4 Exercise. Let $a < b$. Show that the closed interval $[a, b] \subseteq \mathbb{R}$ is not homeomorphic to the half-closed interval $(a, b]$.

E7.5 Exercise. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *strictly increasing* is for all $x, y \in \mathbb{R}$ such that $x > y$ we have $f(x) > f(y)$, and is it *strictly decreasing* is for all $x, y \in \mathbb{R}$ such that $x > y$ we have $f(x) < f(y)$. Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous 1-1 function then f is either strictly increasing or strictly decreasing.

E7.6 Exercise. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) \cdot f(f(x)) = 1$ for all $x \in \mathbb{R}$ and that $f(10) = 9$. Find the value of $f(5)$. Justify your answer.

E7.7 Exercise. Let $f\colon S^n \to \mathbb{R}$ be a continuous function. Show that there exists a point $x \in S^n$ such that $f(x) = f(-x)$. Here if $x = (x_1, \ldots, x_n) \in S^n$ then $-x = (-x_1, \ldots, -x_n)$.

E7.8 Exercise. Let $a < b$. Show that there does not exist a continuous bijection $f\colon (a, b) \to [a, b]$. Remember that a continuous bijection need not be a homeomorphism since the inverse function may be not continuous (see 6.12).

E7.9 Exercise. Prove Proposition 7.14.

E7.10 Exercise. Let X be a topological space. Show that the following conditions are equivalent:

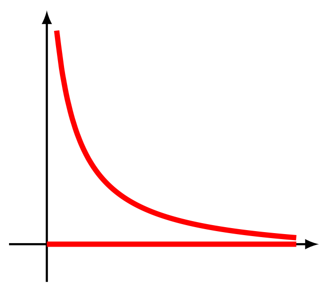
1) X is connected
2) if $A \subseteq X$ is any set such that $A \neq X$ and $A \neq \varnothing$ then $\mathrm{Bd}(A) \neq \varnothing$.

E7.11 Exercise. Let X be a topological space. Show that every connected component of X is closed in X.

E7.12 Exercise. Let (X, ϱ) be a metric space. Assume for some $x_0 \in X$ and $r > 0$ the open ball $B(x_0, r)$ consists of countably many points. Show that X is not connected.

E7.13 Exercise. Let X be the subspace of $\mathbb{R}^2$ consisting of the positive x-axis and of the graph of the function $f(x) = \frac{1}{x}$ for $x > 0$:

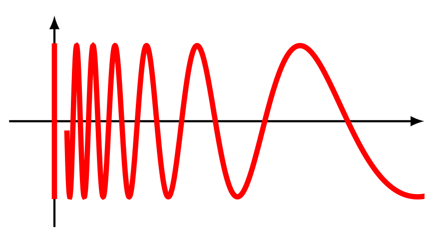
$$X := \{(x, 0) \in \mathbb{R}^2 \mid x \geq 0\} \cup \{(x, \tfrac{1}{x}) \in \mathbb{R}^2 \mid x > 0\}$$



Show that X is not connected.

E7.14 Exercise. The *topologist's sine curve* is the subspace Y of $\mathbb{R}^2$ that consists of a segment of the y-axis and of the graph of the function $f(x) = \sin(\frac{1}{x})$:

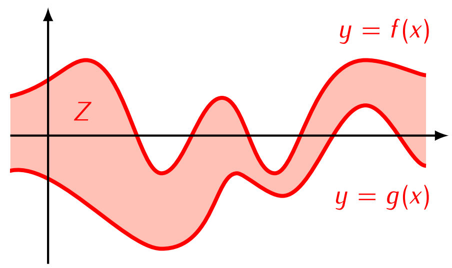
$$Y := \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\} \cup \{(x, \sin(\tfrac{1}{x})) \in \mathbb{R}^2 \mid x > 0\}$$



Show that Y is connected.

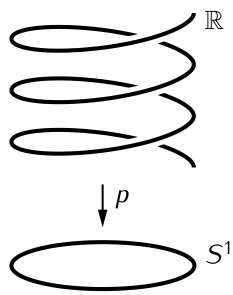
E7.15 Exercise. Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be continuous functions such that $g(x) < f(x)$ for all $x \in \mathbb{R}$. Let Z be the subspace of $\mathbb{R}^2$ given by

$$Z = \{(x, y) \mid g(x) \leq y \leq f(x)\}$$



Show that Z is connected.

E7.16 Exercise. Consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $p\colon \mathbb{R} \to S^1$ denote the function given by $p(t) = (\sin 2\pi t, \cos 2\pi t)$. Geometrically speaking this function wraps $\mathbb{R}$ infinitely many times around the circle:



Show that there does not exist a continuous function $g\colon S^1 \to \mathbb{R}$ such that $pg = \mathrm{id}_{S^1}$.

E7.17 Exercise. A space X is *totally disconnected* if every connected component of X consists of a single point. Obviously every discrete topological space is totally disconnected. Consider the set ot rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Show that $\mathbb{Q}$ is totally disconnected. Note that by Exercise 6.1 $\mathbb{Q}$ is not a discrete space.

E7.18 Exercise. Show that metrizability is a topological invariant. That is, it X and Y are homeomorphic spaces, and X metrizable then so is Y.