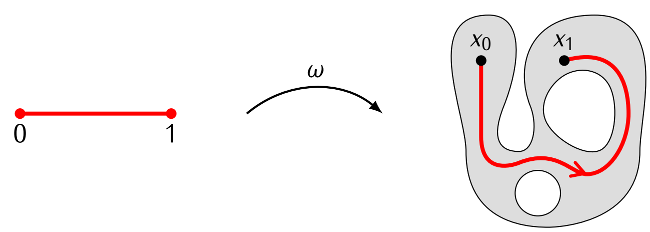
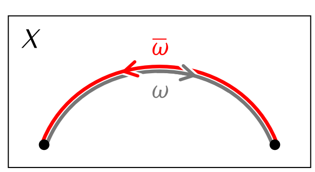
8 | Path Connectedness

The notion of connectedness of a space was invented to define rigorously what it means that a space is "in one piece". In this chapter we introduce path connectedness which is designed to capture a similar property but in a different way. It turns out that these two notions are not the same: while every path connected space is connected, the opposite is not true. In effect path connectedness gives us a new topological invariant of spaces. Additional related invariants are obtained by considering local connectedness and local path connectedness of spaces.

8.1 Definition. Let X be a topological space. A *path* in X is a continuous function $\omega\colon [0,1] \to X$. If $\omega(0) = x_0$ and $\omega(1) = x_1$ then we say that ω joins x_0 with x_1.

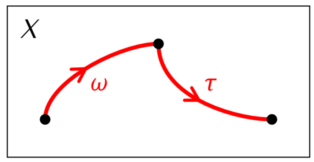


8.2 Definition. 1) If $\omega\colon [0,1] \to X$ is a path in X then the *inverse* of ω is the path $\overline{\omega}$ given by $\overline{\omega}(t) = \omega(1-t)$ for $t \in [0,1]$.



2) If $\omega.\tau\colon [0,1] \to X$ are paths such that $\omega(1) = \tau(0)$ then the *concatenation* of ω and τ if the path $\omega * \tau$ given by

$$(\omega * \tau)(t) = \begin{cases} \omega(2t) & \text{for } t \in [0, 1/2] \\ \tau(2t-1) & \text{for } t \in [1/2, 1] \end{cases}$$



8.3 Definition. A space X is *path connected* if for every $x_0, x_1 \in X$ there is a path joining x_0 with x_1.

8.4 Example. For any $n \geq 1$ the space $\mathbb{R}^n$ is path connected. Indeed, if $x_0, x_1 \in \mathbb{R}^n$ then define $\omega\colon [0,1] \to \mathbb{R}^n$ by

$$\omega(t) = (1-t)x_0 + tx_1$$

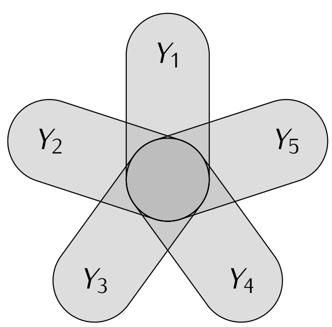
We have $\omega(0) = x_0$ and $\omega(1) = x_1$.

8.5 Proposition. *Every path connected space is connected.*

Proof. Exercise. □

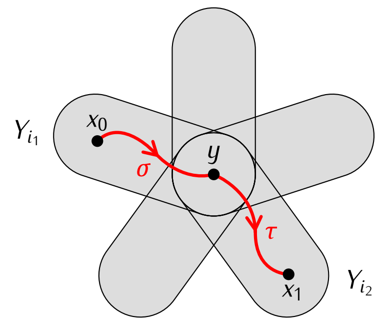
8.6 Note. It is not true that a connected space must be path connected. For example, let Y be the topologist's sine curve (7.14). This is a connected space. On the other hand Y is not path connected (exercise).

8.7 Proposition. *Let X be a topological space and for $i \in I$ let Y_i be a subspace of X. Assume that $\bigcup_{i \in I} Y_i = X$ and $\bigcap_{i \in I} Y_i \neq \varnothing$. If Y_i is path connected for each $i \in I$ then X is also path connected.*



Proof. Let $x_0, x_1 \in X$. We need to show that there exists a path $\omega\colon [0,1] \to X$ such that $\omega(0) = x_0$ and $\omega(1) = x_1$.

Since $X = \bigcup_{i \in I} Y_i$ we have $x_0 \in Y_{i_0}$ and $x_1 \in Y_{i_1}$ for some $i_0, i_1 \in I$. Let $y \in \bigcap_{i \in I} Y_i$. Since Y_{i_0} is path connected and $x_0, y \in Y_{i_0}$ there is a path $\sigma\colon [0,1] \to Y_{i_0}$ such that $\sigma(0) = x_0$ and $\sigma(1) = y$. Also, since Y_{i_1} is path connected and $x_1, y \in Y_{i_1}$ there is a path $\tau\colon [0,1] \to Y_{i_1}$ such that $\tau(0) = y$ and $\tau(1) = x_1$. The concatenation $\sigma * \tau$ gives a path joining x_0 with x_1.



□

8.8 Definition. Let X be a topological space. A *path connected component* of X is a subspace $Y \subseteq X$ such that

1) Y is path connected
2) if $Y \subseteq Z \subseteq X$ and Z is path connected then $Y = Z$.

8.9 Proposition. *Let X be a topological space.*

1) For every point $x_0 \in X$ there exist a path connected component $Y \subseteq X$ such that $x_0 \in Y$.
2) If Y, Y' are path connected components of X then either $Y \cap Y' = \varnothing$ or $Y = Y'$.

Proof. Similar to the proof of Proposition 7.18. □

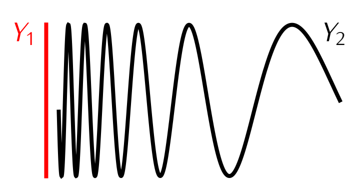
8.10 Proposition. *Let $x_0 \in X$ The path connected component $Y \subseteq X$ that contains x_0 is given by:*

$$Y = \{x \in X \mid \text{there exists a path joining } x \text{ with } x_0\}$$

Proof. Exercise. □

8.11 Example. Let Y be the topologist's sine curve. The space Y has only one connected component (since Y is connected). On the other hand it has two path connected components:

$$Y_1 = \{(0, y) \mid -1 \leq y \leq 1\} \quad \text{and} \quad Y_2 = \{(x, \sin(\tfrac{1}{x})) \mid x > 0\}$$



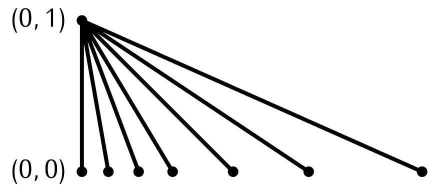
8.12 Definition. Let X be a topological space.

1) X is *locally connected* if for any $x \in X$ and any open neighborhood U of x there is an open neighborhood V of x such that $V \subseteq U$ and V is connected.

2) X is *locally path connected* if for any $x \in X$ and any open neighborhood U of x there is an open neighborhood V of x such that $V \subseteq U$ and V is path connected.

8.13 Example. Let $X = (0,1) \cup (2,3) \subseteq \mathbb{R}$. The space X is neither connected nor path connected but it is both locally connected and locally path connected.

8.14 Example. Let X be the subspace of $\mathbb{R}^2$ consisting of the intervals joining points $(0,0)$ and $(1/n, 0)$ for $n = 1, 2, \dots$ with the point $(0,1)$:



The space X is called the *harmonic broom*. This space is connected and path connected. It is neither locally connected nor locally path connected since any neighborhood of the point $(0,0)$ that does not contain the point $(0,1)$ is not connected.

8.15 Proposition. *If X is locally path connected then it is locally connected.*

Proof. Exercise. $\square$

8.16 Proposition. *If a space X is locally connected then connected components of X are open in X.*

Proof. Exercise. $\square$

8.17 Proposition. *If a space X is locally path connected then path connected components of X are open in X.*

Proof. Exercise. $\square$

8.18 Proposition. *If X is a connected and locally path connected space then X is path connected.*

Proof. It is enough to show that X has only one path connected component. Assume, by contradiction, that X has at least two distinct path connected components. Let Y be some path connected component of X and let Y' be the union of all other path connected components. By Proposition 8.17 both Y and Y' are open sets. Also $Y \cap Y' = \varnothing$ and $Y \cup Y' = X$. This contradicts the assumption that X is connected. $\square$

Exercises to Chapter 8

E8.1 Exercise. Prove Proposition 8.5.

E8.2 Exercise. Prove Proposition 8.10.

E8.3 Exercise. The goal of this exercise is to verify that the statement of Note 8.6 holds. Show that the topologist sine curve (Exercise 7.14) is not path connected.

E8.4 Exercise. Let X be a topological space whose elements are integers, and such that $U \subseteq X$ is open if either $U = \varnothing$ or $U = X \smallsetminus S$ for some finite set S. Show that X is locally connected but not locally path connected.

E8.5 Exercise. Prove Proposition 8.16.

E8.6 Exercise. Prove Proposition 8.17.

E8.7 Exercise. Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with coefficients in $\mathbb{R}$. Since each matrix consists of n^2 real numbers the set $M_n(\mathbb{R})$ can be identified with $\mathbb{R}^{n^2}$. Using this identification we can consider $M_n(\mathbb{R})$ as a metric space. Let $GL_n(\mathbb{R})$ be the subspace of $M_n(\mathbb{R})$ consisting of all invertible matrices. Equivalently:

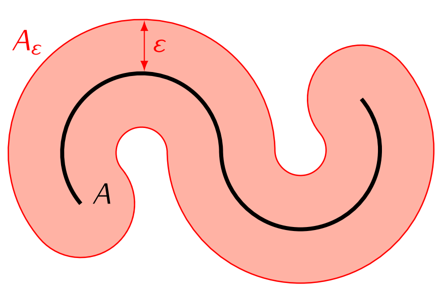
$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$$

where $\det A$ is the determinant of A. Show that $GL_n(R)$ has exactly two path connected components: $GL_n^+(\mathbb{R})$ and $GL_n^-(\mathbb{R})$ where

$$GL_n^+(\mathbb{R}) = \{A \in GL_n(R) \mid \det A > 0\}, \qquad GL_n^-(\mathbb{R}) = \{A \in GL_n(R) \mid \det A < 0\}$$

E8.8 Exercise. Let X be a subspace of $\mathbb{R}^n$. Show that if X is connected and it is open in $\mathbb{R}^n$ then X is path connected.

E8.9 Exercise. For $A \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ define $A_\varepsilon := \{x \in \mathbb{R}^n \mid d(x, y) < \varepsilon \text{ for some } y \in A\}$.



Show that if A is connected then A_ε is path connected for any $\varepsilon > 0$.

E8.10 Exercise. Let A be a countable set of points in $\mathbb{R}^2$. Show that the space $\mathbb{R}^2 \smallsetminus A$ is path connected.

E8.11 Exercise. Let X be a topological space and let $U, V \subseteq X$ be open sets such that $U \cup V$ and $U \cap V$ are path connected. Show that U and V are path connected.