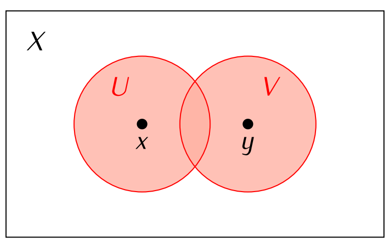
9 | Separation Axioms

Separation axioms are a family of topological invariants that give us new ways of distinguishing between various spaces. The idea is to look how open sets in a space can be used to create “buffer zones” separating pairs of points and closed sets. Separations axioms are denoted by T_1, T_2, etc., where T comes from the German word *Trennungsaxiom*, which just means “separation axiom”. Separation axioms can be also seen as a tool for identifying how close a topological space is to being metrizable: spaces that satisfy an axiom T_i can be considered as being closer to metrizable spaces than spaces that do not satisfy T_i.

9.1 Definition. A topological space X *satisfies the axiom* T_1 if for every points $x, y \in X$ such that $x \neq y$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.



9.2 Example. If X is a space with the antidiscrete topology and X consists of more than one point then X does not satisfy T_1.

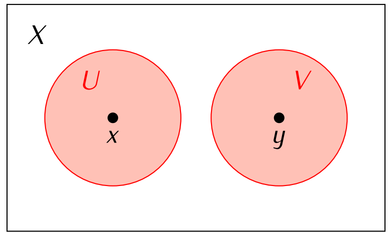
9.3 Proposition. *Let X be a topological space. The following conditions are equivalent:*

1) *X satisfies T_1.*
2) *For every point $x \in X$ the set $\{x\} \subseteq X$ is closed.*

Proof. Exercise. □

9.4 Definition. A topological space X *satisfies the axiom* T_2 if for any points $x, y \in X$ such that $x \neq y$

there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$.



A space that satisfies the axiom T_2 is called a *Hausdorff space*.

9.5 Note. Any metric space satisfies T_2. Indeed, for $x, y \in X$, $x \neq y$ take $U = B(x, \varepsilon)$, $V = B(y, \varepsilon)$ where $\varepsilon < \frac{1}{2}\varrho(x, y)$. Then U, V are open sets, $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

9.6 Note. If X satisfies T_2 then it satisfies T_1.

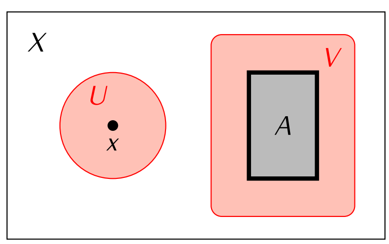
9.7 Example. The real line $\mathbb{R}$ with the Zariski topology satisfies T_1 but not T_2.

The following is a generalization of Proposition 5.13

9.8 Proposition. *Let X be a Hausdorff space and let $\{x_n\}$ be a sequence in X. If $x_n \to y$ and $x_n \to z$ for some then $y = z$.*

Proof. Exercise. □

9.9 Definition. A topological space X *satisfies the axiom* T_3 if X satisfies T_1 and if for each point $x \in X$ and each closed set $A \subseteq X$ such that $x \notin A$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $A \subseteq V$, and $U \cap V = \varnothing$.



A space that satisfies the axiom T_3 is called a *regular space*.

9.10 Note. Since in spaces satisfying T_1 sets consisting of a single point are closed (9.3) it follows that if a space satisfies T_3 then it satisfies T_2.

9.11 Example. Here is an example of a space X that satisfies T_2 but not T_3. Take the set

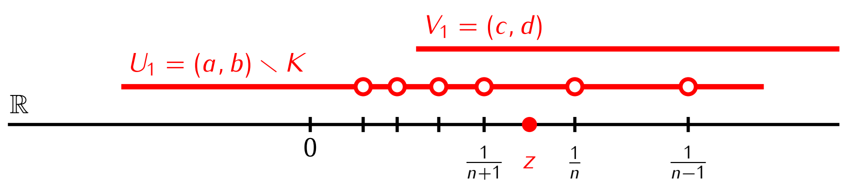
$$K = \{\tfrac{1}{n} \mid n = 1, 2, \dots\} \subseteq \mathbb{R}$$

Define a topological space X as follows. As a set $X = \mathbb{R}$. A basis $\mathcal{B}$ of the topology on X is given by

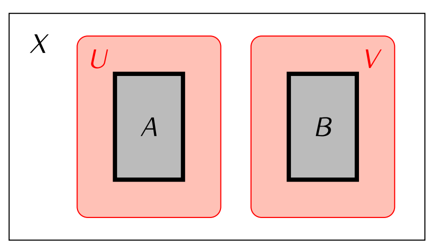
$$\mathcal{B} = \{U \subseteq \mathbb{R} \mid U = (a, b) \text{ or } U = (a, b) \smallsetminus K \text{ for some } a < b\}$$

Notice that the set $X \smallsetminus K$ is open in X, so K is a closed set.

The space X satisfies T_2 since any two points can be separated by some open intervals. On the other hand we will see that X does not satisfy T_3. Take $x = 0 \in X$ and let $U, V \subseteq X$ be open sets such that $x \in U$ and $K \subseteq V$. We will show that $U \cap V \neq \varnothing$. Since $x \in U$ and U is open there exists a basis element $U_1 \in \mathcal{B}$ such that $x \in U_1$ and $U_1 \subseteq U$. By assumption $U_1 \cap K = \varnothing$, so $U_1 = (a, b) \smallsetminus K$ for some $a < 0 < b$. Take n such that $\frac{1}{n} < b$. Since $\frac{1}{n} \in V$ and V is open there is a basis element $V_1 \in \mathcal{B}$ such that $\frac{1}{n} \in V_1$ and $V_1 \subseteq V$. Since $V_1 \cap K \neq \varnothing$ we have $V_1 = (c, d)$ for some $c < \frac{1}{n} < d$. For any $z \in \mathbb{R}$ such that $c < z < \frac{1}{n}$ and $z \notin K$ we have $z \in U_1 \cap V_1$, and so $z \in U \cap V$.



9.12 Definition. A topological space X *satisfies the axiom* T_4 if X satisfies T_1 and if for any closed sets $A, B \subseteq X$ such that $A \cap B = \varnothing$ there exist open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$.



A space that satisfies the axiom T_4 is called a *normal space*.

9.13 Note. If X satisfies T_4 then it satisfies T_3.

9.14 Theorem. *Every metric space is normal.*

The proof of this theorem will rely on the following fact:

9.15 Proposition. *Let X be a topological space satisfying T_1. If for any pair of closed sets $A, B \subseteq X$ satisfying $A \cap B = \varnothing$ there exists a continuous function $f\colon X \to [0, 1]$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$ then X is a normal space.*

Proof. Exercise. □

9.16 Definition. Let (X, ϱ) be a metric space. The *distance between a point $x \in X$ and a set $A \subseteq X$* is the number

$$\varrho(x, A) := \inf\{\varrho(x, a) \mid a \in A\}$$

9.17 Lemma. *If (X, ϱ) is a metric space and $A \subseteq X$ is a closed set then $\varrho(x, A) = 0$ if and only if $x \in A$.*

Proof. Exercise. □

9.18 Lemma. *Let (X, ϱ) be a metric space and $A \subseteq X$. The function $\varphi\colon X \to \mathbb{R}$ given by*

$$\varphi(x) = \varrho(x, A)$$

is continuous.

Proof. Let $x \in X$. We need to check that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varrho(x, x') < \delta$ then $|\varphi(x) - \varphi(x')| < \epsilon$. It will be enough to show that

$$|\varphi(x) - \varphi(x')| \leq \varrho(x, x')$$

for all $x, x' \in X$ since then we can take $\delta = \varepsilon$.

For $a \in A$ we have

$$\varrho(x, A) \leq \varrho(x, a) \leq \varrho(x, x') + \varrho(x', a)$$

This gives

$$\varrho(x, A) \leq \varrho(x, x') + \varrho(x', A)$$

and so

$$\varphi(x) - \varphi(x') = \varrho(x, A) - \varrho(x', A) \leq \varrho(x, x')$$

In the same way we obtain $\varphi(x') - \varphi(x) \leq \varrho(x', x)$, and so $|\varphi(x) - \varphi(x')| \leq \varrho(x, x')$.

□

Proof of Theorem 9.14. Let (X, ϱ) be a metric space and let $A, B \subseteq X$ be closed sets such that $A \cap B = \varnothing$. By Proposition 9.15 it will suffice to show that there exists a continuous function $f\colon X \to [0, 1]$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$. Take f to be the function given by

$$f(x) = \frac{\varrho(x, A)}{\varrho(x, A) + \varrho(x, B)}$$

By Lemma 9.17 $\varrho(x, A) = 0$ only if $x \in A$, and $\varrho(x, B) = 0$ only if $x \in B$. Since $A \cap B = \varnothing$ we have $\varrho(x, A) + \varrho(x, B) \neq 0$ for all $x \in X$, so f is well defined. From Lemma 9.18 it follows that f is a continuous function. Finally, for any $x \in A$ we have

$$f(x) = \frac{\varrho(x, A)}{\varrho(x, A) + \varrho(x, B)} = \frac{0}{0 + \varrho(x, B)} = 0$$

and for any $x \in B$ we have

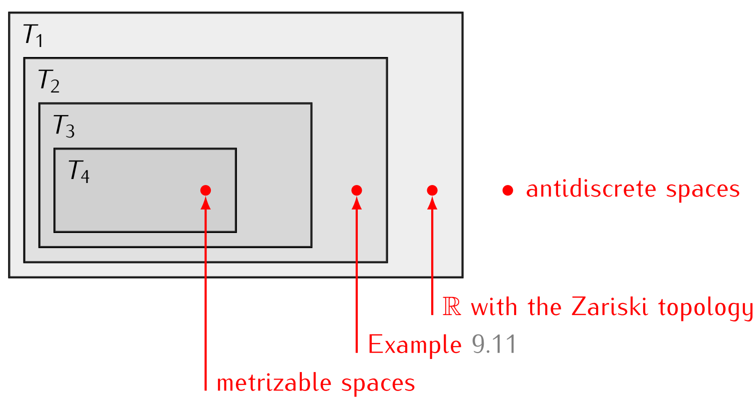
$$f(x) = \frac{\varrho(x, A)}{\varrho(x, A) + \varrho(x, B)} = \frac{\varrho(x, A)}{\varrho(x, A) + 0} = 1$$

□

Notice that the function f constructed in the proof of Theorem 9.14 satisfies a condition that is stronger than the assumption of Proposition 9.15: we have $f(x) = 0$ if and only if $x \in A$ and $f(x) = 1$ if and only if $x \in B$. Thus we obtain:

9.19 Corollary. *If (X, ϱ) is a metric space and $A, B \subseteq X$ are closed sets such that $A \cap B = \varnothing$ then there exists a continuous function $f\colon X \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.*

9.20 Note. The results described above can be summarized by the following picture:



Each rectangle represents the class of topological spaces satisfying the corresponding separation axiom. No area of this diagram is empty. Even though we have not seen here an example of a space that satisfies T_3 but not T_4 such spaces do exist.

Exercises to Chapter 9

E9.1 Exercise. Prove Proposition 9.3.

E9.2 Exercise. Prove Proposition 9.8.

E9.3 Exercise. Prove Proposition 9.15.

E9.4 Exercise. Prove Lemma 9.17.

E9.5 Exercise. Let X be a topological space and let Y be a subspace of X.

a) Show that if X satisfies T_1 then Y satisfies T_1.
b) Show that if X satisfies T_2 then Y satisfies T_2.
c) Show that if X satisfies T_3 then Y satisfies T_3.

Note: It may happen that X satisfies T_4 but Y does not.

E9.6 Exercise. Show that if X is a normal space and Y is a closed subspace of X then Y is a normal space.

E9.7 Exercise. Let X be a Hausdorff space. Show that the following conditions are equivalent:

(i) every subspace of X is a normal space.
(ii) for any two sets $A, B \subseteq X$ such that $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$ there exists open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \varnothing$.

E9.8 Exercise. This is a generalization of Exercise 6.9. Recall that a retract of a topological space X is a subspace $Y \subseteq X$ for which there exists a continuous function $r\colon X \to Y$ such that $r(x) = x$ for all $x \in Y$. Show that if X is a Hausdorff space and $Y \subseteq X$ is a retract of X then Y is a closed in X.

E9.9 Exercise. Let X be a space satisfying T_1. Show that the following conditions are equivalent:

(i) X is a normal space.
(ii) For any two disjoint closed sets $A, B \subseteq X$ there exist closed sets $A', B' \subseteq X$ such that $A \cap A' = \varnothing$, $B \cap B' = \varnothing$ and $A' \cup B' = X$.

E9.10 Exercise. Let X be a topological space and Y be a Hausdorff space. Let $f, g\colon X \to Y$ be continuous functions and let $A \subseteq X$ be given by

$$A = \{x \in X \mid f(x) = g(x)\}$$

Show that A is closed in X.

E9.11 Exercise. Let X be a topological space, Y be a Hausdorff space, and let A be a set dense in X. Let $f, g\colon X \to Y$ be continuous functions. Show that if $f(x) = g(x)$ for all $x \in A$ then $f(x) = g(x)$ for all $x \in X$

E9.12 Exercise. Let $f\colon X \to Y$ be a continuous function. Assume that f is onto and that for any closed set $A \subseteq X$ the set $f(A) \subseteq Y$ is closed. Show that if X is a normal space then Y is also normal.