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Inverse scattering transform for the vector nonlinear Schrödinger equation with nonvanishing boundary conditions

Barbara Prinari
Dipartimento di Fisica and Sezione INFN, Università di Lecce 73100, Lecce, Italy

Mark J. Ablowitz
Department of Applied Mathematics, University of Colorado, Boulder, Colorado 80309-0526

Gino Biondini
Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14260-2900

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The inverse scattering transform for the vector defocusing nonlinear Schrödinger (NLS) equation with nonvanishing boundary values at infinity is constructed. The direct scattering problem is formulated on a two-sheeted covering of the complex plane. Two out of the six Jost eigenfunctions, however, do not admit an analytic extension on either sheet of the Riemann surface. Therefore, a suitable modification of both the direct and the inverse problem formulations is necessary. On the direct side, this is accomplished by constructing two additional analytic eigenfunctions which are expressed in terms of the adjoint eigenfunctions. The discrete spectrum, bound states and symmetries of the direct problem are then discussed. In the most general situation, a discrete eigenvalue corresponds to a quartet of zeros (poles) of certain scattering data. The inverse scattering problem is formulated in terms of a generalized Riemann-Hilbert (RH) problem in the upper/lower half planes of a suitable uniformization variable. Special soliton solutions are constructed from the poles in the RH problem, and include dark-dark soliton solutions, which have dark solitonic behavior in both components, as well as dark-bright soliton solutions, which have one dark and one bright component. The linear limit is obtained from the RH problem and is shown to correspond to the Fourier transform solution obtained from the linearized vector NLS system. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2209169]

I. INTRODUCTION

The inverse scattering transform (IST) for the scalar nonlinear Schrödinger (NLS) equation

$$iq_t = q_{xx} - 2\sigma|q|^2 q \tag{1.1}$$

(subscripts x and t denote partial differentiation throughout) has been extensively studied in the literature, both in the focusing ($\sigma=-1$) and in the defocusing ($\sigma=1$) cases.[1–3] In particular, the defocusing case with nonvanishing boundary conditions was first studied in 1973;[4] the problem was subsequently clarified and generalized in various works,[5–10] and a detailed study can be found in the monograph.[11] Equation (1.1) with $\sigma=1$ admits soliton solutions with nontrivial boundary conditions, the so-called dark/gray solitons, which have the form

$$q(x,t) = q_0 e^{2iq_0^2 t}[\cos\alpha + i\sin\alpha\tanh[\sin\alpha q_0(x - 2q_0\cos\alpha t - x_0)]] \tag{1.2}$$

with q_0, α and x_0 arbitrary real parameters. Such solutions satisfy the boundary conditions

$$q(x,t) \to q_\pm(t) = q_0 e^{2iq_0^2 t \pm i\alpha} \quad \text{as } x \to \pm\infty$$

and appear as localized dips of intensity $q_0^2 \sin^2 \alpha$ on the background field q_0.

While the IST for the scalar NLS equation was developed many years ago, both with vanishing and nonvanishing boundary conditions, the basic formulation of IST has not been fully developed for the vector nonlinear Schrödinger (VNLS) equation

$$i\mathbf{q}_t = \mathbf{q}_{xx} - 2\sigma\|\mathbf{q}\|^2\mathbf{q}, \tag{1.3}$$

where $\mathbf{q}=\mathbf{q}(x,t)$ is, in general, an M-component vector and $\|\cdot\|$ is the standard Euclidean norm. The focusing case ($\sigma=-1$) with vanishing boundary conditions in two components was developed by Manakov in 1974.[12] However, the IST for the VNLS with nonzero boundary conditions has been open for over 30 years (partial results can be found in Ref. 13). It is worth noting that Ref. 14 provides an elegant direct and inverse scattering theory for decaying potentials on the real line. The extension to nondecaying potentials, however, is not straightforward and therefore here we employ a different approach. We should also remark that direct methods have been applied to VNLS as a way to derive explicit bright and dark soliton solutions, see for instance Refs. 17–20 and the review article Ref. 21.

In this work we present the IST for the two-component defocusing VNLS equation [namely, Eq. (1.3) with $\sigma=1$ and $M=2$] with nonvanishing boundary conditions as $x\to\pm\infty$. In Sec. II we discuss the direct scattering problem. Section II A is devoted to the study of the analyticity of the scattering eigenfunctions. Similar to the scalar equation, the spectral parameter of the associated block-matrix scattering problem for the VNLS is an element of a two-sheeted Riemann surface. The vector problem however presents additional difficulties due, in part, to the fact that two out of the six scattering eigenfunctions, defined via their asymptotics at infinity, do not admit an analytic extension on either sheet of the surface. Therefore a suitable modification both of the direct and of the inverse problem is necessary. On the direct side, this is achieved by defining in Sec. II B an "adjoint" scattering problem, which provides two additional analytic solutions of the original scattering problem. In Sec. II C we study the symmetries, and in Sec. II D we introduce a uniformization variable. In Sec. II E we study the asymptotic behavior of the eigenfunctions for large values of the scattering parameter, and in Sec. II F we discuss the discrete spectrum. The inverse problem is formulated in Sec. III as a Riemann-Hilbert (RH) problem associated with analytic eigenfunctions. The RH problem is then transformed into a closed linear system of algebraic-integral equations. The time evolution of the scattering data and the conserved quantities are discussed in Sec. IV. Explicit solutions are obtained in Sec. V; they include vector generalization of the dark and gray soliton solutions of the scalar case as well as more exotic dark-bright soliton solutions. Finally, in Sec. VI the linearized solution of the VNLS equation is obtained and found to be consistent with that of the RH formulation, and in the Appendix we discuss the WKB expansion of the eigenfunctions at large values of the scattering parameter.

II. DIRECT PROBLEM

It is well-known[12] that the two-component defocusing VNLS equation (1.3) with $\sigma=1$ and $M=2$ is associated to the Lax pair

$$v_x = (ik\mathbf{J} + \mathbf{Q})v, \tag{2.1a}$$

$$v_t = \begin{pmatrix} 2ik^2 + i\mathbf{q}^T\mathbf{r} & -2k\mathbf{q}^T - i\mathbf{q}_x^T \\ -2k\mathbf{r} + i\mathbf{r}_x & -2ik^2\mathbf{I}_2 - i\mathbf{r}\mathbf{q}^T \end{pmatrix} v, \tag{2.1b}$$

where $v(x,t,k)=(v^{(1)}(x,t,k),v^{(2)}(x,t,k),v^{(3)}(x,t,k))^T$ is the scattering eigenfunction, k is the scattering parameter, $\mathbf{q}(x,t)=(q^{(1)}(x,t),q^{(2)}(x,t))^T$ and $\mathbf{r}(x,t)=(r^{(1)}(x,t),r^{(2)}(x,t))^T=\mathbf{q}^*(x,t)$ are the scattering potentials, $\mathbf{I}_N$ is the $N\times N$ identity matrix, the superscript T denotes matrix transpose, and where

$$\mathbf{J} = \mathrm{diag}(-1,1,1), \quad \mathbf{Q}(x,t) = \begin{pmatrix} 0 & \mathbf{q}^T \\ \mathbf{r} & 0_{2\times 2} \end{pmatrix}. \tag{2.2}$$

Explicitly, the compatibility of the system of equations (2.1) (i.e., the equality of the mixed derivatives of the 3-component vector v with respect to x and t), together with the constraint $\mathbf{r} = \mathbf{q}^*$, is equivalent to the requirement that $\mathbf{q}(x,t)$ satisfy Eq. (1.3) with $\sigma=1$. Throughout this work, we consider potentials with the same time-independent amplitudes at both space infinities, which we can write without loss of generality as

$$\mathbf{q}(x,t) \sim \mathbf{q}_\pm(t) = e^{i\Theta_\pm(t)}\mathbf{q}_0, \quad \mathbf{r}(x,t) \sim \mathbf{r}_\pm(t) = e^{-i\Theta_\pm(t)}\mathbf{q}_0, \quad x \to \pm\infty, \tag{2.3}$$

where $\Theta_\pm(t)=\mathrm{diag}(\theta_\pm^{(1)},\theta_\pm^{(2)})$ and $\mathbf{q}_0=(q_0^{(1)},q_0^{(2)})^T \in \mathbb{R}^+\times\mathbb{R}^+$, and where $\|\mathbf{q}_0\|=\sqrt{(q_0^{(1)})^2+(q_0^{(2)})^2}$ is assumed to be non-zero. For brevity, in the following we will use $q_0=\|\mathbf{q}_0\|$.

A. Eigenfunctions, integral equations and analyticity

The eigenfunctions for the scattering problem (2.1a) with boundary conditions (2.3) are introduced by fixing the large-x asymptotics for $k\in\mathbb{R}$ with $|k|\geqslant q_0$,

$$\phi_1(x,k) \sim w_1^-(k)e^{-i\lambda x}, \quad \phi_2(x,k) \sim w_2^-(k)e^{ikx}, \quad \phi_3(x,k) \sim w_3^-(k)e^{i\lambda x}, \quad x \to -\infty, \tag{2.4a}$$

$$\psi_1(x,k) \sim w_1^+(k)e^{-i\lambda x}, \quad \psi_2(x,k) \sim w_2^+(k)e^{ikx}, \quad \psi_3(x,k) \sim w_3^+(k)e^{i\lambda x}, \quad x \to +\infty, \tag{2.4b}$$

where $\lambda(k)=\sqrt{k^2-q_0^2}$, the eigenvectors $w_1^\pm(k), w_2^\pm(k), w_3^\pm(k)$ are given by

$$w_1^-(k) = \begin{pmatrix} \lambda + k \\ i\mathbf{r}_- \end{pmatrix}, \quad w_2^-(k) = \begin{pmatrix} 0 \\ -i\mathbf{q}_-^\perp \end{pmatrix}, \quad w_3^-(k) = \begin{pmatrix} \lambda - k \\ -i\mathbf{r}_- \end{pmatrix}, \tag{2.5a}$$

$$w_1^+(k) = \begin{pmatrix} \lambda + k \\ i\mathbf{r}_+ \end{pmatrix}, \quad w_2^+(k) = \begin{pmatrix} 0 \\ -i\mathbf{q}_+^\perp \end{pmatrix}, \quad w_3^+(k) = \begin{pmatrix} \lambda - k \\ -i\mathbf{r}_+ \end{pmatrix}, \tag{2.5b}$$

and where we introduced a notation which we will use throughout this work: for any two-component vector $\mathbf{p}=(p^{(1)},p^{(2)})^T$ we write $\mathbf{p}^\perp=(p^{(2)},-p^{(1)})^T$. Note that for brevity we will omit the time dependence of the potentials and eigenfunctions throughout the discussion of the direct problem.

The Wronskian of a set $\{v_1,v_2,v_3\}$ of solutions of the scattering problem (2.1a) is defined in the usual way as

$$\mathrm{Wr}(v_1,v_2,v_3) = \det(v_1,v_2,v_3),$$

and satisfies the equation $d[\mathrm{Wr}(v_1,v_2,v_3)]/\mathrm{d}x = ik\,\mathrm{Wr}(v_1,v_2,v_3)$. Taking into account the asymptotic behavior of the solutions in Eq. (2.4) we then have

$$\mathrm{Wr}(\phi_1,\phi_2,\phi_3) = \mathrm{Wr}(\psi_1,\psi_2,\psi_3) = -2\lambda q_0^2 e^{ikx}. \tag{2.6}$$

Hence, for any nondecaying potential $\mathbf{q}(x,t)$, the two Wronskians in Eq. (2.6) are nonzero for all $x\in\mathbb{R}$ and all k such that $\lambda(k)\neq 0$ (i.e., everywhere except at the branch points of λ). We also introduce the solutions with fixed (with respect to x) boundary conditions

$$M_1(x,k) = e^{i\lambda x}\phi_1(x,k), \quad M_2(x,k) = e^{-ikx}\phi_2(x,k), \quad M_3(x,k) = e^{-i\lambda x}\phi_3(x,k), \tag{2.7a}$$

$$N_1(x,k) = e^{i\lambda x}\psi_1(x,k), \quad N_2(x,k) = e^{-ikx}\psi_2(x,k), \quad N_3(x,k) = e^{-i\lambda x}\psi_3(x,k), \tag{2.7b}$$

which can be represented in terms of the integral equations

$$M_j(x,k) = w_j^-(k) + \int_{-\infty}^{\infty} \mathbf{G}_j^-(x-x',k)(\mathbf{Q}(x') - \mathbf{Q}_-)M_j(x',k)\mathrm{d}x', \tag{2.8a}$$

$$N_j(x,k) = w_j^+(k) + \int_{-\infty}^{\infty} \mathbf{G}_j^+(x-x',k)(\mathbf{Q}(x') - \mathbf{Q}_+)N_j(x',k)\mathrm{d}x' \tag{2.8b}$$

for $j=1,2,3$, where

$$\mathbf{Q}_\pm = \begin{pmatrix} 0 & \mathbf{q}_\pm^T \\ \mathbf{r}_\pm & 0_{2\times 2} \end{pmatrix}, \tag{2.9}$$

and where the matrix Green's functions $\mathbf{G}_j^\pm(x,k)$ are defined below. The choice of Green's functions, together with the choice of the inhomogeneous terms in Eqs. (2.8), determine the analytic properties of the corresponding eigenfunctions. The superscripts $\pm$ in the Green's functions, like in the inhomogeneous terms, refer to the corresponding eigenfunctions being defined in terms of their asymptotics as $x \to \pm\infty$.

Using the Fourier transform technique, one can show that

$$\mathbf{G}_1^\mp(x,k) = \pm\,\theta(\pm x)\left\{ \frac{1}{2\lambda(\lambda+k)}[(\lambda+k)(\lambda\mathbf{I}_3 - k\mathbf{J}) + i(\lambda+k)\mathbf{Q}_\mp + \tilde{\mathbf{Q}}_\mp] + \frac{e^{2i\lambda x}}{2\lambda(\lambda-k)}[(\lambda-k)(\lambda\mathbf{I}_3 + k\mathbf{J}) - i(\lambda-k)\mathbf{Q}_\mp + \tilde{\mathbf{Q}}_\mp] + \frac{e^{i(\lambda+k)x}}{q_0^2}\tilde{\mathbf{Q}}_\mp \right\}, \tag{2.10a}$$

$$\mathbf{G}_3^\mp(x,k) = \pm\,\theta(\pm x)\left\{ \frac{1}{2\lambda(\lambda-k)}[(\lambda-k)(\lambda\mathbf{I}_3 + k\mathbf{J}) - i(\lambda-k)\mathbf{Q}_\mp + \tilde{\mathbf{Q}}_\mp] + \frac{e^{-2i\lambda x}}{2\lambda(\lambda+k)}[(\lambda+k)(\lambda\mathbf{I}_3 - k\mathbf{J}) + i(\lambda+k)\mathbf{Q}_\mp + \tilde{\mathbf{Q}}_\mp] + \frac{e^{-i(\lambda-k)x}}{q_0^2}\tilde{\mathbf{Q}}_\mp \right\}, \tag{2.10b}$$

$$\mathbf{G}_2^\mp(x,k) = \pm\,\theta(\pm x)\left\{ \frac{e^{-i(\lambda+k)x}}{2\lambda(\lambda+k)}[(\lambda+k)(\lambda\mathbf{I}_3 - k\mathbf{J}) + i(\lambda+k)\mathbf{Q}_\mp + \tilde{\mathbf{Q}}_\mp] + \frac{e^{i(\lambda-k)x}}{2\lambda(\lambda-k)}[(\lambda-k)(\lambda\mathbf{I}_3 + k\mathbf{J}) - i(\lambda-k)\mathbf{Q}_\mp + \tilde{\mathbf{Q}}_\mp] + \frac{1}{q_0^2}\tilde{\mathbf{Q}}_\mp \right\}, \tag{2.10c}$$

where

$$\tilde{\mathbf{Q}}_\pm = \begin{pmatrix} 0 & \mathbf{0}_{1\times 2} \\ \mathbf{0}_{2\times 1} & \mathbf{q}_\pm^\perp(\mathbf{r}_\pm^\perp)^T \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & q_\pm^{(2)} r_\pm^{(2)} & -r_\pm^{(1)} q_\pm^{(2)} \\ 0 & -r_\pm^{(2)} q_\pm^{(1)} & r_\pm^{(1)} q_\pm^{(1)} \end{pmatrix}. \tag{2.11}$$

Note that Eqs. (2.10) are significantly more complicated than the case of the vector system with zero boundary conditions (e.g., see Ref. 22).

So far, the integral equations and Green's functions are only defined for real k and λ. In order to extend the eigenfunctions to complex values of k, we note that, for instance, the Green's function $\mathbf{G}_1^-(x,k)$ does not grow exponentially as $|k| \to \infty$ if and only if

$$\operatorname{Im}\lambda \geqslant 0 \quad \text{and } \operatorname{Im}(\lambda+k) \geqslant 0. \tag{2.12a}$$

Similarly, $\mathbf{G}_1^+(x,k)$ does not grow exponentially as $|k| \to \infty$ if and only if

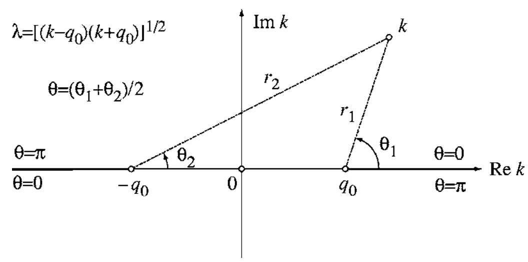

FIG. 1. The choice of branch cut for $\lambda=(k^2-q_0^2)^{1/2}$ in the complex k-plane. Here $\theta=(\theta_1+\theta_2)/2$.

$$\operatorname{Im}\lambda \leqslant 0 \quad \text{and } \operatorname{Im}(\lambda+k) \leqslant 0. \tag{2.12b}$$

It is therefore natural to introduce the Riemann surface of equation $\lambda^2=k^2-q_0^2$ obtained by gluing together two copies of the extended complex k-plane, which we will call $\mathbb{C}_1$ and $\mathbb{C}_2$, cut along the semilines $(-\infty,-q_0)$ and (q_0,∞).

On $\mathbb{C}_1$ one can introduce the local polar coordinates

$$k-q_0=r_1e^{i\theta_1}, \quad 0\leqslant\theta_1<2\pi,$$

$$k+q_0=r_2e^{i\theta_2}, \quad -\pi\leqslant\theta_2<\pi$$

with the magnitudes r_1 and r_2 uniquely fixed by the location of the point $k: r_1=|k-q_0|$ and $r_2=|k+q_0|$ (cf. Fig. 1). Then one can define

$$\lambda(k)=(r_1r_2)^{1/2}e^{i(\theta_1+\theta_2)/2}. \tag{2.13}$$

If $\theta=(\theta_1+\theta_2)/2$, then θ varies continuously between 0 and π both in the upper and in the lower k-planes, with a cut in the region $(-\infty,-q_0)\cup(q_0,\infty)$, and one has $\operatorname{Im}\lambda\geqslant 0$ and $\operatorname{Im}(\lambda\pm k)\geqslant 0$ for all $k\in\mathbb{C}_1$. Conversely, on $\mathbb{C}_2$ one defines

$$\lambda(k)=-(r_1r_2)^{1/2}e^{i(\theta_1+\theta_2)/2}, \tag{2.14}$$

which will give $\operatorname{Im}\lambda\leqslant 0$ and also $\operatorname{Im}(\lambda\pm k)\leqslant 0$, again with a cut in the region $(-\infty,-q_0)\cup(q_0,\infty)$. The upper branches of the cuts on sheet $\mathbb{C}_1$ are then glued with the lower branches on sheet $\mathbb{C}_2$ and vice versa as shown in Fig. 2(a).

With the above definitions, both conditions (2.12a) are satisfied if and only if k is on the upper sheet of the Riemann surface, and both conditions (2.12b) if and only if k is on the lower sheet. For potentials that rapidly approach $\mathbf{Q}_\pm$ as $x\to\pm\infty$, the Green's function $\mathbf{G}_1^-(x,k)$ then defines via Eq. (2.8a) an eigenfunction $M_1(x,k)$ which admits analytic extension on the entire upper sheet of the Riemann surface. Similarly, for suitable potentials the eigenfunction $N_1(x,k)$ defined by $\mathbf{G}_1^+(x,k)$ via Eq. (2.8b) admits analytic extension on the entire lower sheet. In a similar way one can investigate the properties of the remaining Green's functions. Overall we conclude that the eigenfunctions $M_1(x,k)=\phi_1(x,k)e^{i\lambda x}$ and $N_3(x,k)=\psi_3(x,k)e^{-i\lambda x}$ *are analytic on the upper sheet*, and $M_3(x,k)=\phi_3(x,k)e^{-i\lambda x}$ and $N_1(x,k)=\psi_1(x,k)e^{i\lambda x}$ *are analytic on the lower sheet*. Unlike the case of vanishing boundaries, however, the remaining two eigenfunctions, namely $M_2(x,k)$ and $N_2(x,k)$, in general are analytic neither on the upper nor on the lower sheet.

Equation (2.6) shows that for all real $k\neq\pm q_0$, the two matrices $\Phi(x,k)=(\phi_1,\phi_2,\phi_3)$ and $\Psi(x,k)=(\psi_1,\psi_2,\psi_3)$ each contain a set of three linearly independent solutions of the third-order scattering problem (2.4a). Thus it must be possible to express one set of solutions as a linear combination of the other, where the coefficients depend on k but are independent of x:

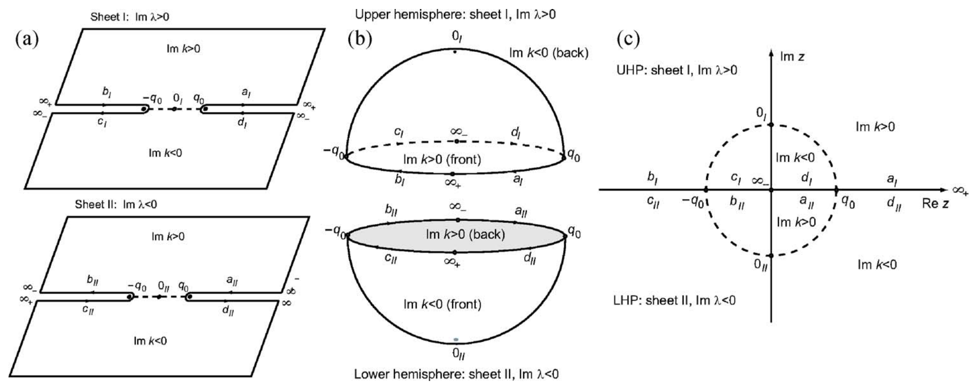

FIG. 2. (a) The two-sheeted covering of the complex plane defined by the scattering parameters (k,λ). (b) The topologically equivalent genus-0 Riemann sphere. (c) The corresponding complex plane for the uniformization variable $z=k+\lambda$ (which will be introduced in Sec. II D).

$$\Phi(x,k) = \Psi(x,k)\mathbf{A}^T(k), \tag{2.15a}$$

where $\mathbf{A}(k)=(a_{ij})$ is the 3×3 matrix of scattering coefficients. That is, $\phi_1(x,k)=a_{11}(k)\psi_1(x,k)+a_{12}(k)\psi_2(x,k)+a_{13}(k)\psi_3(x,k)$, with similar expressions for $\phi_2(x,k)$ and $\phi_3(x,k)$. Note that Eqs. (2.6) imply $\det(\mathbf{A})=1$. We can also express the "right" eigenfunctions in terms of the "left" ones,

$$\Psi(x,k) = \Phi(x,k)\mathbf{B}^T(k), \tag{2.15b}$$

where $\mathbf{B}(k)=(b_{ij})=\mathbf{A}^{-1}(k)$. Note that the scattering coefficients $a_{ij}(k)$ and $b_{ij}(k)$ are in general only defined where all of the eigenfunctions are, i.e., for $k\in\mathbb{R}$ and $|k|>q_0$, or, more precisely, on the oriented half-lines defined in Fig. 2, namely on $a_{\rm I}\equiv d_{\rm II}$, $b_{\rm I}\equiv c_{\rm II}$, $c_{\rm I}\equiv b_{\rm II}$, and $d_{\rm I}\equiv a_{\rm II}$. Note also that upper and lower banks of the cut are not equivalent, because both $\lambda(k)$ and the scattering eigenfunctions are discontinuous along the cut. These semilines define the contour $\mathcal{L}=a_{\rm I}\cup b_{\rm I}\cup c_{\rm I}\cup d_{\rm I}\equiv d_{\rm II}\cup c_{\rm II}\cup b_{\rm II}\cup a_{\rm II}$ namely $\mathcal{L}=(q_0+i0,\infty+i0)\cup(-\infty+i0,-q_0+i0)\cup-(-q_0-i0,-\infty-i0)\cup-(\infty-i0,q_0-i0)$ on the upper sheet.

Some of the scattering coefficients can be analytically extended off the real axis. From Eqs. (2.15) one can derive Wronskian representations for the scattering coefficients. Unlike the scalar case, however, such representations are not definitive in order to establish analyticity, since they all involve either $\phi_2(x,k)$ and/or $\psi_2(x,k)$, which do not admit analytic continuation. However, one can derive alternative representations for the scattering coefficients that provide the analytic extension sought for. For instance, using the first column of Eq. (2.15a) and the asymptotics (2.4b), one can check that

$$a_{11}(k) = \frac{1}{2\lambda(\lambda+k)}\lim_{x\to+\infty} e^{i\lambda x}[(\lambda+k)\phi_1^{(1)}(x,k) + iq_+^{(1)}\phi_1^{(2)}(x,k) + iq_+^{(2)}\phi_1^{(3)}(x,k)], \tag{2.16}$$

and since this expression for $a_{11}(k)$ only depends on the components of the vector $e^{i\lambda x}\phi_1(x,k)$, it indicates that for suitable potentials $a_{11}(k)$ can be analytically extended on the upper sheet of the Riemann surface. Similarly one finds that $a_{33}(k)$ and $b_{11}(k)$ can be analytically extended on the lower sheet of the Riemann surface, and $b_{33}(k)$ can be extended on the upper sheet of the Riemann surface. In general, however, the remaining scattering coefficients do not have any special analyticity properties.

The problem of determining the class of potentials for which a limit like (2.16) (with respect to a parameter, here x) of an analytic function of k is still an analytic function of k, is beyond the scope of this paper. We point out that this result is true for all the special solutions considered in this work.

B. Adjoint problem and auxiliary eigenfunctions

In order to formulate and solve the inverse scattering problem, one needs two independent sets of analytic eigenfunctions. The main issue at this stage is eliminating the nonanalytic eigenfunctions ϕ_2 and ψ_2. The approach introduced by Kaup in Ref. 15 for investigating the three-wave interaction is generalized here in order to obtain a representation of the nonanalytic eigenfunctions in terms of analytic eigenfunctions and scattering data. The key idea is to consider the "adjoint" eigenvalue problem

$$v_x^{\mathrm{ad}} = (-ik\mathbf{J} + \mathbf{Q}^T)v^{\mathrm{ad}} \tag{2.17}$$

where $\mathbf{Q}$ and $\mathbf{J}$ are defined in Eq. (2.2). One then recalls the well-known fact (see, for instance Ref. 16) that if $u^{\mathrm{ad}}(x,k)$ and $w^{\mathrm{ad}}(x,k)$ are two arbitrary solutions of the adjoint problem (2.17), then

$$v(x,k) = -\mathbf{J}(u^{\mathrm{ad}}(x,k) \wedge w^{\mathrm{ad}}(x,k))e^{ikx}, \tag{2.18}$$

where $\wedge$ denotes the vector product, is a solution of the original scattering problem (2.1a). As before, one defines two sets of solutions of Eq. (2.17), i.e., as $x \to -\infty$

$$\phi_1^{\mathrm{ad}}(x,k) \sim \begin{pmatrix} \lambda + k \\ -i\mathbf{q}_- \end{pmatrix} e^{i\lambda x}, \quad \phi_2^{\mathrm{ad}}(x,k) \sim \begin{pmatrix} 0 \\ i\mathbf{r}_-^{\perp} \end{pmatrix} e^{-ikx}, \quad \phi_3^{\mathrm{ad}}(x,k) \sim \begin{pmatrix} \lambda - k \\ i\mathbf{q}_- \end{pmatrix} e^{-i\lambda x} \tag{2.19a}$$

and as $x \to +\infty$

$$\psi_1^{\mathrm{ad}}(x,k) \sim \begin{pmatrix} \lambda + k \\ -i\mathbf{q}_+ \end{pmatrix} e^{i\lambda x}, \quad \psi_2^{\mathrm{ad}}(x,k) \sim \begin{pmatrix} 0 \\ i\mathbf{r}_+^{\perp} \end{pmatrix} e^{-ikx}, \quad \psi_3^{\mathrm{ad}}(x,k) \sim \begin{pmatrix} \lambda - k \\ i\mathbf{q}_+ \end{pmatrix} e^{-i\lambda x}. \tag{2.19b}$$

With techniques identical to those used to derive the integral equations and the Green's functions associated to the eigenfunctions of the scattering problem (2.1a), one can then show that $e^{i\lambda x}\phi_3^{\mathrm{ad}}(x,k)$ and $e^{-i\lambda x}\psi_1^{\mathrm{ad}}(x,k)$ are analytic in the upper sheet of the Riemann surface, $e^{-i\lambda x}\phi_1^{\mathrm{ad}}(x,k)$ and $e^{i\lambda x}\psi_3^{\mathrm{ad}}(x,k)$ are analytic on the lower sheet and $e^{ikx}\phi_2^{\mathrm{ad}}(x,k)$ and $e^{ikx}\psi_2^{\mathrm{ad}}(x,k)$ on neither sheet. Analogues of Eqs. (2.15) also exist,

$$\Phi^{\mathrm{ad}}(x,k) = \Psi^{\mathrm{ad}}(x,k)\widetilde{\mathbf{B}}^T(k), \quad \Psi^{\mathrm{ad}}(x,k) = \Phi^{\mathrm{ad}}(x,k)\widetilde{\mathbf{A}}^T(k), \tag{2.20}$$

where $\Phi^{\mathrm{ad}}(x,k)=(\phi_1^{\mathrm{ad}},\phi_2^{\mathrm{ad}},\phi_3^{\mathrm{ad}})$ and $\Psi^{\mathrm{ad}}(x,k)=(\psi_1^{\mathrm{ad}},\psi_2^{\mathrm{ad}},\psi_3^{\mathrm{ad}})$, and where $\widetilde{\mathbf{A}}(k)=(\widetilde{a}_{ij})$ and $\widetilde{\mathbf{B}}(k)=(\widetilde{b}_{ij})=\widetilde{\mathbf{A}}^{-1}(k)$ are the adjoint scattering matrices.

From these adjoint states, we can now use Eqs. (2.19) to define via (2.18) two new solutions of the original scattering problem (2.1a), namely,

$$\bar{\chi}(x,k) = -e^{ikx}\mathbf{J}(\phi_1^{\mathrm{ad}}(x,k) \wedge \psi_3^{\mathrm{ad}}(x,k)), \tag{2.21a}$$

$$\chi(x,k) = -e^{ikx}\mathbf{J}(\phi_3^{\mathrm{ad}}(x,k) \wedge \psi_1^{\mathrm{ad}}(x,k)). \tag{2.21b}$$

By construction, $\bar{\chi}(x,k)e^{-ikx}$ is analytic in the lower sheet [where $\phi_1^{\mathrm{ad}}(x,k)e^{-i\lambda x}$ and $\psi_3^{\mathrm{ad}}(x,k)e^{i\lambda x}$ are], and $\chi(x,k)e^{-ikx}$ is analytic in the upper sheet [where $\phi_3^{\mathrm{ad}}(x,k)e^{i\lambda x}$ and $\psi_1^{\mathrm{ad}}(x,k)e^{-i\lambda x}$ are]. Moreover, by comparing the asymptotic behavior as $x \to \pm\infty$ of eigenfunctions and adjoint eigenfunctions, one can check that, for all cyclic indices j,l,m,

$$\phi_j(x,k) = -e^{ikx}\mathbf{J}(\phi_l^{\mathrm{ad}}(x,k) \wedge \phi_m^{\mathrm{ad}}(x,k))/\Gamma_j(k), \tag{2.22a}$$

$$\psi_j(x,k) = -e^{ikx}\mathbf{J}(\psi_l^{\mathrm{ad}}(x,k) \wedge \psi_m^{\mathrm{ad}}(x,k))/\Gamma_j(k), \tag{2.22b}$$

and reciprocally

$$\phi_j^{\mathrm{ad}}(x,k) = -e^{-ikx}\mathbf{J}(\phi_l(x,k) \wedge \phi_m(x,k))/\Gamma_j(k), \tag{2.22c}$$

$$\psi_j^{\mathrm{ad}}(x,k) = -e^{-ikx}\mathbf{J}(\psi_l(x,k) \wedge \psi_m(x,k))/\Gamma_j(k), \tag{2.22d}$$

where

$$\Gamma_1(k) = \lambda - k, \quad \Gamma_2(k) = 2\lambda, \quad \Gamma_3(k) = \lambda + k. \tag{2.23}$$

From Eqs. (2.22) and (2.15), (2.20) it then follows that

$$\tilde{\mathbf{A}}^T(k) = \Gamma(k)\mathbf{A}(k)\Gamma^{-1}(k), \quad \tilde{\mathbf{B}}^T(k) = \Gamma(k)\mathbf{B}(k)\Gamma^{-1}(k), \tag{2.24}$$

where $\Gamma(k)=\mathrm{diag}(\Gamma_1(k),\Gamma_2(k),\Gamma_3(k))$. Substituting the first of Eq. (2.20) into Eq. (2.21) and using (2.22b) yields

$$\chi(x,k) = 2\lambda[b_{33}(k)\psi_2(x,k) - b_{23}(k)\psi_3(x,k)], \tag{2.25a}$$

$$\bar{\chi}(x,k) = 2\lambda[b_{21}(k)\psi_1(x,k) - b_{11}(k)\psi_2(x,k)]. \tag{2.25b}$$

Each of these two relations provides a decomposition of the nonanalytic eigenfunction $\psi_2(x,k)$,

$$\psi_2(x,k) = \frac{b_{21}(k)}{b_{11}(k)}\psi_1(x,k) - \frac{1}{2\lambda}\frac{\bar{\chi}(x,k)}{b_{11}(k)} = \frac{b_{23}(k)}{b_{33}(k)}\psi_3(x,k) + \frac{1}{2\lambda}\frac{\chi(x,k)}{b_{33}(k)}. \tag{2.26}$$

Similar relations hold for the eigenfunction $\phi_2(x,k)$, where now the scattering coefficients $a_{ij}(k)$ are involved. Precisely, one finds

$$\bar{\chi}(x,k) = 2\lambda[a_{23}(k)\phi_3(x,k) - a_{33}(k)\phi_2(x,k)], \tag{2.27a}$$

$$\chi(x,k) = 2\lambda[a_{11}(k)\phi_2(x,k) - a_{21}(k)\phi_1(x,k)] \tag{2.27b}$$

and consequently one obtains similar representations for $\phi_2(x,k)$:

$$\phi_2(x,k) = \frac{a_{23}(k)}{a_{33}(k)}\phi_3(x,k) - \frac{1}{2\lambda}\frac{\bar{\chi}(x,k)}{a_{33}(k)} = \frac{a_{21}(k)}{a_{11}(k)}\phi_1(x,k) + \frac{1}{2\lambda}\frac{\chi(x,k)}{a_{11}(k)}. \tag{2.28}$$

These expressions will be key to define the inverse scattering problem in Sec. III.

C. Symmetries

Importantly, the scattering problem admits two symmetries, which relate the value of the eigenfunctions on different sheets of the Riemann surface. These symmetries translate into compatibility conditions (constraints) on the scattering data, and will play a fundamental role in the formulation of the inverse problem.

First symmetry $(k,\lambda)\to(k^*,\lambda^*)$*:* When the potential satisfies the symmetry condition $\mathbf{r}=\mathbf{q}^*$, one has $\mathbf{Q}^H=\mathbf{Q}$, and therefore from Eq. (2.17) it follows that

$$\frac{\partial}{\partial x}[v^{\mathrm{ad}}(k^*)]^* = (ik\mathbf{J} + \mathbf{Q}^H)(v^{\mathrm{ad}}(k^*))^* = (ik\mathbf{J} + \mathbf{Q})(v^{\mathrm{ad}}(k^*))^*.$$

Hence, taking into account the boundary conditions (2.4) and (2.19), we have

$$\phi_j^{\mathrm{ad}}(k,\lambda) = (\phi_j(k^*,\lambda^*))^*, \quad \psi_j^{\mathrm{ad}}(k,\lambda) = (\psi_j(k^*,\lambda^*))^*, \quad j=1,2,3 \tag{2.29}$$

and, as a consequence of Eqs. (2.24), (2.15), and (2.20)

$$\Gamma(k,\lambda)\mathbf{B}(k,\lambda)\Gamma^{-1}(k,\lambda) = \mathbf{A}^H(k^*,\lambda^*), \tag{2.30}$$

where $\Gamma(k,\lambda)=\mathrm{diag}(\Gamma_1,\Gamma_2,\Gamma_3)$ as before. In particular, Eqs. (2.30) give

$$b_{11}(k,\lambda) = a_{11}^*(k^*,\lambda^*), \quad b_{33}(k,\lambda) = a_{33}^*(k^*,\lambda^*) \tag{2.31}$$

showing that $a_{11}(k,\lambda)$ [respectively, $b_{33}(k,\lambda)$] has a zero on the upper sheet of the Riemann surface at a point $(k_n,\lambda(k_n))$ if and only if $b_{11}(k,\lambda)$ [respectively, $a_{33}(k,\lambda)$] has a zero at the conjugate point $(k_n^*,\lambda^*(k_n^*))$ on the lower sheet.

Second symmetry $(k,\lambda)\to(k,-\lambda)$*:* This involution relates the values of the eigenfunctions on the two sheets, and in particular across the cuts, for arbitrary fixed k on either sheet and $\lambda\to-\lambda$. Indeed, the scattering problem is clearly invariant with respect to the exchange $(k,\lambda)\to(k,-\lambda)$, and by looking at the boundary conditions (2.4) and (2.5) one can check that

$$\psi_1(x,k,-\lambda) = -\psi_3(x,k,\lambda), \quad \phi_1(x,k,-\lambda) = -\phi_3(x,k,\lambda) \tag{2.32a}$$

while ψ_2 and ϕ_2 are invariant with the respect to the symmetry $\lambda\leftrightarrow-\lambda$, i.e.,

$$\psi_2(x,k,-\lambda) = \psi_2(x,k,\lambda), \quad \phi_2(x,k,-\lambda) = \phi_2(x,k,\lambda). \tag{2.32b}$$

Therefore, from the equations (2.15a) defining the scattering coefficients one has

$$a_{11}(k,-\lambda) = a_{33}(k,\lambda), \quad a_{22}(k,-\lambda) = a_{22}(k,\lambda), \tag{2.33a}$$

$$a_{12}(k,-\lambda) = -a_{32}(k,\lambda), \quad a_{13}(k,-\lambda) = a_{31}(k,\lambda), \quad a_{21}(k,-\lambda) = -a_{23}(k,\lambda). \tag{2.33b}$$

The same symmetry relations hold for the coefficients $b_{ij}(k)$, i.e.,

$$b_{11}(k,-\lambda) = b_{33}(k,\lambda), \quad b_{12}(k,-\lambda) = -b_{32}(k,\lambda), \quad b_{13}(k,-\lambda) = b_{31}(k,\lambda) \tag{2.34a}$$

$$b_{22}(k,-\lambda) = b_{22}(k,\lambda), \quad b_{21}(k,-\lambda) = -b_{23}(k,\lambda). \tag{2.34b}$$

Note that Eq. (2.33a) implies that $(k_n,\lambda(k_n))$ is a zero of $a_{11}(k,\lambda)$ in the upper sheet if and only if $(k_n,-\lambda(k_n))$ is a zero for $a_{33}(k,\lambda)$ in the lower sheet, and the same for $b_{11}(k,\lambda)$ and $b_{33}(k,\lambda)$. Finally, note that, taking into account Eqs. (2.32) and (2.34), comparing Eqs. (2.25a) and (2.25b) yields

$$\chi(x,k,\lambda) = \bar{\chi}(x,k,-\lambda). \tag{2.35}$$

D. Uniformization coordinate

In a similar way as for the scalar problem (e.g., see Ref. 11), we can introduce a uniformization variable z (global uniformizing parameter) defined by the conformal mapping

$$z = k + \lambda(k). \tag{2.36a}$$

The inverse mapping is given by

$$k = \tfrac{1}{2}(z + \hat{z}^*), \quad \lambda = z - k = \tfrac{1}{2}(z - \hat{z}^*), \tag{2.36b}$$

where we have introduced the shorthand notation

$$\hat{z} = q_0^2/z^*, \tag{2.36c}$$

which we will use throughout the rest of this work. (Note $\lambda - k = -\hat{z}^*$, which will also be useful later on.) With regard to the mapping $(k,\lambda)\to z$, it should be observed that (cf. Fig. 2(a),(c)):

(i) The branch cuts on the two sheets of the Riemann surface are mapped onto the real z-axis.
(ii) The two sheets $\mathbb{C}_1$ and $\mathbb{C}_2$ of the Riemann surface are, respectively, mapped onto the upper and lower half-planes of the complex z-plane.
(iii) A neighborhood of $k=\infty$ on either sheet is mapped onto a neighborhood of $z=\infty$ or $z=0$ depending on the sign of $k_{\rm im}$ (cf. Sec. II E).

(iv) The symmetry $k-i0 \to k+i0$ on the contours (giving the discontinuity of eigenfunctions and scattering data on the banks of the cut) transforms into $z \to \hat{z}^* = q_0^2/z$ on the real z-axis.

According to the discussion in Secs. I A and I B, the eigenfunctions $\phi_1(x,z)e^{i\lambda(z)x}$, $\psi_3(x,z)e^{-i\lambda(z)x}$ and $\chi(x,z)e^{-ik(z)x}$ are analytic in the upper half-plane of z, while $\phi_3(x,z)e^{-i\lambda(z)x}$, $\psi_1(x,z)e^{i\lambda(z)x}$ and $\bar{\chi}(x,z)e^{-ik(z)x}$ are analytic in the lower half-plane. Similarly, the scattering coefficients $a_{11}(z)$ and $b_{33}(z)$ are analytic in the upper half-plane of z, while $a_{33}(z)$ and $b_{11}(z)$ are analytic in the lower half-plane.

It should be noted that although the uniformization coordinate will be important in the inverse problem, it is not essential in our formulation of the direct problem. We introduce it here because it turns out to be convenient when discussing the location of the discrete eigenvalues, which is done in Sec. II F.

In terms of the global parameter z, the first symmetry becomes $z \to z^*$. Under this transformation, the symmetry relations (2.29) and (2.30) are then, respectively, written as

$$\phi_j^{\mathrm{ad}}(x,z) = (\phi_j(x,z^*))^*, \quad \psi_j^{\mathrm{ad}}(x,z) = (\psi_j(x,z^*))^*, \quad j = 1,2,3, \tag{2.37}$$

$$b_{\ell j}^*(z^*) = \Gamma_j(z) a_{j\ell}(z) \Gamma_\ell^{-1}(z), \quad \ell, j = 1,2,3, \tag{2.38}$$

where

$$\Gamma_1(z) = -\hat{z}^*, \quad \Gamma_2(z) = 2\lambda(z), \quad \Gamma_3(z) = z. \tag{2.39}$$

Equation (2.38) can also be written compactly as

$$\Gamma(z)\mathbf{B}(z)\Gamma^{-1}(z) = \mathbf{A}^H(z^*), \tag{2.40}$$

where $\Gamma(z) = \mathrm{diag}(\Gamma_1, \Gamma_2, \Gamma_3)$ as before. Taking into account Eq. (2.22), the symmetries (2.37) can be written in terms of eigenfunctions as follows:

$$\phi_j^*(x,z^*) = -e^{-ik(z)x}\mathbf{J}(\phi_l(x,z) \wedge \phi_m(x,z))/\Gamma_j(z) \tag{2.41a}$$

and

$$\psi_j^*(x,z^*) = -e^{-ik(z)x}\mathbf{J}(\psi_l(x,z) \wedge \psi_m(x,z))/\Gamma_j(z) \tag{2.41b}$$

where j, l, m are cyclic indices.

The second symmetry relates values of eigenfunctions and scattering coefficients at points (k, λ) and $(k, -\lambda)$ on the two sheets or at the cuts. In terms of the uniformization variable z, the transformation then becomes $z \to \hat{z}^* = q_0^2/z$. Hence the symmetry relations (2.33) can be written as

$$a_{11}(\hat{z}^*) = a_{33}(z), \quad a_{12}(\hat{z}^*) = -a_{32}(z), \tag{2.42a}$$

$$a_{13}(\hat{z}^*) = a_{31}(z), \quad a_{21}(\hat{z}^*) = -a_{23}(z), \tag{2.42b}$$

and the same relations hold for the coefficients $b_{ij}(z)$. Also note that the symmetry relations (2.32) between the auxiliary eigenfunctions can be written as

$$\phi_1(x,z) = -\phi_3(x,\hat{z}^*), \quad \psi_1(x,z) = -\psi_3(x,\hat{z}^*), \tag{2.43a}$$

$$\chi(x,z) = \bar{\chi}(x,\hat{z}^*). \tag{2.43b}$$

Taking into account Eq. (2.30) and recalling that $\mathbf{B}(z) = \mathbf{A}^{-1}(z)$ and that both matrices have unit determinant, on either side of the real z-axis we find

$$|a_{11}(z)|^2 + |a_{12}(z)|^2 \frac{\Gamma_1(z)}{\Gamma_2(z)} + |a_{13}(z)|^2 \frac{\Gamma_1(z)}{\Gamma_3(z)} = 1,$$

where $z \in \mathbb{R}$. Combining this with Eq. (2.39) we then obtain

$$|a_{11}(z)|^2 = 1 + \frac{q_0^2}{z^2}|a_{13}(z)|^2 + \frac{q_0^2}{z^2 - q_0^2}|a_{12}(z)|^2 \quad \forall\, z \in \mathbb{R}. \tag{2.44}$$

The second term on the right-hand side is non-negative; the last term, however, can be either positive or negative and therefore one cannot *a priori* exclude real zeros of $a_{11}(z)$ if $z \in (-q_0, q_0)$. Similar results follow for zeros of $a_{33}(z)$, $b_{11}(z)$, and $b_{33}(z)$, taking into account the symmetry relations (2.38) and (2.42).

Note that both symmetry transformations relate values in the upper half z-plane to values in the lower half z-plane, since both z^* and $\hat{z}^*$ are in the opposite half-plane as z. In the following we will assume that the scattering coefficients $a_{11}(z)$, etc., have no zeros on the real z-axis.

E. Asymptotic behavior of eigenfunctions and scattering data

In order to determine the asymptotic behavior of the eigenfunctions for large values of the scattering parameter k, we first note the following: in the upper sheet of the Riemann surface (i.e., when $\lambda_{\rm im} \geqslant 0$), one has, above the cut (i.e., when $k_{\rm im} > 0$)

$$\lambda + k \sim 2k + O(1), \quad \lambda - k \sim -\frac{q_0^2}{2k} + o(1/k) \quad \text{as } |k| \to \infty, \tag{2.45a}$$

and below the cut (i.e., when $k_{\rm im} < 0$)

$$\lambda + k \sim \frac{q_0^2}{2k} + o(1/k), \quad \lambda - k \sim -2k + O(1) \quad \text{as } |k| \to \infty. \tag{2.45b}$$

Similar relations hold in the lower sheet of the Riemann surface (i.e., when $\lambda_{\rm im} \leqslant 0$). Using these relations we can obtain the large-k expansion of the eigenfunctions on each sheet. It is more convenient however to express this behavior in terms of the uniformization variable z, which will be used in the inverse problem. To this aim, we note that (cf. Fig. 2)

(i) $|k| \to \infty$ in the upper-half-plane of sheet I corresponds to $z \to \infty$ in the upper-half z-plane,
(ii) $|k| \to \infty$ in the lower-half-plane of sheet II corresponds to $z \to \infty$ in the lower-half z-plane,
(iii) $|k| \to \infty$ in the lower-half-plane of sheet I corresponds to $z \to 0$ in the upper-half z-plane,
(iv) $|k| \to \infty$ in the upper-half-plane of sheet II corresponds to $z \to 0$ in the lower-half z-plane.

It should be noted here that there is no conceptual distinction between the points $z=0$ and $z=\infty$ in the z-plane, and one can change one into the other by simply defining $z=k-\lambda$ instead of $z=k+\lambda$.

Taking Eqs. (2.45) into account and using both the integral equations (2.8) and the WKB expansions of the eigenfunctions (see the Appendix) we obtain that as $z \to \infty$ in the upper-half z-plane one has

$$\phi_1(x,z)e^{i\lambda x} \sim \begin{pmatrix} z \\ i\mathbf{r}(x) \end{pmatrix}, \quad \psi_3(x,z)e^{-i\lambda x} \sim -\begin{pmatrix} \mathbf{q}^T(x)\mathbf{r}_+/z \\ i\mathbf{r}_+ \end{pmatrix}, \tag{2.46a}$$

while as $z \to 0$ in the upper-half z-plane one has

$$\phi_1(x,z)e^{i\lambda x} \sim \begin{pmatrix} \mathbf{q}^T(x)\mathbf{r}_-/\hat{z}^* \\ i\mathbf{r}_- \end{pmatrix}, \quad \psi_3(x,z)e^{-i\lambda x} \sim -\begin{pmatrix} \hat{z}^* \\ i\mathbf{r}(x) \end{pmatrix}. \tag{2.46b}$$

Similarly, as $z \to 0$ in the lower-half z-plane one has

$$\phi_3(x,z)e^{-i\lambda x} \sim -\begin{pmatrix} \hat{z}^* \\ i\mathbf{r}(x) \end{pmatrix}, \quad \psi_1(x,z)e^{i\lambda x} \sim \begin{pmatrix} \mathbf{q}^T(x)\mathbf{r}_+/\hat{z}^* \\ i\mathbf{r}_+ \end{pmatrix}, \tag{2.46c}$$

while as $z\to\infty$ in the lower-half z-plane one has

$$\phi_3(x,z)e^{-i\lambda x} \sim -\begin{pmatrix} \mathbf{q}^T(x)\mathbf{r}_-/z \\ i\mathbf{r}_- \end{pmatrix}, \quad \psi_1(x,z)e^{i\lambda x} \sim \begin{pmatrix} z \\ i\mathbf{r}(x) \end{pmatrix}. \tag{2.46d}$$

Asymptotic expansions for the adjoint eigenfunctions can also be obtained. Then, using the asymptotics of $\phi_j^{\mathrm{ad}}(x,z)$ and $\psi_j^{\mathrm{ad}}(x,z)$ as well as Eqs. (2.21), one can obtain the asymptotic expansions for the auxiliary eigenfunctions $\bar{\chi}(x,z)$ and $\chi(x,z)$. Explicitly, in the upper-half z-plane one has

$$\chi(x,z)e^{-ikx} \sim -\begin{pmatrix} \mathbf{q}^T(x)\mathbf{q}_-^\perp \\ i\mathbf{q}_-^\perp z \end{pmatrix} \quad \text{as } z\to\infty, \tag{2.47a}$$

$$\chi(x,z)e^{-ikx} \sim \begin{pmatrix} \mathbf{q}^T(x)\mathbf{q}_+^\perp \\ i\mathbf{q}_+^\perp \hat{z}^* \end{pmatrix} \quad \text{as } z\to 0, \tag{2.47b}$$

whereas in the lower-half z-plane

$$\bar{\chi}(x,z)e^{-ikx} \sim -\begin{pmatrix} \mathbf{q}^T(x)\mathbf{q}_- \\ i\mathbf{q}_-^\perp \hat{z}^* \end{pmatrix} \quad \text{as } z\to 0, \tag{2.47c}$$

$$\bar{\chi}(x,z)e^{-ikx} \sim \begin{pmatrix} \mathbf{q}^T(x)\mathbf{q}_+^\perp \\ i\mathbf{q}_+^\perp z \end{pmatrix} \quad \text{as } z\to\infty. \tag{2.47d}$$

Equations (2.16) and (2.46) also allow us to obtain the asymptotic behavior of the scattering coefficients. For example, in the upper-half z-plane, as $z\to\infty$ one has

$$a_{11}(z) \sim 1, \quad b_{33}(z) \sim \mathbf{q}_-^T\mathbf{r}_+/q_0^2, \tag{2.48a}$$

while as $z\to 0$ one has

$$a_{11}(z) \sim \mathbf{q}_+^T\mathbf{r}_-/q_0^2, \quad b_{33}(z) \sim 1. \tag{2.48b}$$

Similar expressions hold for $b_{11}(z)$ and $a_{33}(z)$ in the lower-half z-plane: namely, as $z\to 0$ one has

$$a_{33}(z) \sim 1, \quad b_{11}(z) \sim \mathbf{q}_-^T\mathbf{r}_+/q_0^2 \tag{2.48c}$$

while as $z\to\infty$ one has

$$a_{33}(z) \sim \mathbf{q}_+^T\mathbf{r}_-/q_0^2, \quad b_{11}(z) \sim 1. \tag{2.48d}$$

Note that $\mathbf{q}_+^T\mathbf{r}_- = (\mathbf{q}_-^T\mathbf{r}_+)^* = e^{i\Delta\theta^{(1)}}|q_0^{(1)}|^2 + e^{i\Delta\theta^{(2)}}|q_0^{(2)}|^2$, where we have introduced the asymptotic phase differences for the potentials, $\Delta\theta^{(1)} = \theta_+^{(1)} - \theta_-^{(1)}$ and $\Delta\theta^{(2)} = \theta_+^{(2)} - \theta_-^{(2)}$ (cf. Eq. (2.3)). Hereafter, we will assume that these asymptotic phase differences are the same in both components, namely

$$\Delta\theta^{(1)} = \Delta\theta^{(2)} =: \Delta\theta. \tag{2.49}$$

If Eqs. (2.49) are satisfied, then

$$\mathbf{q}_+^T\mathbf{r}_- = (\mathbf{q}_-^T\mathbf{q}_+)^* = e^{i\Delta\theta}q_0^2, \tag{2.50}$$

and the asymptotic behaviors of the scattering coefficients in Eqs. (2.48) simplify correspondingly.

F. Discrete eigenvalues and bound states

Recall that in the 2×2 scattering problem for the nondecaying scalar NLS equation there is a one-to-one correspondence between poles of the transmission coefficients [here, zeros of $a_{11}(z)$ etc] and eigenvalues, which, in turn, are related to bound states. Hence, the unitarity relation [i.e., the analog of Eq. (2.44)], together with the self-adjointness of the scattering problem, ensure that the transmission coefficients can only have poles at $k=k_n\in(-q_0,q_0)$, i.e., for $z=z_n$ on the circle C_0 of radius q_0 centered at the origin (e.g., see Ref. 11). As we will see in the following, in the case of vector NLS equation with nondecaying boundary conditions, the decay properties of the eigenfunctions at a pole of the transmission coefficients are not sufficient to give a bound state.

Importantly, when $\mathbf{r}=\mathbf{q}^*$ any solution $v(x,k)$ of the scattering problem (2.1a) satisfies the relation

$$-i(k-k^*)\|v(x,k)\|^2=\frac{\partial}{\partial x}[|v^{(1)}(x,k)|^2-|v^{(2)}(x,k)|^2-|v^{(3)}(x,k)|^2]. \qquad (2.51)$$

Equation (2.51) shows that in order for $k=k_n$ to be an eigenvalue corresponding to a square integrable eigenfunction, k_n must be real (i.e., $k_n=k_n^*$). For $k_n\in\mathbb{R}$ with $|k_n|<q_0$ (i.e., for $z\in C_0$) one has $\lambda(k_n)=\pm i\sqrt{q_0^2-k_n^2}$ (with the upper/lower sign on sheet I/II of the Riemann surface, respectively). Correspondingly, $\phi_1(x,k_n,\lambda(k_n))$ and $\phi_3(x,k_n,\lambda(k_n))$ are exponentially decaying as $x\to-\infty$ while $\psi_1(x,k_n,\lambda(k_n))$ and $\psi_3(x,k_n,\lambda(k_n))$ are exponentially decaying as $x\to+\infty$. As we will see in Sec. II F 1, poles of the transmission coefficient at these points then give rise to bound states. It should be noted that, unlike the scalar case, the unitarity conditions [e.g., see Eq. (2.44)] are not enough to exclude poles of the transmission coefficients for $k\in\mathbb{R}$ with $|k|>q_0$ (i.e., for real values of z). In these cases, however, all eigenfunctions are oscillating as $x\to\pm\infty$. Hence, the only eigenvalues $k=k_n$ corresponding to square integrable eigenfunctions lie in the segment $(-q_0,q_0)$. In terms of the uniformization variable z, this means that *any* eigenfunctions belonging to $L_2(\mathbb{R})$ correspond to discrete eigenvalues on the circle C_0 of radius q_0. Therefore, if the scattering coefficients $a_{11}(z)$, etc., have a zero off the circle C_0, then the corresponding eigenfunctions cannot form a bound state, that is, either they are not decaying rapidly enough at both space infinities, or they are singular, which prevents the eigenfunction from being $L_2(\mathbb{R})$. We will see that both situations can in principle occur, the first case corresponding to zeros z_n of $a_{11}(z)$ inside the circle, while the second case to zeros outside the circle.

In order to locate discrete eigenvalues as it will apply to the inverse problem, it is convenient to introduce the 3×3 matrices

$$E_+(x,z)=(\phi_1,\chi,\psi_3),\quad E_-(x,z)=(\psi_1,\bar{\chi},\phi_3).$$

With this notation, $E_+(x,z)$ collects three eigenfunctions which are analytic in the upper-half z-plane, and $E_-(x,z)$ three eigenfunctions analytic in the lower-half-plane. Then we note that Eqs. (2.6), (2.25), and (2.27) together imply

$$\det(E_+(x,z))=\mathrm{Wr}(\phi_1(x,z),\chi(x,z),\psi_3(x,z))=-4q_0^2\lambda^2(z)a_{11}(z)b_{33}(z)e^{ik(z)x}, \qquad (2.52a)$$

$$\det(E_-(x,z))=\mathrm{Wr}(\psi_1(x,z),\bar{\chi}(x,z),\phi_3(x,z))=4q_0^2\lambda^2(z)a_{33}(z)b_{11}(z)e^{ik(z)x}. \qquad (2.52b)$$

Equation (2.52a) shows that the Wronskian vanishes (i.e., the three solutions which comprise E_+ become linearly dependent) at the zeros of $a_{11}(z)$ and $b_{33}(z)$. Due to the symmetries (2.38) and (2.42) among the scattering coefficients, however, we have

$$a_{11}(z_n)=0\Leftrightarrow b_{11}(z_n^*)=0\Leftrightarrow b_{33}(\hat{z}_n)=0\Leftrightarrow a_{33}(\hat{z}_n^*)=0 \qquad (2.53)$$

[where as before we used the notation (2.36c), i.e., $\hat{z}=q_0^2/z^*$]. If the zero z_n of $a_{11}(z)$ is on the circle C_0 of radius q_0, then $\hat{z}_n\equiv z_n$, and therefore $a_{11}(z)$ and $b_{33}(z)$ vanish at the same point. Hence, the Wronskian (2.52a) will have a double zero at $z=z_n$ in this case. However, if $a_{11}(z)$ admits a simple zero at a point $z=z_n$ off the circle C_0 (i.e., $|z_n|\neq q_0$ and $\mathrm{Im}\,z_n>0$), then such zeros appear in quartets (cf. Fig. 3), and the Wronskian (2.52a) will have a simple zero both at z_n and at $\hat{z}_n = q_0^2/z_n^*$ in the upper-half-plane. Similarly, with regard to Eq. (2.52b), $a_{33}(z)$ and $b_{11}(z)$ can either both vanish at the conjugate points $z_n^* \equiv \hat{z}_n^*$, on the lower-half circle, or vanish individually at two different points in the lower-half-plane ($\hat{z}_n^*$ and $z_n^* = q_0^2/z_n$, respectively). Hereafter we will use ζ_n to denote zeros of $a_{11}(z)$ *on the circle* C_0, and we will reserve the notation z_n for the zeros of $a_{11}(z)$ *off the circle* C_0.

1. Zeros on the circle

Let us first consider the case of zeros on the circle of radius q_0 and assume that $a_{11}(z)$ and $b_{33}(z)$ both have a simple zero at the point $z = \zeta_n = k_n + i\nu_n$, with $|k_n| < q_0$ and $\nu_n = \sqrt{q_0^2 - k_n^2} > 0$. As we noted earlier, in this case the Wronskians (2.52a) and (2.52b) each have a double zero, respectively, at $z = \zeta_n$ and at $z = \zeta_n^*$. In principle there are two possibilities: either $\chi(x, \zeta_n) = 0$ or $\chi(x, \zeta_n) \neq 0$. If $\chi(x, \zeta_n) = 0$, then also $\bar{\chi}(x, \zeta_n^*) = 0$, due to the symmetry (2.43b) (since in this case $\zeta_n^* = q_0^2/\zeta_n$). If $\chi(x, \zeta_n) \neq 0$ instead, one also has $\bar{\chi}(x, \zeta_n^*) \neq 0$. In the following we show that in fact it is always the case that $\chi(x, \zeta_n) = \chi(x, \zeta_n^*) = 0$.

Indeed, let ζ_n be a zero of $a_{11}(z)$ and $b_{33}(z)$ on the circle of radius q_0. Then, according to Eq. (2.21b), $\chi(x, \zeta_n) = 0$ if and only if $\phi_3^{\mathrm{ad}}(x, \zeta_n) \wedge \psi_1^{\mathrm{ad}}(x, \zeta_n) = 0$. Since $\phi_3^{\mathrm{ad}}(x, z)$ and $\psi_1^{\mathrm{ad}}(x, z)$ are eigenfunctions whose asymptotic behavior is fixed, they cannot vanish identically for all x. Hence for $\chi(x, \zeta_n)$ to be zero $\phi_3^{\mathrm{ad}}(x, \zeta_n)$ and $\psi_1^{\mathrm{ad}}(x, \zeta_n)$ must be proportional to each other. Then, due to the symmetry (2.29), it follows that $\phi_3(x, \zeta_n^*) \propto \psi_1(x, \zeta_n^*)$. Moreover, Eq. (2.43b) implies that $\chi(x, \zeta_n) = \bar{\chi}(x, \zeta_n^*)$, and therefore [recalling the definition (2.21a)] we conclude that

$$\chi(x, \zeta_n) = \bar{\chi}(x, \zeta_n^*) = 0 \quad \text{iff } \phi_3(x, \zeta_n^*) \propto \psi_1(x, \zeta_n^*) \text{ and } \phi_1(x, \zeta_n) \propto \psi_3(x, \zeta_n). \tag{2.54}$$

Suppose now that $\chi(x, \zeta_n) \neq 0$ [and hence also $\bar{\chi}(x, \zeta_n^*) \neq 0$]. If Eqs. (2.25a) and (2.27b) can be continued off the real z-axis, then it follows that

$$\chi(x, \zeta_n) \propto \psi_3(x, \zeta_n), \quad \chi(x, \zeta_n) \propto \phi_1(x, \zeta_n)$$

[with nonzero proportionality coefficients because by assumption $\chi(x, \zeta_n) \neq 0$]. If this is the case, then $\phi_1(x, \zeta_n) \propto \psi_3(x, \zeta_n)$, and [due to the symmetry (2.29)] one also has $\phi_1^{\mathrm{ad}}(x, \zeta_n^*) \propto \psi_3^{\mathrm{ad}}(x, \zeta_n^*)$. But then it follows that $\bar{\chi}(x, \zeta_n^*) = 0$, which contradicts the hypothesis. In conclusion, if ζ_n and ζ_n^* are a pair of zeros on the circle, then $\chi(x, \zeta_n) = \bar{\chi}(x, \zeta_n^*) = 0$, Eq. (2.54) holds and one can write

$$\phi_1(x, \zeta_n) = b_n^{(1)} \psi_3(x, \zeta_n), \tag{2.55a}$$

$$\phi_3(x, \zeta_n^*) = \bar{b}_n^{(1)} \psi_1(x, \zeta_n^*), \tag{2.55b}$$

corresponding to a bound state. Note that due to the symmetry (2.32a) between the eigenfunctions, from Eqs. (2.55) it follows that

$$\bar{b}_n^{(1)} = b_n^{(1)}. \tag{2.56}$$

Since $\chi(x, \zeta_n) = \bar{\chi}(x, \zeta_n^*) = 0$ for all zeros ζ_n of $a_{11}(z)$ and ζ_n^* of $a_{33}(z)$ on the circle of radius q_0, it is then natural in this case to rescale the Wronskians in Eq. (2.52a) as

$$\mathrm{Wr}\left(\phi_1(x, z), \frac{\chi(x, z)}{2\lambda(z) b_{33}(z)}, \psi_3(x, z) \right) = -2q_0^2 \lambda(z) a_{11}(z) e^{ik(z)x}, \tag{2.57a}$$

$$\mathrm{Wr}\left(\psi_1(x, z), \frac{\bar{\chi}(x, z)}{2\lambda(z) b_{11}(z)}, \phi_3(x, z) \right) = 2q_0^2 \lambda(z) a_{33}(z) e^{ik(z)x}. \tag{2.57b}$$

The rescaled Wronskians will then have simple zeros at ζ_n and ζ_n^*.

2. Zeros off the circle

Suppose that $a_{11}(z)$, which is analytic in the upper-half z-plane, has a simple zero at a point $z=z_n=k_n+i\nu_n$, with $\nu_n>0$ and $|z_n|\neq q_0$. First of all, note that, according to (2.36b),

$$k(z_n)=\tfrac{1}{2}[k_n(1+|\hat{z}_n|^2/q_0^2)+i\nu_n(1-|\hat{z}_n|^2/q_0^2)],$$

$$\lambda(z_n)=\tfrac{1}{2}[k_n(1-|\hat{z}_n|^2/q_0^2)+i\nu_n(1+|\hat{z}_n|^2/q_0^2)].$$

Thus, the behavior of $e^{ik(z_n)x}$ at large x depends on whether $|z_n|\lesseqgtr q_0$ (recall that $\hat{z}=q_0^2/z^*$; hence, one has $|\hat{z}_n|\lesseqgtr q_0$ for $|z_n|\gtreqless q_0$), and it will be exponentially decaying at one space infinity and exponentially growing at the other one. On the other hand, $\phi_1(x,z_n)\sim e^{-i\lambda(z_n)x}$ will be decaying as $x\to-\infty$ and $\psi_3(x,z_n)\sim e^{i\lambda(z_n)x}$ will be decaying as $x\to+\infty$ irrespective of whether z_n is inside or outside the circle.

If the zero z_n of $a_{11}(z)$ is off the circle C_0, then $b_{33}(z)$ will have a zero at point $\hat{z}_n=q_0^2/z_n^*\neq z_n$, but $b_{33}(z_n)\neq 0$ *in general*. Then from Eq. (2.52a) it follows that z_n is a simple zero of the Wronskian. In this case we assume that $\chi(x,z_n)$ does not vanish. Then from Eq. (2.25a) we deduce that

$$\chi(x,z_n)\sim -2i\lambda(z_n)b_{33}(z_n)\begin{pmatrix}0\\ \mathbf{q}_+^{\perp}\end{pmatrix}\exp\left\{\tfrac{1}{2}[ik_n(1+|z_n|^2/q_0^2)+\nu_n(1-|z_n|^2/q_0^2)]x\right\},\quad x\to+\infty, \tag{2.58}$$

where we note that the other contribution formally obtained from $b_{23}(z)\psi_3(x,z)$ in Eq. (2.25a) is exponentially small (and in any case, smaller than the contribution of the remaining term). On the other hand, from Eq. (2.27b) it follows that, at a zero z_n of $a_{11}(z)$, the eigenfunction $\chi(x,z_n)$ is proportional to $\phi_1(x,z_n)$:

$$\phi_1(x,z_n)=b_n^{(2)}\chi(x,z_n). \tag{2.59}$$

If $|z_n|>q_0$ (i.e., if the zero is *outside* the circle C_0), we would obtain a bound state, since the eigenfunctions $\chi(x,z_n)$ and $\phi_1(x,z_n)$ would be decaying at both space infinities. Therefore zeros of $a_{11}(z)$ outside the circle C_0 cannot occur for a smooth eigenfunction, since this would violate the eigenvalue relation (2.51). On the other hand, if $|z_n|<q_0$ (i.e., if the zero is *inside* C_0), the relation (2.59) still holds, but the eigenfunctions $\chi(x,z_n)$ and $\phi_1(x,z_n)$ will be exponentially growing as $x\to+\infty$, according to Eq. (2.58), and this does not contradict Eq. (2.51). Hence zeros z_n inside the circle C_0 are not forbidden.

Similarly, the Wronskian (2.52a) vanishes at the zero of $b_{33}(z)$ corresponding to z_n, that is [according to Eq. (2.53)], at the point $\hat{z}_n=q_0^2/z_n^*$. If z_n is inside the circle C_0 of radius q_0, then $\hat{z}_n$ will be outside the same circle, and *vice versa*. Also, in general $a_{11}(\hat{z}_n)\neq 0$, and consequently from Eq. (2.27b) it follows

$$\chi(x,\hat{z}_n)\sim -2i\lambda(\hat{z}_n)a_{11}(\hat{z}_n)\begin{pmatrix}0\\ \mathbf{q}_-^{\perp}\end{pmatrix}\exp\left\{\tfrac{1}{2}[ik_n(1+|z_n|^2/q_0^2)-\nu_n(1-|z_n|^2/q_0^2)]x\right\},\quad x\to-\infty. \tag{2.60}$$

From Eq. (2.25a), however, one deduces that $\chi(x,\hat{z}_n)$ is proportional to $\psi_3(x,\hat{z}_n)$,

$$\chi(x,\hat{z}_n)=\hat{b}_n^{(2)}\psi_3(x,\hat{z}_n). \tag{2.61}$$

Therefore, if $|z_n|>q_0$ (i.e., if $\hat{z}_n$ is *inside* C_0), this would be a bound state, since $\psi_3(x,\hat{z}_n)$ decays as $x\to+\infty$ and $\chi(x,\hat{z}_n)$ as $x\to-\infty$, according to Eq. (2.60). Hence, as before, this situation cannot occur for a smooth eigenfunction, in accordance with Eq. (2.51). On the other hand, if $|z_n|<q_0$ (i.e., if $\hat{z}_n$ is *outside* C_0), the eigenfunctions $\chi(x,z_n)$ and $\psi_3(x,z_n)$ are exponentially growing as $x\to+\infty$. Hence such situations do not contradict Eq. (2.51).

Finally, one has analogous results for the eigenfunctions in the lower-half-plane in correspondence to the points $z_n^* = k_n - i\nu_n$ [zeros of $b_{11}(z)$ off the circle C_0] and $\hat{z}_n^* = q_0^2/z_n$ [zeros of $a_{33}(z)$ off the circle C_0]. Specifically, at points z_n^*, where $b_{11}(z_n^*) = 0$, one has

$$\bar{\chi}(x, z_n^*) = \bar{b}_n^{(2)} \psi_1(x, z_n^*). \tag{2.62}$$

Again, if $|z_n| > q_0$, this would correspond to a bound state. On the other hand, if $|z_n| < q_0$ the eigenfunctions in (2.62) will be growing as $x \to -\infty$. Also, at points $q_0^2/z_n \equiv \hat{z}_n^*$, where $a_{33}(\hat{z}_n^*) = 0$, one has $\bar{\chi}(x, \hat{z}_n^*)$ proportional to $\phi_3(x, \hat{z}_n^*)$,

$$\phi_3(x, \hat{z}_n^*) = \check{b}_n^{(2)} \bar{\chi}(x, \hat{z}_n^*). \tag{2.63}$$

Summarizing, in the case of a pair of zeros z_n and $\hat{z}_n$ in the upper-half-plane such that $a_{11}(z_n) = 0$ and $b_{33}(\hat{z}_n) = 0$ (with z_n inside the circle C_0 of radius q_0 and $\hat{z}_n = q_0^2/z_n^*$ outside C_0), the eigenfunctions are related to each other as

$$\chi(x, \hat{z}_n) = \hat{b}_n^{(2)} \psi_3(x, \hat{z}_n), \tag{2.64a}$$

$$\phi_1(x, z_n) = b_n^{(2)} \chi(x, z_n) \equiv b_n^{(2)} \bar{\chi}(x, \hat{z}_n^*) \tag{2.64b}$$

[cf. Eqs. (2.61) and (2.59)], but neither $\chi(x, \hat{z}_n)$ nor $\phi_1(x, z_n)$ are bound states. In Eq. (2.64b) we used the symmetry (2.43b) to express $\chi(x, z)$ in terms of $\bar{\chi}(x, z)$. At the corresponding pair of zeros in the conjugate points in the lower half plane it is $a_{33}(\hat{z}_n^*) = 0$ and $b_{11}(z_n^*) = 0$, and one has the following relations:

$$\phi_3(x, \hat{z}_n^*) = \check{b}_n^{(2)} \bar{\chi}(x, \hat{z}_n^*) \equiv -b_n^{(2)} \bar{\chi}(x, \hat{z}_n^*), \tag{2.64c}$$

$$\bar{\chi}(x, z_n^*) = \bar{b}_n^{(2)} \psi_1(x, \hat{z}_n^*) \equiv -\hat{b}_n^{(2)} \psi_1(x, z_n^*) \tag{2.64d}$$

[cf. Eqs. (2.63) and (2.62)], where we have used the symmetries (2.43) for the eigenfunctions in order to express the proportionality constants in terms those appearing in Eqs. (2.64).

Finally, it should be noted that there is no conceptual difference between the interior and the exterior of the circle C_0. The reason why the z_n are only allowed to be inside C_0 is because they are defined as the zeros of $a_{11}(z)$. One could equivalently define z_n as the zeros of $b_{33}(z)$ (which amounts to switching $z_n \leftrightarrow \hat{z}_n$), in which case one would obtain that z_n are only allowed to be outside C_0.

G. Symmetries in the norming constants

Eigenvalues on the circle: We first consider a pair of zeros $\{\zeta_n, \zeta_n^*\}$ on the circle C_0 of radius q_0. At these points, Eqs. (2.55) hold, with $\bar{b}_n^{(1)} = b_n^{(1)}$, according to Eq. (2.56). Moreover, from symmetries Eqs. (2.41a) and (2.55) if follows

$$\psi_3^*(x, \zeta_n) = (1/b_n^{(1)})^* \phi_1^*(x, \zeta_n) = -(1/b_n^{(1)})^* e^{-ik(\zeta_n^*)x} \mathbf{J}(\phi_2(x, \zeta_n^*) \wedge \phi_3(x, \zeta_n^*)) / \Gamma_1(\zeta_n^*)$$

and, on the other hand, Eq. (2.41b) implies

$$\psi_3^*(x, \zeta_n) = -e^{-ik(\zeta_n^*)x} \mathbf{J}(\psi_1(x, \zeta_n^*) \wedge \psi_2(x, \zeta_n^*)) / \Gamma_3(\zeta_n^*).$$

Then observe that from Eqs. (2.26) and (2.28) it follows

$$\psi_1(x, \zeta_n^*) \wedge \psi_2(x, \zeta_n^*) = -\frac{1}{2\lambda(\zeta_n^*)} \psi_1(x, \zeta_n^*) \wedge \frac{\bar{\chi}(x, \zeta_n^*)}{b_{11}(\zeta_n^*)}, \tag{2.65a}$$

$$\phi_2(x,\zeta_n^*) \wedge \phi_3(x,\zeta_n^*) = -\frac{1}{2\lambda(\zeta_n^*)}\frac{\bar{\chi}(x,\zeta_n^*)}{a_{33}(\zeta_n^*)} \wedge \phi_3(x,\zeta_n^*), \tag{2.65b}$$

and from the discussion in Sec. II F 1 we have $\bar{\chi}(x,\zeta_n^*)/a_{33}(\zeta_n^*) \neq 0$. Back-substituting, we finally obtain

$$(b_n^{(1)})^* = -\frac{\Gamma_3(\zeta_n^*)}{\Gamma_1(\zeta_n^*)}\frac{b_{11}(\zeta_n^*)}{a_{33}(\zeta_n^*)}\bar{b}_n^{(1)}.$$

In order to simplify the above relation, first note that from Eq. (2.39) it follows $-\Gamma_1(\zeta_n^*)/\Gamma_3(\zeta_n^*) = \zeta_n/\zeta_n^*$. For the special case of reflectionless potential, with only one pair of eigenvalues (zeros) $\{\zeta_n,\zeta_n^*\}$ on the circle C_0 of radius q_0, one has

$$a_{11}(z) = \frac{z-\zeta_n}{z-\zeta_n^*}, \quad a_{33}(z) = a_{11}(\hat{z}^*) = \frac{\zeta_n}{\zeta_n^*}\cdot\frac{z-\zeta_n^*}{z-\zeta_n}, \quad b_{11}(z) = a_{11}^*(z^*) = \frac{z-\zeta_n^*}{z-\zeta_n}$$

so that $b_{11}(z)/a_{33}(z) \equiv \zeta_n^*/\zeta_n$. In conclusion, one has

$$(b_n^{(1)})^* = \left(\frac{\zeta_n^*}{\zeta_n}\right)^2 \bar{b}_n^{(1)} \equiv \left(\frac{\zeta_n^*}{\zeta_n}\right)^2 b_n^{(1)} \tag{2.66}$$

which, in particular, implies that

$$\frac{\zeta_n^*}{\zeta_n} b_n^{(1)} \in \mathbb{R}.$$

Eigenvalues off the circle: We now consider the case of zeros off the circle C_0, and establish a relation between the norming constants $b_n^{(2)}$ and $\hat{b}_n^{(2)}$ in Eq. (2.64). Recall that

$$\chi(x,\hat{z}_n) = \hat{b}_n^{(2)}\psi_3(x,\hat{z}_n), \quad \phi_1(x,z_n) = b_n^{(2)}\chi(x,z_n)$$

and, instead of the second relation, we could as well make use of the symmetry relations (2.43) and consider

$$\phi_3(x,\hat{z}_n^*) = -b_n^{(2)}\bar{\chi}(x,\hat{z}_n^*). \tag{2.67}$$

Then we can write

$$\chi^*(x,\hat{z}_n) = (\hat{b}_n^{(2)})^*\psi_3^*(x,\hat{z}_n) = -\frac{1}{2\lambda(\hat{z}_n^*)\Gamma_3(\hat{z}_n^*)b_{11}(\hat{z}_n^*)}\frac{(\hat{b}_n^{(2)})^*}{b_n^{(2)}}\chi^*(x,\hat{z}_n)$$

[where Eqs. (2.41b), the first of Eqs. (2.26) and Eqs. (2.67), (2.37), and (2.21b) were used in turn]. As a result we obtain

$$(\hat{b}_n^{(2)})^* = -2\lambda(\hat{z}_n^*)\Gamma_3(\hat{z}_n^*)b_{11}(\hat{z}_n^*)b_n^{(2)}. \tag{2.68}$$

The previous relation can be simplified by taking into account that $-2\lambda(\hat{z}_n^*) = (z_n^2 - q_0^2)/z_n$ and $\Gamma_3(\hat{z}_n^*) = q_0^2/z_n$, and that in the reflectionless case, with only one quartet of eigenvalues $\{z_n, z_n^*, \hat{z}_n, \hat{z}_n^*\}$ (cf. Fig. 3), one has

$$b_{11}(z) = \frac{z-z_n^*}{z-z_n}, \quad b_{11}(\hat{z}_n^*) = \frac{q_0^2 - |z_n|^2}{q_0^2 - z_n^2}$$

(again recall $\hat{z} = q_0^2/z^*$), so that

$$(\hat{b}_n^{(2)})^* = \frac{q_0^2}{z_n^2}(|z_n|^2 - q_0^2)b_n^{(2)}. \tag{2.69}$$

III. INVERSE PROBLEM

In order to formulate the inverse scattering in terms of a Riemann-Hilbert (RH) problem, one needs a representation of eigenfunctions that are meromorphic in the upper-half z-plane in terms of a combination of eigenfunctions that are meromorphic in the lower-half-plane via suitably defined jump conditions. In this case one employs the two sets of analytic eigenfunctions $E_+(x,z) = (\phi_1, \chi, \psi_3)$ and $E_-(x,z)=(\psi_1, \bar{\chi}, \phi_3)$, which have already been used in Sec. II F. One then uses Eqs. (2.15a), which define the scattering coefficients, together with Eq. (2.25a) [which gives $\chi(x,z)$ in terms of $\psi_2(x,z)$ and $\psi_3(x,z)$] and Eqs. (2.26) [which give $\psi_2(x,z)$ in terms of $\psi_1(x,z)$ and $\bar{\chi}(x,z)$], to obtain for all $z \in \mathbb{R}$,

$$\frac{\phi_3(x,z)}{a_{33}(z)}e^{-i\lambda(z)x} = \psi_3(x,z)e^{-i\lambda(z)x} - \left[\frac{b_{31}(z)}{b_{11}(z)}\psi_1(x,z) + \frac{a_{32}(z)}{a_{33}(z)}\frac{\bar{\chi}(x,z)}{2\lambda(z)b_{11}(z)}\right]e^{-i\lambda(z)x}, \tag{3.1a}$$

$$\frac{\phi_1(x,z)}{a_{11}(z)}e^{i\lambda(z)x} = \psi_1(x,z)e^{i\lambda(z)x} + \left[\frac{a_{12}(z)}{a_{11}(z)}\frac{\chi(x,z)}{2\lambda(z)b_{33}(z)} - \frac{b_{13}(z)}{b_{33}(z)}\psi_3(x,z)\right]e^{i\lambda(z)x}, \tag{3.1b}$$

$$\frac{\chi(x,z)}{2\lambda(z)b_{33}(z)}e^{-ik(z)x} = -\frac{\bar{\chi}(x,z)}{2\lambda(z)b_{11}(z)}e^{-ik(z)x} + \left[\frac{b_{21}(z)}{b_{11}(z)}\psi_1(x,z) - \frac{b_{23}(z)}{b_{33}(z)}\psi_3(x,z)\right]e^{-ik(z)x}. \tag{3.1c}$$

Note that in the equations above we have used the relation $\mathbf{A}(z)=\mathbf{B}(z)^{-1}$ among the scattering coefficients. Recalling the symmetries (2.42) and (2.43b), the system of Eqs. (3.1) can be written as

$$\frac{\phi_3(x,z)}{a_{33}(z)}e^{-i\lambda(z)x} = \psi_3(x,z)e^{-i\lambda(z)x} - \left[\rho_1(z)\psi_1(x,z) - \rho_2(\hat{z}^*)\frac{\bar{\chi}(x,z)}{2\lambda(z)b_{11}(z)}\right]e^{-i\lambda(z)x}, \tag{3.2a}$$

$$\frac{\phi_1(x,z)}{a_{11}(z)}e^{i\lambda(z)x} = \psi_1(x,z)e^{i\lambda(z)x} - \left[\rho_1(\hat{z}^*)\psi_3(x,z) - \rho_2(z)\frac{\bar{\chi}(x,\hat{z}^*)}{2\lambda(z)b_{11}(\hat{z}^*)}\right]e^{i\lambda(z)x}, \tag{3.2b}$$

$$\frac{\chi(x,z)}{2\lambda(z)b_{33}(z)}e^{-ik(z)x} = -\frac{\bar{\chi}(x,z)}{2\lambda(z)b_{11}(z)}e^{-ik(z)x} + [\bar{\rho}_2(z)\psi_1(x,z) + \bar{\rho}_2(\hat{z}^*)\psi_3(x,z)]e^{-ik(z)x}, \tag{3.2c}$$

where again $\hat{z}=q_0^2/z^*$, and where we have introduced the analogs of reflection coefficients

$$\rho_1(z) = \frac{b_{31}(z)}{b_{11}(z)}, \quad \rho_2(z) = \frac{a_{12}(z)}{a_{11}(z)}, \quad \bar{\rho}_2(z) = \frac{b_{21}(z)}{b_{11}(z)}. \tag{3.3}$$

Note that only two of the above three coefficients are independent, since according to Eq. (2.40) one has

$$\bar{\rho}_2^*(z^*) = \frac{q_0^2}{q_0^2 - z^2}\rho_2(z). \tag{3.4}$$

A. Riemann-Hilbert problem

The system of Eqs. (3.2) can be considered as a generalized matrix Riemann-Hilbert problem on the real z-axis in the variables z, $\hat{z}^* \equiv q_0^2/z$, with poles in correspondence with the zeros of $a_{11}(z)$ and $b_{33}(z)$ in the upper-half-plane, as well as the zeros of $b_{11}(z)$ and $a_{33}(z)$ in the lower-half-plane. The next task is to solve the above RH problem by expressing the solutions in terms of a linear system of algebraic-integral equations.

Let us consider the first equation, namely Eq. (3.2a). From the asymptotic expansions (2.46) it follows

$$\frac{\phi_3(x,z)}{a_{33}(z)}e^{-i\lambda(z)x} \sim \psi_3(x,z)e^{-i\lambda(z)x} \sim \begin{pmatrix} 0 \\ -i\mathbf{r}_+ \end{pmatrix}, \quad z \to \infty, \tag{3.5a}$$

$$\frac{\phi_3(x,z)}{a_{33}(z)}e^{-i\lambda(z)x} \sim \psi_3(x,z)e^{-i\lambda(z)x} \sim \begin{pmatrix} -\hat{z}^* \\ -i\mathbf{r}(x) \end{pmatrix}, \quad z \to 0. \tag{3.5b}$$

Therefore, in Eq. (3.2a) we subtract from both sides the behavior at infinity and the pole at zero (which are the same for the left-hand side and the first term on the right-hand side). Also, note that the left-hand side is meromorphic in the lower-half-plane, with (simple) poles at the zeros of $a_{33}(z)$ (which we have denoted by $\zeta_n^*, \hat{z}_n^*$), while the first term on the right-hand side is analytic in the upper-half-plane. Hence, we also subtract from both sides of the equation the residues at the poles. We then introduce the Cauchy projectors,

$$P_\pm(f)(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - (z \pm i0)}\mathrm{d}\zeta, \tag{3.6}$$

which are well defined for any function $f(\zeta)$ that is integrable on the real line (e.g., see Ref. 23). Applying P_+ to Eq. (3.2a) after the above-mentioned subtractions, we then get

$$\psi_3(x,z)e^{-i\lambda(z)x} = -\begin{pmatrix} \hat{z}^* \\ i\mathbf{r}_+ \end{pmatrix} + \sum_{\substack{n=1 \\ |\zeta_n|=q_0}}^{N_1} \frac{\phi_3(x,\zeta_n^*)e^{-i\lambda(\zeta_n^*)x}}{a'_{33}(\zeta_n^*)(z-\zeta_n^*)} + \sum_{\substack{n=1 \\ |z_n|<q_0}}^{N_2} \frac{\phi_3(x,\hat{z}_n^*)e^{-i\lambda(\hat{z}_n^*)x}}{a'_{33}(\hat{z}_n^*)(z-\hat{z}_n^*)} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{\zeta-(z+i0)}\left[\rho_1(\zeta)\psi_1(x,\zeta) - \rho_2(\hat{\zeta}^*)\frac{\bar{\chi}(x,\zeta)}{2\lambda(\zeta)b_{11}(\zeta)}\right]e^{-i\lambda(\zeta)x}, \tag{3.7}$$

where N_1 and N_2 are, respectively, the number of zeros ζ_n of $a_{11}(z)$ on the circle C_0 of radius q_0 and of zeros z_n inside the circle C_0 (cf. section II F 2). Regarding the contribution of the discrete spectrum we now take into account that for any zero ζ_n on the circle C_0, according to Eq. (2.55b) we can write $\phi_3(x,\zeta_n^*)=\bar{b}_n^{(1)}\psi_1(x,\zeta_n^*)$, while for any zero z_n off the circle C_0, Eq. (2.64c) gives $\phi_3(x,\hat{z}_n^*)=-b_n^{(2)}\bar{\chi}(x,\hat{z}_n^*)$. Therefore, from Eq. (3.7) we obtain

$$\psi_3(x,z)e^{-i\lambda(z)x} = -\begin{pmatrix} \hat{z}^* \\ i\mathbf{r}_+ \end{pmatrix} + \sum_{\substack{n=1 \\ |\zeta_n|=q_0}}^{N_1} \bar{C}_n^{(1)}\frac{\psi_1(x,\zeta_n^*)e^{-i\lambda(\zeta_n^*)x}}{z-\zeta_n^*} + \sum_{\substack{n=1 \\ |\zeta_n|<q_0}}^{N_2} C_n^{(2)}\frac{\bar{\chi}(x,\hat{z}_n^*)e^{-i\lambda(\hat{z}_n^*)x}}{\hat{z}_n^*(z-\hat{z}_n^*)} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{\zeta-(z+i0)}\left[\rho_1(\zeta)\psi_1(x,\zeta) - \rho_2(\hat{\zeta}^*)\frac{\bar{\chi}(x,\zeta)}{2\lambda(\zeta)b_{11}(\zeta)}\right]e^{-i\lambda(\zeta)x}. \tag{3.8a}$$

In a similar way one can treat Eqs. (3.2b) and (3.2c). Applying a projector P_- and using (2.55a), (2.64b), and (2.64d) yields in these cases

$$\psi_1(x,z)e^{i\lambda(z)x} = \begin{pmatrix} z \\ i\mathbf{r}_+ \end{pmatrix} + \sum_{\substack{n=1 \\ |\zeta_n|=q_0}}^{N_1} \frac{z}{z-\zeta_n} C_n^{(1)} \psi_3(x,\zeta_n) e^{i\lambda(\zeta_n)x} + \sum_{\substack{n=1 \\ |\zeta_n|<q_0}}^{N_2} \frac{z}{z-z_n} C_n^{(2)} \bar{\chi}(x,\hat{z}_n^*) e^{i\lambda(z_n)x}$$
$$- \frac{z}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{\zeta-(z-i0)} \left[\rho_1(\hat{\zeta}^*)\psi_3(x,\zeta) - \rho_2(\zeta)\frac{\bar{\chi}(x,\hat{\zeta}^*)}{2\lambda(\zeta)b_{11}(\hat{\zeta}^*)} \right] \frac{e^{i\lambda(\zeta)x}}{\zeta}, \tag{3.8b}$$

$$\frac{\bar{\chi}(x,z)e^{-ik(z)x}}{2\lambda(z)b_{11}(z)} = \begin{pmatrix} 0 \\ i\mathbf{q}_+^{\perp} \end{pmatrix} - \sum_{\substack{n=1 \\ |z_n|<q_0}}^{N_2} \bar{C}_n^{(2)} \psi_1(x,z_n^*) e^{-ik(z_n^*)x} \frac{z}{(z-\hat{z}_n)(z-z_n^*)}$$
$$- \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{\zeta-(z-i0)} [\bar{\rho}_2(\zeta)\psi_1(x,\zeta) + \bar{\rho}_2(\hat{\zeta}^*)\psi_3(x,\zeta)] e^{-ik(\zeta)x}. \tag{3.8c}$$

Note that in Eqs. (3.8) we have introduced the norming constants

$$\bar{C}_n^{(1)} = \frac{\bar{b}_n^{(1)}}{a_{33}'(\zeta_n^*)}, \quad C_n^{(1)} = \frac{b_n^{(1)}}{\zeta_n a_{11}'(\zeta_n)}, \quad C_n^{(2)} = \frac{b_n^{(2)}}{z_n a_{11}'(z_n)}, \quad \bar{C}_n^{(2)} = -\frac{\bar{b}_n^{(2)}}{z_n^* b_{11}'(z_n^*)} \equiv \frac{\hat{b}_n^{(2)}}{z_n^* b_{11}'(z_n^*)}. \tag{3.9}$$

From the symmetry (2.42), whenever $|\zeta_n| = q_0$, it follows

$$a_{11}'(\zeta_n) = -\frac{q_0^2}{\zeta_n^2} a_{33}'(\zeta_n^*) = -\frac{\zeta_n^*}{\zeta_n} a_{33}'(\zeta_n^*). \tag{3.10}$$

Hence, recalling Eq. (3.9) and (2.56), one has

$$\bar{C}_n^{(1)} = \frac{\bar{b}_n^{(1)}}{a_{33}'(\zeta_n^*)} = -\frac{b_n^{(1)}\zeta_n^*}{\zeta_n a_{11}'(\zeta_n)} \equiv -\zeta_n^* C_n^{(1)}. \tag{3.11a}$$

Also, symmetry (2.38) implies $(b_{11}'(z_n^*))^* = a_{11}'(z_n)$ and therefore

$$(\bar{C}_n^{(2)})^* = \frac{q_0^2}{z_n^2}(|z_n|^2 - q_0^2) C_n^{(2)}. \tag{3.11b}$$

Equations (3.8) are the fundamental equations for the inverse scattering problem. They contain the $N_1 + N_2$ independent (complex) norming constants $C_n^{(1)}$ and $C_n^{(2)}$. In the absence of discrete eigenvalues (that is, when $N_1 = N_2 = 0$), Eqs. (3.8) are a linear system of three vector integral equations for the three eigenfunctions $\psi_1(x,z)$, $\psi_3(x,z)$, and $\bar{\chi}(x,z)$. In general (that is, when $N_1 \neq 0$ or $N_2 \neq 0$), the system is consistently closed by evaluating the first equation at $z = \zeta_n$, for $n = 1, \ldots, N_1$, the second at $z = \zeta_n^*$ for $n = 1, \ldots, N_1$ and $z = z_n^*$ for $n = 1, \ldots, N_2$ and the last one at $z = \hat{z}_n^*$, $n = 1, \ldots, N_2$.

It should be noted that, using the WKB expansions for the eigenfunctions (see the Appendix) and the Wronskian relations for the scattering coefficients, one can show that the reflection coefficients (3.3) decay as appropriate powers of z both as $z \to 0$ and as $z \to \infty$ so as to make the integrals in Eqs. (3.8) convergent.

B. Trace formula

From the definition of the reflection coefficients (3.3) and the symmetries (2.38), we can write Eq. (2.44) as

$$|a_{11}(z)|^{-2} = 1 - \frac{z^2}{q_0^2}|\rho_1(z)|^2 - \frac{q_0^2}{z^2 - q_0^2}|\rho_2(z)|^2. \tag{3.12}$$

Recall that $a_{11}(z)$ is analytic in the upper-half z-plane, with $a_{11}(z) \sim 1$ as $|z| \to \infty$, and that it has (simple) zeros at the points $\{\zeta_n\}_{n=1}^{N_1}$ on the circle C_0 of radius q_0, and $\{z_n\}_{n=1}^{N_2}$ off the circle C_0. Therefore, assuming that it does not vanish for any $z \in \mathbb{R}$, one can explicitly write

$$a_{11}(z) = \prod_{n=1}^{N_1} \frac{z - \zeta_n}{z - \zeta_n^*} \prod_{n=1}^{N_2} \frac{z - z_n}{z - z_n^*} \exp\left\{ -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log[1 - \zeta^2|\rho_1(\zeta)|^2/q_0^2 - q_0^2|\rho_2(\zeta)|^2/(\zeta^2 - q_0^2)]}{\zeta - z} \mathrm{d}\zeta \right\}. \tag{3.13}$$

The scattering coefficients $a_{33}(z)$, $b_{11}(z)$, and $b_{33}(z)$ can obviously be obtained from $a_{11}(z)$ by symmetry [cf. Eqs. (2.38) and (2.42)]. In fact, it is worth noting that all other entries in the scattering matrix $\mathbf{A}(z) = (a_{ij}(z))$ and its inverse $\mathbf{B}(z) = (b_{ij}(z))$ can be reconstructed in terms of the reflection coefficients (3.3) and of the elements of the discrete spectrum, once the symmetries (2.38) and (2.42) are taken into account. In this sense, the reflection coefficients (3.3), together with the discrete eigenvalues and relative norming constants, constitute a minimal set of scattering data.

We also mention that from the asymptotic behavior (2.48b) of $a_{11}(z)$ as $z \to 0$, the following relation between the scattering data and the asymptotic phase differences $\Delta\theta = \theta_+^{(j)} - \theta_-^{(j)}$ in the potentials can be obtained:

$$e^{i\Delta\theta} = \prod_{n=1}^{N_1} \frac{\zeta_n}{\zeta_n^*} \prod_{n=1}^{N_2} \frac{z_n}{z_n^*} \exp\left\{ -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log[1 - \zeta^2|\rho_1(\zeta)|^2/q_0^2 - q_0^2|\rho_2(\zeta)|^2/(\zeta^2 - q_0^2)]}{\zeta} \mathrm{d}\zeta \right\}. \tag{3.14}$$

Equation (3.14) is the analog of the Θ-condition that was obtained in Ref. 11 for the scalar NLS equation.

IV. TIME EVOLUTION

Equation (2.1b) fixes the time evolution of eigenfunctions and scattering data, as well as the asymptotic phases of the potential. Thus, asymptotically, the time dependence of the eigenfunctions is given by

$$\frac{\partial v}{\partial t} \sim \begin{pmatrix} 2ik^2 + iq_0^2 & -2k\mathbf{q}_\pm^T \\ -2k\mathbf{r}_\pm & -2ik^2\mathbf{I}_2 - i\mathbf{r}_\pm\mathbf{q}_\pm^T \end{pmatrix} v \quad \text{as } x \to \pm\infty. \tag{4.1}$$

The eigenfunctions $\phi_j(x,t,k)$ and $\psi_j(x,t,k)$ however are defined at all times t by the asymptotic behavior in Eqs. (2.4) as $x \to \pm\infty$. Those boundary conditions are not compatible with the time evolution prescribed by Eq. (2.1b). To determine the time evolution of $\phi_j(x,t,k)$ and $\psi_j(x,t,k)$, one can introduce modified eigenfunctions which are simultaneously solutions of the x and t part of the Lax pair. For instance, let $\tilde{\phi}_1(x,t,k) = e^{i\omega_\infty^{(1)}t}\phi_1(x,t,k)$, so that

$$\frac{\partial \tilde{\phi}_1}{\partial t} = i\omega_\infty^{(1)}\tilde{\phi}_1 + e^{i\omega_\infty^{(1)}t}\frac{\partial \phi_1}{\partial t}. \tag{4.2}$$

Requiring that $\tilde{\phi}_1(x,t,k)$ be a solution of the time-differential Eq. (2.1b) [and hence, asymptotically as $x\to-\infty$, of Eq. (4.1) with the lower sign], and recalling that $r_-^{(j)}(t)$ depends on t via the phase $\theta_-^{(j)}(t)$ [cf. (2.3)], one then obtains from (2.4) and (2.5),

$$\phi_1 \sim \begin{pmatrix} \lambda+k \\ i\mathbf{r}_-(t) \end{pmatrix} e^{-i\lambda x}, \quad \frac{\partial \phi_1}{\partial t} \sim \begin{pmatrix} 0 \\ \dot{\Theta}_-(t) i\mathbf{r}_-(t) \end{pmatrix} e^{-i\lambda x}, \quad \frac{\partial \phi_1}{\partial x} \sim -i\lambda \begin{pmatrix} \lambda+k \\ i\mathbf{r}_-(t) \end{pmatrix} e^{-i\lambda x},$$

where $\dot{\Theta}_\pm(t)=\mathrm{diag}(\dot{\theta}_\pm^{(1)}(t),\dot{\theta}_\pm^{(2)}(t))$ and the dot denotes differentiation with respect to time. Substituting these into Eq. (4.1) and looking at each of the three components of $\tilde{\phi}_1(x,t,k)$ we then obtain, respectively, from each component,

$$\omega_\infty^{(1)} = 2k\lambda + q_0^2 = \dot{\theta}_-^{(1)} - q_0^2 + 2k\lambda = \dot{\theta}_-^{(2)} - q_0^2 + 2k\lambda.$$

In order for these three expressions to be compatible, it is necessary that $\dot{\theta}_-^{(1)}(t)=\dot{\theta}_-^{(2)}(t)=2q_0^2$, that is,

$$\theta_-^{(j)}(t) = \theta_-^{(j)} + 2q_0^2 t, \quad j=1,2, \tag{4.3}$$

which completely fixes the time evolution of the asymptotic phases $\theta_-^{(j)}$ for the potential. In a similar way one can obtain the evolution of the asymptotic phases as $x\to+\infty$ to show that

$$\theta_\pm^{(j)}(t) = \theta_\pm^{(j)}(0) + 2q_0^2 t, \quad j=1,2. \tag{4.4}$$

[Note that Eq. (4.4) can also be obtained directly from the asymptotics of the VNLS Eq. (1.3) as $x\to\pm\infty$.] Moreover, one finds that all of the eigenfunctions $\phi_j(x,t,k)$ and $\psi_j(x,t,k)$ satisfy a modified version of Eq. (2.1b),

$$\frac{\partial v_j}{\partial t} = \begin{pmatrix} 2ik^2 + i\mathbf{q}^T\mathbf{r} & -2k\mathbf{q}^T - i\mathbf{q}_x^T \\ -2k\mathbf{r} + i\mathbf{r}_x & -2ik^2\mathbf{I}_2 - i\mathbf{r}\mathbf{q}^T \end{pmatrix} v_j - i\omega_\infty^{(j)} v_j, \tag{4.5}$$

where $\omega_\infty=\mathrm{diag}(\omega_\infty^{(1)},\omega_\infty^{(2)},\omega_\infty^{(3)})$, and

$$(\omega_\infty^{(1)},\omega_\infty^{(2)},\omega_\infty^{(3)}) = (2k\lambda + q_0^2, -2k^2 - 2q_0^2, q_0^2 - 2k\lambda).$$

Differentiating the scattering equations (2.15a) with respect to t and taking into account Eq. (4.5), one then obtains the time evolution of the elements of the scattering matrix $\mathbf{A}$,

$$\frac{\partial a_{j\ell}}{\partial t} = i(\omega_\infty^{(\ell)} - \omega_\infty^{(j)}) a_{j\ell}, \quad j,\ell = 1,2,3. \tag{4.6}$$

From Eq. (4.6) it follows immediately that all the diagonal elements $a_{\ell\ell}(k)$ of the scattering matrix are time independent. Since $a_{11}(k)$ and $a_{33}(k)$ [as well as $b_{11}(k)$ and $b_{33}(k)$, which are related to the previous ones by symmetries (2.31)] are constants of the motion, the eigenvalues k_n, being the zeros of $a_{11}(k)$, are also time independent. The same holds for the zeros of $a_{33}(k)$. It is convenient to write explicitly the time dependence of the off-diagonal scattering coefficients

$$a_{13}(k,t) = e^{-4ik\lambda t} a_{13}(k,0), \quad a_{31}(k,t) = e^{4ik\lambda t} a_{31}(k,0), \tag{4.7a}$$

$$a_{23}(k,t) = e^{2i(k^2 - k\lambda + q_0^2)t + iq_0^2 t} a_{23}(k,0), \quad a_{32}(k,t) = e^{-2i(k^2 - k\lambda + q_0^2)t - iq_0^2 t} a_{32}(k,0), \tag{4.7b}$$

$$a_{12}(k,t) = e^{-2i(k^2+k\lambda+q_0^2)t-iq_0^2t}a_{12}(k,0), \quad a_{21}(k,t) = e^{2i(k^2+k\lambda+q_0^2)t+iq_0^2t}a_{21}(k,0). \tag{4.7c}$$

The evolution of the coefficient $b_{j\ell}(k,t)$ is the same as that of the $a_{j\ell}(k,t)$.

In a similar way one can determine the time dependence of the norming constants. Indeed, differentiating (2.55a) and (2.55b) and taking into account Eq. (4.5) we get for an eigenvalue $\zeta_n = k_n + i\nu_n$ with $|\zeta_n| = q_0$,

$$b_n^{(1)}(t) = b_n^{(1)}(0)e^{4k_n\nu_n t}, \quad \bar{b}_n^{(1)}(t) = \bar{b}_n^{(1)}(0)e^{4k_n\nu_n t}. \tag{4.8}$$

Therefore, according to the definitions (3.9)

$$\bar{C}_n^{(1)}(t) = \bar{C}_n^{(1)}(0)e^{4k_n\nu_n t}, \quad C_n^{(1)}(t) = C_n^{(1)}(0)e^{4k_n\nu_n t}, \quad n = 1, \ldots, N_1. \tag{4.9}$$

Similarly, for eigenvalues z_n and $\hat{z}_n$ off the circle (cf. Fig. 3), Eqs. (2.61) and (2.59) yield

$$b_n^{(2)}(t) = b_n^{(2)}(0)\exp[-i(z_n^2 + 4q_0^2)t], \quad \hat{b}_n^{(2)}(t) = \hat{b}_n^{(2)}(0)\exp[i((z_n^*)^2 + 4q_0^2)t], \quad n = 1, \ldots, N_2. \tag{4.10}$$

A. Conserved quantities

According to Eq. (4.6), the scattering coefficient $a_{11}(z)$ is time independent. Since $a_{11}(z)$ is analytic in the upper-half z-plane and $a_{11}(z) \to 1$ as $z \to \infty$, it admits an asymptotic Laurent series expansion whose coefficients are constants of motion. Similarly, the coefficients of the Taylor series expansion of $a_{11}(z)$ about $z=0$ are constant of the motion as well. Moreover, one can write the following expansions of the modified eigenfunction $M_1(x,z)$:

$$M_1^{(j)}(x,z) = zM_{1,\infty}^{(j,-1)}(x) + M_{1,\infty}^{(j,0)}(x) + \frac{1}{z}M_{1,\infty}^{(j,1)}(x) + \frac{1}{z^2}M_{1,\infty}^{(j,2)}(x) + \cdots, \quad j = 1,2,3 \tag{4.11}$$

as $z \to \infty$, and

$$M_1^{(j)}(x,z) = M_{1,0}^{(j,0)}(x) + zM_{1,0}^{(j,1)}(x) + z^2M_{1,0}^{(j,2)}(x) + \cdots, \quad j = 1,2,3 \tag{4.12}$$

as $z \to 0$. Substituting Eqs. (4.11) and (4.12) in Eq. (2.16), we can then obtain two infinite sets of conserved quantities:

$$I_m = M_{1,\infty}^{(1,m)}(+\infty) + iq_+^{(1)}M_{1,\infty}^{(2,m-1)}(+\infty) + iq_+^{(2)}M_{1,\infty}^{(3,m-1)}(+\infty), \quad m = 0,1,2, \ldots, \tag{4.13a}$$

$$K_m = M_{1,0}^{(1,m-2)}(+\infty) + iq_+^{(1)}M_{1,0}^{(2,m)}(+\infty) + iq_+^{(2)}M_{1,0}^{(3,m)}(+\infty), \quad m = 1,2, \ldots, \tag{4.13b}$$

where

$$M_{1,\infty}^{(j,m)}(+\infty) = \lim_{x\to+\infty} M_{1,\infty}^{(j,m)}(x), \quad M_{1,0}^{(j,m)}(+\infty) = \lim_{x\to+\infty} M_{1,0}^{(j,m)}(x), \quad j = 1,2,3$$

and where $M_{1,\infty}^{(2,-1)}(+\infty)$, $M_{1,\infty}^{(3,-1)}(+\infty)$ and $M_{1,0}^{(1,-1)}(+\infty)$, $M_{1,0}^{(1,-2)}(+\infty)$ are all assumed to be identically zero.

The first few coefficients of the asymptotic expansions (4.11) and (4.12) are computed in the Appendix, by means of a WKB expansion. Taking into account (1.2) and (1.3) and (1.10), (1.11), we can write explicitly the first few conserved quantities in (4.13). From Eq. (4.13a) we have

$$I_0 = \int_{-\infty}^{\infty} (\|\mathbf{q}(x,t)\|^2 - q_0^2)\mathrm{d}x, \quad I_1 = \int_{-\infty}^{\infty} \mathbf{q}^T(x,t)\mathbf{r}_x(x,t)\mathrm{d}x, \tag{4.14a}$$

$$I_2 = \int_{-\infty}^{\infty} [\mathbf{q}^T(x,t)\mathbf{r}_{xx}(x,t) - \|\mathbf{q}(x)\|^2(\|\mathbf{q}(x,t)\|^2 - q_0^2)]\mathrm{d}x, \tag{4.14b}$$

or equivalently

$$I_2 = -\int_{-\infty}^{\infty} [\|\mathbf{q}_x(x,t)\|^2 + (\|\mathbf{q}(x,t)\|^4 - q_0^4)]\mathrm{d}x,$$

etc. Note that I_2 is the Hamiltonian of the VNLS equation (1.3). Similarly, from Eq. (4.13b) one obtains

$$K_1 = \mathbf{q}_+^T\mathbf{r}_-, \quad K_2 = \int_{-\infty}^{\infty} \mathbf{q}_+^T(\mathbf{r}(x,t)\mathbf{q}^T(x,t)\mathbf{r}_- - q_0^2\mathbf{r}_-)\mathrm{d}x, \tag{4.15a}$$

and so on and so forth. Note that assuming the asymptotic phase differences are the same in both components [cf. Eq. (2.49)], Eq. (4.15a) becomes

$$K_1 = e^{i\Delta\theta}q_0^2, \quad K_2 = e^{i\Delta\theta}q_0^2 I_0,$$

etc., which show that the asymptotic phase difference is constant, in agreement with Eq. (4.4).

Finally, note that motion constants are also given in terms of the scattering data by the trace formula (3.13). In fact, recalling that $a_{11}(z)$, as well as its zeros z_n, ζ_n (discrete eigenvalues) are time independent, the coefficients of the expansions of $a_{11}(z)$ both as $z \to 0$ and as $z \to \infty$ in the upper-half-plane of z, i.e.,

$$J_n = \int_{-\infty}^{+\infty} \zeta^n \log[1 - \zeta^2|\rho_1(\zeta)|^2/q_0^2 - q_0^2|\rho_2(\zeta)|^2/(\zeta^2 - q_0^2)]\mathrm{d}\zeta, \quad n \in \mathbb{Z} \tag{4.16}$$

provide an infinite set of conserved quantities, assuming all of these integrals are convergent.

V. EXPLICIT SOLUTIONS

Let us discuss the special solutions obtained in the case where there is no continuum spectrum, that is, for reflectionless potentials, $\rho_j(z) = \bar{\rho}_j(z) \equiv 0$ for $j = 1, 2$ and all $z \in \mathbb{R}$.

A. Dark-dark soliton solutions

We first consider the case of a reflectionless potential with one single eigenvalue on the circle C_0 of radius q_0 (i.e., $N_1 = 1$ and $N_2 = 0$), and let $\zeta_1 = k_1 + i\nu_1$ with $-q_0 < k_1 < q_0$ and $\nu_1 = \sqrt{q_0^2 - k_1^2}$. In this case the first two equations of the inverse problem [namely Eqs. (3.8a) and (3.8b)] reduce to the closed system

$$\psi_3(x,z)e^{-i\lambda(z)x} = -\begin{pmatrix} \hat{z}^* \\ i\mathbf{r}_+ \end{pmatrix} + \bar{C}_1^{(1)} \frac{\psi_1(x,\zeta_1^*)e^{-\nu_1 x}}{z - \zeta_1^*}, \tag{5.1a}$$

$$\psi_1(x,z)e^{i\lambda(z)x} = \begin{pmatrix} z \\ i\mathbf{r}_+ \end{pmatrix} + \frac{z}{z - \zeta_1} C_1^{(1)} \psi_3(x,\zeta_1)e^{-\nu_1 x}. \tag{5.1b}$$

Evaluating Eq. (5.1a) at $z = \zeta_1$ and Eq. (5.1b) at $z = \zeta_1^*$, we get a linear system whose solution is given by

$$\psi_3(x,\zeta_1) = -\begin{pmatrix}\zeta_1^* \\ i\mathbf{r}_+\end{pmatrix} e^{-\nu_1 x} \frac{1 + \dfrac{i\bar{C}_1^{(1)}}{2\nu_1} e^{-2\nu_1 x}}{1 - \dfrac{C_1^{(1)}\bar{C}_1^{(1)}}{(2\nu_1)^2}\zeta_1^* e^{-4\nu_1 x}}, \tag{5.2a}$$

$$\psi_1(x,\zeta_1^*) = \begin{pmatrix}\zeta_1^* \\ i\mathbf{r}_+\end{pmatrix} e^{-\nu_1 x} \frac{1 - \dfrac{i\zeta_1^*\bar{C}_1^{(1)}}{2\nu_1} e^{-2\nu_1 x}}{1 - \dfrac{C_1^{(1)}\bar{C}_1^{(1)}}{(2\nu_1)^2}\zeta_1^* e^{-4\nu_1 x}}, \tag{5.2b}$$

where we used the fact that $|\zeta_1|^2 = q_0^2$. We write the common denominator of Eqs. (5.2) as

$$1 - \frac{C_1^{(1)}\bar{C}_1^{(1)}}{(2\nu_1)^2}\zeta_1^* e^{-4\nu_1 x} \equiv (1 + \gamma e^{-2\nu_1 x})(1 - \gamma e^{-2\nu_1 x})$$

with $(2\nu_1\gamma)^2 = C_1^{(1)}\bar{C}_1^{(1)}\zeta_1^*$. Then from (3.11a) it follows

$$\bar{C}_1^{(1)} = -\zeta_1^* C_1^{(1)} \tag{5.3}$$

and therefore

$$\frac{i\bar{C}_1^{(1)}}{2\nu_1} = \mp \frac{\sqrt{C_1^{(1)}\bar{C}_1^{(1)}\zeta_1^*}}{2\nu_1} \equiv \mp \gamma \tag{5.4}$$

so that the eigenfunctions (5.2a) and (5.2b) can be written as

$$\psi_1(x,\zeta_1^*) = -\psi_3(x,\zeta_1) = \begin{pmatrix}\zeta_1^* \\ i\mathbf{r}_+\end{pmatrix} e^{-i\nu_1 x} \frac{1}{1 \pm \gamma e^{-2\nu_1 x}}. \tag{5.5}$$

Recalling the definitions (3.9) and the symmetry relation (3.10), we get

$$C_1^{(1)}\bar{C}_1^{(1)}\zeta_1^* = -\bar{b}_1^{(1)} b_1^{(1)}/(a_{33}'(\zeta_1^*))^2.$$

Furthermore, in the pure one-soliton case, one has $\zeta_1^* a_{33}'(\zeta_1^*) = \zeta_1/(\zeta_1^* - \zeta_1)$ and hence the previous relation becomes $C_1^{(1)}\bar{C}_1^{(1)}\zeta_1^* = (2\nu_1)^2 \bar{b}_1^{(1)} b_1^{(1)} (\zeta_1^*/\zeta_1)^2$ so that $\gamma^2 = \bar{b}_1^{(1)} b_1^{(1)} (\zeta_1^*/\zeta_1)^2$. Finally, using the symmetry (2.66) we have

$$\gamma^2 = |b_1^{(1)}|^2,$$

that is $\gamma = |b_1^{(1)}|$ assuming without loss of generality that $\gamma > 0$. Then, from Eq. (5.4) it follows that $\bar{C}_1^{(1)} = \pm i(2\nu_1\gamma)$ that is, $\bar{C}_1^{(1)}$ is purely imaginary. In the following, in order to exclude singular solutions from the IST procedure, we assume the imaginary part of $\bar{C}_1^{(1)}$ is positive, i.e., corresponding to the upper sign. Then from (5.1a) we obtain

$$\psi_3(x,z)e^{-i\lambda(z)x} = -\begin{pmatrix}\hat{z}^* \\ i\mathbf{r}_+\end{pmatrix} + \frac{2i\nu_1\gamma}{z - \zeta_1^*}\begin{pmatrix}\zeta_1^* \\ i\mathbf{r}_+\end{pmatrix}\frac{e^{-2\nu_1 x}}{1 + \gamma e^{-2\nu_1 x}}. \tag{5.6}$$

According to (2.46b), from the last two components of (5.6) in the limit $z \to 0$ it follows

$$\mathbf{r}(x) = \mathbf{r}_+ \left[1 + \frac{2i\nu_1\gamma}{\zeta_1^*} \frac{e^{-2\nu_1 x}}{1+\gamma e^{-2\nu_1 x}} \right]. \tag{5.7}$$

Taking into account the time dependence of the norming constant as given in Eq. (4.9) and then taking the complex conjugate to get $\mathbf{q}(x,t)$ one obtains a solution of the VNLS equation

$$\mathbf{q}(x,t) = \mathbf{q}_+(0) e^{2iq_0^2 t} \left[1 + (e^{2i\alpha}-1) \frac{e^{2q_0 (\sin\alpha)(x-2q_0(\cos\alpha)t-x_0)}}{1+e^{2q_0(\sin\alpha)(x-2q_0(\cos\alpha)t-x_0)}} \right], \tag{5.8a}$$

$$\zeta_1 = k_1 + i\nu_1 = q_0 e^{-i\alpha}, \quad e^{2\nu_1 x_0} = \gamma(0) \equiv |b_1^{(1)}(0)| \tag{5.8b}$$

which is of the same type as Eq. (1.2) in both components, multiplied by the constant polarization (i.e., unit magnitude) vector $\mathbf{p}_+ = \mathbf{q}_+(0)/q_0$. Let us also mention that from Eq. (5.7) it follows that $\mathbf{r}(x) \to \mathbf{r}_+$ as $x \to +\infty$. Also, as $x \to -\infty$ one has

$$\mathbf{r}(x) \sim \left(1 + \frac{2i\nu_1}{\zeta_1^*}\right)\mathbf{r}_+ = \frac{\zeta_1}{\zeta_1^*}\mathbf{r}_+$$

therefore the asymptotic behavior satisfies the analog of the Θ-condition for the scalar NLS equation (cf. Ref. 11), that is,

$$\frac{r_-^{(j)}}{r_+^{(j)}} = \frac{\zeta_1}{\zeta_1^*}, \quad j=1,2 \tag{5.9}$$

in agreement with Eq. (3.14) with $\rho_j = 0$, $N_1 = 1$ and $N_2 = 0$. Note that the right-hand side of Eq. (5.9) is independent of j, which is consistent with the assumption that the asymptotic phase difference is the same in both components.

B. Dark-bright soliton solutions

We now consider one quarter of eigenvalues off the circle C_0 of radius q_0 (cf. Fig. 3) and no continuous spectrum (i.e., $N_1 = 0$ and $N_2 = 1$). The system of equation (3.8) for the inverse problem then reduces to

$$\psi_3(x,z)e^{-i\lambda(z)x} = -\begin{pmatrix} \hat{z}^* \\ i\mathbf{r}_+ \end{pmatrix} + \frac{C_1^{(2)}}{\hat{z}_1^*} \frac{\bar{\chi}(x,\hat{z}_1^*)e^{-i\lambda(\hat{z}_1^*)x}}{z-\hat{z}_1^*}, \tag{5.10a}$$

$$\psi_1(x,z)e^{i\lambda(z)x} = \begin{pmatrix} z \\ i\mathbf{r}_+ \end{pmatrix} + \frac{z}{z-z_1} C_1^{(2)} \bar{\chi}(x,\hat{z}_1^*) e^{i\lambda(z_1)x}, \tag{5.10b}$$

$$\frac{\bar{\chi}(x,z)e^{-ik(z)x}}{2\lambda(z)b_{11}(z)} = \begin{pmatrix} 0 \\ i\mathbf{q}_+^\perp \end{pmatrix} - \bar{C}_1^{(2)} \frac{z}{(z-\hat{z}_1)(z-z_1^*)} \psi_1(x,z_1^*) e^{-ik(z_1^*)x}, \tag{5.10c}$$

where $\bar{C}_1^{(2)}$ and $C_1^{(2)}$ are given by Eqs. (3.9). To obtain a closed system, we evaluate the second equation at point $z = z_1^*$ and the third equation at $z = \hat{z}_1^*$, which gives a system of two equations for two unknowns, $\psi_1(x,z_1^*)$ and $\bar{\chi}(x,\hat{z}_1^*)$. Then, back-substituting, we obtain the expression of all the (z-dependent) eigenfunctions. Indeed, from Eqs. (5.10b) and (5.10c) one obtains

$$\bar{\chi}(x,\hat{z}_1^*) = \frac{e^{ik(\hat{z}_1^*)x}}{1+\beta_1\gamma_1 e^{-2\nu_1 x}} \left[\alpha_1 \begin{pmatrix} 0 \\ i\mathbf{q}_+^\perp \end{pmatrix} - \beta_1 \begin{pmatrix} z_1^* \\ i\mathbf{r}_+ \end{pmatrix} e^{-i(k(z_1^*)+\lambda(z_1^*))x} \right],$$

where

$$\alpha_1 = 2\lambda(\hat{z}_1^*)b_{11}(\hat{z}_1^*), \quad \beta_1 = \alpha_1 \bar{C}_1^{(2)} \frac{\hat{z}_1^*}{(\hat{z}_1^* - \hat{z}_1)(\hat{z}_1^* - z_1^*)}, \quad \gamma_1 = C_1^{(2)} \frac{z_1^*}{z_1^* - z_1} \tag{5.11}$$

and, substituting into Eq. (5.10b),

$$\psi_1(x,z)e^{i\lambda(z)x} = \begin{pmatrix} z \\ i\mathbf{r}_+ \end{pmatrix} + \frac{z}{z-z_1} C_1^{(2)} \frac{e^{iz_1 x}}{1+\beta_1\gamma_1 e^{-2\nu_1 x}} \left[\alpha_1 \begin{pmatrix} 0 \\ i\mathbf{q}_+^{\perp} \end{pmatrix} - \beta_1 \begin{pmatrix} z_1^* \\ i\mathbf{r}_+ \end{pmatrix} e^{-iz_1^* x} \right].$$

To find a dark-bright soliton solution, we take $r_+^{(1)}=0$ (and consequently $q_+^{(1)}=0$) with $r_+^{(2)} = (q_+^{(2)})^* \neq 0$, and we look at the second and third components of $\psi_1(x,t)e^{i\lambda x}$, which, according to (2.46d), in the limit $z\to\infty$ reconstruct the potential $\mathbf{r}(x)$. Explicitly, we get

$$r^{(1)}(x) = \alpha_1 q_+^{(2)} C_1^{(2)} e^{ik_1 x} \frac{e^{-\nu_1 x}}{1+\beta_1\gamma_1 e^{-2\nu_1 x}}, \tag{5.12a}$$

$$r^{(2)}(x) = r_+^{(2)} \left[1 - C_1^{(2)}\beta_1 \frac{e^{-2\nu_1 x}}{1+\beta_1\gamma_1 e^{-2\nu_1 x}} \right]. \tag{5.12b}$$

In the pure one-soliton case, using the analyticity properties we can write explicitly the scattering coefficients $b_{11}(z)$ and $a_{11}(z)$ and their derivatives. Recalling that $a_{11}(z)$ is analytic in the upper-half-plane, that it goes to 1 as $z\to\infty$, and assuming that it has a single, simple zero at $z=z_1$ [cf. Eq. (2.53)], we get

$$a_{11}(z) = \frac{z-z_1}{z-z_1^*}, \quad a_{11}'(z_1) = \frac{1}{z_1 - z_1^*}.$$

Thus, recalling that $b_{11}(z)=a_{11}^*(z^*)$ and substituting into Eq. (5.11), we obtain

$$\alpha_1 = \frac{q_0^2 - |z_1|^2}{z_1}, \quad \beta_1 = \bar{C}_1^{(2)} \frac{z_1^*}{z_1^* - z_1}, \quad \gamma_1 = C_1^{(2)} \frac{z_1^*}{z_1^* - z_1}. \tag{5.13}$$

Note that α_1 vanishes if $|z_1|=q_0$ so that for zeros on the circle C_0 the bright component becomes trivial. Note also that from Eq. (3.11b) it follows that

$$\beta_1\gamma_1 \equiv \frac{q_0^2}{4\nu_1^2}(q_0^2 - |z_1|^2)|C_1^{(2)}|^2$$

which is real and positive for any eigenvalue z_1 inside the circle C_0 of radius q_0. Note that having $|z_1| > q_0$ (i.e., an eigenvalue outside C_0) would produce a singular potential.

Inserting the time dependence (4.10) into the expressions for the potential (5.12), we finally obtain the dark-bright soliton solution of the VNLS equation (1.3),

$$r^{(1)}(x,t) = \nu_1(q_0^2/|z_1|^2 - 1)^{1/2} e^{i\varphi_1 - 2iq_0^2 t + ik_1 x - i(k_1^2 - \nu_1^2)t} \operatorname{sech}[\nu_1(x - 2k_1 t) + x_0], \tag{5.14a}$$

$$r^{(2)}(x,t) = q_0 e^{i\varphi_2 - 2iq_0^2 t} \left[1 + \frac{2i\nu_1}{z_1^*} \frac{\exp[-2\nu_1 x + 4k_1\nu_1 t + 2x_0]}{1+\exp[-2\nu_1 x + 4k_1\nu_1 t + 2x_0]} \right], \tag{5.14b}$$

where

$$e^{2x_0} = \frac{q_0^2}{4\nu_1^2}(q_0^2 - |z_1|^2)|C_1^{(2)}(0)|^2, \quad \varphi_1 = \arg C_1^{(2)}(0) + \theta_+^{(2)}(0), \quad \varphi_2 = -\theta_+^{(2)}(0). \tag{5.15}$$

As usual, the solution $\mathbf{q}(x,t)$ of Eq. (1.3) is obtained taking the complex conjugate of Eq. (5.14). The dark-bright solution (5.14) can be written in the more compact form

$$q^{(1)}(x,t) = -\nu_1 \sin\alpha\sqrt{q_0^2 - |z_1|^2}\,\mathrm{sech}[\nu_1(x-2k_1t)+x_0]e^{-ik_1x+i[2q_0^2+(k_1^2-\nu_1^2)]t-i\varphi_1}, \tag{5.16a}$$

$$q^{(2)}(x,t) = q_0\{\cos\alpha + i\sin\alpha\tanh[\nu_1(x-2k_1t)+x_0]\}e^{2iq_0^2t-i\varphi_2}, \tag{5.16b}$$

where

$$k_1 = |z_1|\cos\alpha, \quad \nu_1 = -|z_1|\sin\alpha. \tag{5.17}$$

Again, note that the condition $k_1^2+\nu_1^2 \equiv |z_1|^2 < q_0^2$ (i.e., the requirement that the discrete eigenvalue z_1 is inside the circle C_0 of radius q_0) is necessary and sufficient to ensure the regularity of the solution at all times.

Equation (5.16) describes a two-component solution in which the second component $q^{(2)}(x,t)$ represents a dark soliton similar to that in Eq. (1.2) (but with a different relation between amplitude and velocity), while the first component $q^{(1)}(x,t)$ describes a bright soliton similar to that of the scalar focusing NLS (but with a different relation between amplitude and phase). The two components travel together at the same speed $2k_1$. Note that the amplitude of the bright soliton component and that of the intensity dip in the dark soliton component are related by the condition $k_1^2+\nu_1^2 < q_0^2$, and the amplitude of the bright component goes to zero as the eigenvalue approaches the circle (i.e., in the limit $|z_1| \to q_0$). With proper identification of the parameters, Eqs. (5.16) also coincide with the dark-bright soliton solution given in Ref. 19 in the case of x-independent asymptotic boundaries and with $x_0=0$.

VI. SMALL AMPLITUDE LIMIT

It is useful to consider the limit in which the solution $\mathbf{q}(x,t)$ of Eq. (1.3) is a small perturbation of the background field.

A. Linearization

Recall that $\mathbf{q}(x,t) \to \mathbf{q}_\pm(t) = e^{i\Theta_\pm(t)}\mathbf{q}_0$ as $x \to \pm\infty$, with $\Theta_\pm(t) = \mathrm{diag}(\theta_\pm^{(1)}, \theta_\pm^{(2)})$, and $\theta_\pm^{(j)}(t) = \theta_\pm^{(j)}(0) + 2iq_0^2 t$, and with $q_0 = \|\mathbf{q}_0\|$ as usual. We then consider the "normalized" vector NLS equation

$$i\tilde{\mathbf{q}}_t = \tilde{\mathbf{q}}_{xx} + 2(q_0^2 - \|\tilde{\mathbf{q}}\|^2)\tilde{\mathbf{q}}, \tag{6.1}$$

for the rescaled field $\tilde{\mathbf{q}}(x,t) = \mathbf{q}(x,t)e^{-2iq_0^2t}$, and we define

$$\tilde{\mathbf{q}}(x,t) = e^{i\Theta_+(0)}(\mathbf{q}_0 + \mathbf{u}(x,t)), \tag{6.2}$$

with $\|\mathbf{u}(x,t)\| \ll q_0$, so that $\mathbf{u}(x,t)$ represents a small perturbation of the background field $\mathbf{q}_+(t)$. Inserting Eq. (6.2) into the rescaled VNLS equation (6.1) and neglecting higher powers of $\mathbf{u}$ we then obtain a linearization of the VNLS equation around the background solution,

$$i\mathbf{u}_t = \mathbf{u}_{xx} - 2\mathbf{q}_0\mathbf{q}_0^T(\mathbf{u} + \mathbf{u}^*). \tag{6.3}$$

We now look for solutions of Eq. (6.3) employing standard Fourier transforms, where for convenience we write the transform pair as follows:

$$\mathbf{u}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\mathbf{u}}(k,t)e^{2ikx}\,\mathrm{d}k, \quad \hat{\mathbf{u}}(k,t) = 2\int_{-\infty}^{\infty}\mathbf{u}(x,t)e^{-2ikx}\,\mathrm{d}x. \tag{6.4}$$

Inserting the first of Eqs. (6.4) into Eq. (6.3) with (6.2), one finds a system of four first-order differential equations in time for the functions $\hat{\mathbf{u}}(k,t) \equiv (\hat{u}_1(k,t), \hat{u}_2(k,t))$ and $\hat{\mathbf{u}}^*(-k,t)^T \equiv (\hat{u}_1^*(-k,t), \hat{u}_2^*(-k,t))^T$, which can then be solved to obtain

$$\hat{\mathbf{u}}(k,t) = A_1(k)e^{4ik^2t}\mathbf{q}_0^{\perp} + (k - \sqrt{k^2+q_0^2})A_2(k)e^{-4ik\sqrt{k^2+q_0^2}t}\mathbf{q}_0 + (k + \sqrt{k^2+q_0^2})A_3(k)e^{4ik\sqrt{k^2+q_0^2}t}\mathbf{q}_0, \tag{6.5}$$

where $\mathbf{q}_0^{\perp} = (q_0^{(2)}, -q_0^{(1)})^T \in \mathbb{R}^2$. The functions $A_1(k), A_2(k), A_3(k)$ satisfy the symmetry conditions

$$A_2^*(-k) = -A_2(k), \quad A_3^*(-k) = -A_3(k), \tag{6.6}$$

and can be written in terms of the Cauchy data as follows:

$$A_1(k) = \frac{\hat{\mathbf{u}}_0^T(k)\mathbf{q}_0^{\perp}}{q_0^2}, \tag{6.7a}$$

$$A_2(k) = \frac{1}{4kq_0^2\sqrt{k^2+q_0^2}}[(-k + \sqrt{k^2+q_0^2})\mathbf{q}_0^T\hat{\mathbf{u}}_0(k) + (k + \sqrt{k^2+q_0^2})\mathbf{q}_0^T\hat{\mathbf{u}}_0^*(-k)], \tag{6.7b}$$

$$A_3(k) = \frac{1}{4kq_0^2\sqrt{k^2+q_0^2}}[(-k + \sqrt{k^2+q_0^2})\mathbf{q}_0^T\hat{\mathbf{u}}_0^*(-k) + (k + \sqrt{k^2+q_0^2})\mathbf{q}_0^T\hat{\mathbf{u}}_0(k)], \tag{6.7c}$$

where $\hat{\mathbf{u}}_0(k) = \hat{\mathbf{u}}(k,0)$. Together, Eqs. (6.5) and (6.7) yield the solution of the linearized VNLS Eq. (6.3) in terms of given Cauchy data, which in turn provides an approximation of the solution $\mathbf{q}(x,t)$ of the VNLS equation (1.3) in the small amplitude limit.

B. Small amplitude limit from the inverse problem

If we consider the equations of the inverse problem (3.8a), (3.8b), and (3.8c) with no solitons, in the small amplitude limit we can approximate each term on the left-hand side with a series in powers of $\rho_j(z,t)$. Keeping only linear terms in $\rho_j(z,t)$, according to Eq. (2.46b), the expansion as $z \to 0$ of the last two components of $\psi_3(x,z)e^{-i\lambda(z)x}$ yields

$$\mathbf{q}(x,t) = \mathbf{q}_+(t)\left[1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{\zeta}\rho_1^*(\zeta,t)e^{2i\lambda(\zeta)x}\right] - \mathbf{r}_+^{\perp}(t)\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{\zeta}\rho_2^*(\hat{\zeta}^*,t)e^{-i(k(\zeta)-\lambda(\zeta))x} \tag{6.8}$$

(with $\hat{\zeta} = q_0^2/\zeta^*$ as usual). In order to compare with the Fourier transform solutions obtained in the preceding sections, we recall that $\mathbf{q}_{\pm}(t) = e^{i\Theta_{\pm}(t)}\mathbf{q}_0$ and $\mathbf{r}_{\pm}(t) = \exp[-i\Theta_{\pm}(t)]\ \mathbf{q}_0$, and we consider again the normalization $\tilde{\mathbf{q}}(x,t) = \mathbf{q}(x,t)e^{-2iq_0^2t}$. Then, taking into account the time dependence of the scattering coefficients [cf. Eqs. (4.7)], from Eqs. (6.8) we get

$$\tilde{\mathbf{q}}(x,t) = \mathbf{q}_+\left[1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{\zeta}\rho_1^*(\zeta,0)e^{2i\lambda(\zeta)x - 4ik(\zeta)\lambda(\zeta)t}\right] - \mathbf{r}_+^{\perp}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{\zeta}\rho_2^*(\hat{\zeta}^*,0)e^{-i(q_0^2/\zeta)x + i(q_0^2/\zeta)^2t}, \tag{6.9}$$

where now $\mathbf{q}_+ \equiv \mathbf{q}_+(0)$ and $\mathbf{r}_+ \equiv \mathbf{r}_+(0)$. In order to compare with the results in the preceding section, we then perform appropriate changes of variables. Consider the term in square brackets in Eq. (6.9). First, we revert from ζ to the original coordinates $k, \lambda(k)$, so that k runs over the contour $\mathcal{L}$ given by the branch cuts in Fig. 1 and defined in Sec. II A. Then we introduce the variable

$$\xi = \sqrt{k^2 - q_0^2},$$

so that $\xi\,\mathrm{d}\xi = k\,\mathrm{d}k$, obtaining

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{\zeta}\rho_1^*(\zeta,0)e^{2i\lambda(\zeta)x}e^{-4ik(\zeta)\lambda(\zeta)t}=\int_{\mathcal{L}}\frac{\mathrm{d}k}{\lambda(k)}\rho_1^*(k,\lambda(k),0)e^{2i\lambda(k)x}e^{-4ik\lambda(k)t}$$
$$=\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\sqrt{\xi^2+q_0^2}}e^{2i\xi x}[\rho_1^*(\sqrt{\xi^2+q_0^2},\xi,0)e^{-4i\xi\sqrt{\xi^2+q_0^2}t}$$
$$-\rho_1^*(-\sqrt{\xi^2+q_0^2},\xi,0)e^{4i\xi\sqrt{\xi^2+q_0^2}t}]. \tag{6.10}$$

On the other hand, the second term in Eq. (6.9) can be written as an ordinary Fourier transform by simply performing the change of variable $2\xi=-q_0^2/\zeta$. By comparison, one can then show that Eq. (6.9), obtained solving the inverse problem in the limit of small amplitude, indeed coincides with the solution obtained via linearization, i.e., Eq. (6.2) with $\mathbf{u}(x,t)$ given by Eq. (6.4) and $\hat{\mathbf{u}}(k,t)$ by Eq. (6.5). More precisely, one has

$$A_1(\xi)=-\frac{1}{2k}e^{-i(\theta_+^{(1)}+\theta_+^{(2)})}\rho_2^*(-2\xi,\lambda(-2\xi),0), \tag{6.11a}$$

$$A_2(\xi)=\frac{i}{\sqrt{\xi^2+q_0^2}(\sqrt{\xi^2+q_0^2}-\xi)}\rho_1^*(\sqrt{\xi^2+q_0^2},\xi,0), \tag{6.11b}$$

$$A_3(\xi)=\frac{i}{\sqrt{\xi^2+q_0^2}(\sqrt{\xi^2+q_0^2}+\xi)}\rho_1^*(-\sqrt{\xi^2+q_0^2},\xi,0). \tag{6.11c}$$

Then, as a consequence of the symmetry conditions (6.6), it follows that

$$\rho_1(\sqrt{\xi^2+q_0^2},-\xi,0)=\frac{\sqrt{\xi^2+q_0^2}+\xi}{\sqrt{\xi^2+q_0^2}-\xi}\rho_1^*(\sqrt{\xi^2+q_0^2},\xi,0),$$

that is, in terms of the uniformization variable z,

$$\rho_1(\hat{z}^*)=\frac{z^2}{q_0^2}\rho_1^*(z),\quad z\in\mathbb{R}. \tag{6.12}$$

Note that Eq. (6.12) arises from the scattering data relations as well. Indeed, from the definitions (3.3) and symmetry (2.38) it follows

$$\rho_1^*(z)=\frac{b_{31}^*(z)}{b_{11}^*(z)}=\Gamma_1(z)\frac{a_{13}(z)}{a_{11}(z)}\Gamma_3^{-1}(z)\equiv-\frac{q_0^2}{z^2}\frac{a_{13}(z)}{a_{11}(z)} \tag{6.13a}$$

and the analog of symmetries (2.42) for the coefficients $b_{ij}(z)$ yields

$$\rho_1(z)=\frac{b_{31}(z)}{b_{11}(z)}=\frac{b_{13}(\hat{z}^*)}{b_{33}(\hat{z}^*)}. \tag{6.13b}$$

Recalling that $\mathbf{B}(z)=(b_{ij}(z))$ is the inverse matrix of $\mathbf{A}(z)=(a_{ij}(z))$, one can write

$$b_{13}(z)=a_{12}(z)a_{23}(z)-a_{13}(z)a_{22}(z),\quad b_{33}(z)=a_{11}(z)a_{22}(z)-a_{12}(z)a_{21}(z). \tag{6.14}$$

Then, since in the small amplitude limit terms $a_{ij}(z)$ with $i\neq j$ are $o(1)$ while $a_{jj}(z)=O(1)$, one has

$$\rho_1(z)=\frac{b_{13}(\hat{z}^*)}{b_{33}(\hat{z}^*)}\sim-\frac{a_{13}(\hat{z}^*)a_{22}(\hat{z}^*)}{a_{11}(\hat{z}^*)a_{22}(\hat{z}^*)},$$

and consequently Eq. (6.12) follows from Eq. (6.13).

VII. CONCLUSION

We have presented the inverse scattering transform (IST) for the defocusing VNLS equation (1.3) with nonvanishing boundary conditions as $|x|\to\infty$. The direct problem is constructed in terms of scattering eigenfunctions and adjoint eigenfunctions. The six scattering eigenfunctions provide four analytic functions, and the adjoint problem is used to construct two additional analytic functions. A global uniformizing parameter, z, is introduced in order to simplify and elucidate the analysis. The discrete eigenvalues are studied and it is found that one can have pairs of eigenvalues on a circle and/or quartets of eigenvalues symmetrically located inside and outside the circle. The inverse problem is formulated as a generalized Riemann-Hilbert (RH) problem for meromorphic functions in the complex plane of the uniformizing parameter z. The RH problem is transformed into a closed linear system of algebraic-integral equations. The trace formula, conservation laws, and explicit solutions (dark-dark and dark-bright solitons) are obtained. The solution in the small amplitude limit is studied by direct Fourier transform methods and it is shown to agree with the linearized reduction of the inverse problem.

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APPENDIX: WKB EXPANSION OF THE EIGENFUNCTIONS

Consider the following ansatz for the expansion of the eigenfunction $M_1(x,z)$ as $z\to\infty$:

$$M_1^{(1)}(x,z) = zM_{1,\infty}^{(1,-1)}(x) + M_{1,\infty}^{(1,0)}(x) + z^{-1}M_{1,\infty}^{(1,1)}(x) + z^{-2}M_{1,\infty}^{(1,2)}(x) + \cdots , \tag{A1a}$$

$$M_1^{(2)}(x,z) = M_{1,\infty}^{(2,0)}(x) + z^{-1}M_{1,\infty}^{(2,1)}(x) + z^{-2}M_{1,\infty}^{(2,2)}(x) + \cdots , \tag{A1b}$$

$$M_1^{(3)}(x,z) = M_{1,\infty}^{(3,0)}(x) + z^{-1}M_{1,\infty}^{(3,1)}(x) + z^{-2}M_{1,\infty}^{(3,2)}(x) + \cdots . \tag{A1c}$$

Substituting these expressions into the scattering problem (2.1) with $k=(z+q_0^2/z)/2$ and matching the terms with the same order in z^{-n} for $n=-1,0,1,2,\ldots$ yields $M_{1,\infty}^{(1,-1)}(x)=\text{const}$, and the integral equation (2.8a) allows one to fix this constant value to

$$M_{1,\infty}^{(1,-1)}(x) = 1. \tag{A2a}$$

Proceeding further gives

$$M_{1,\infty}^{(2,0)}(x) = ir^{(1)}(x), \quad M_{1,\infty}^{(3,0)}(x) = ir^{(2)}(x), \quad \partial_x M_{1,\infty}^{(1,0)}(x) = i(\|\mathbf{q}(x)\|^2 - q_0^2), \tag{A2b}$$

that is,

$$M_{1,\infty}^{(1,0)}(x) = i\int_{-\infty}^{x} (\|\mathbf{q}(x')\|^2 - q_0^2)\mathrm{d}x'. \tag{A2c}$$

Similarly, at higher orders one has

$$M_{1,\infty}^{(j,m+1)}(x) = ir^{(j-1)}(x)M_{1,\infty}^{(1,m)}(x) - i\partial_x M_{1,\infty}^{(j,m)}(x), \quad m = 0,1,2,\ldots \tag{A3a}$$

for $j=2,3$, as well as

$$\partial_x M_{1,\infty}^{(1,m)}(x) = - iq_0^2 M_{1,\infty}^{(1,m-1)}(x) + q^{(1)}(x)M_{1,\infty}^{(2,m)}(x) + q^{(2)}(x)M_{1,\infty}^{(3,m)}(x), \quad m = 1,2,\ldots \tag{A3b}$$

which allow one to calculate iteratively all coefficients of the asymptotic expansion, with the recurrence relations in Eqs. (A3) anchored by Eqs. (A2). For instance, from Eq. (A3a) with $m=0$ we obtain

$$M_{1,\infty}^{(1,1)}(x) = I_1(x) - \tfrac{1}{2}(I_0(x))^2, \tag{A4a}$$

$$M_{1,\infty}^{(2,1)}(x) = r_x^{(1)}(x) - r^{(1)}(x)I_0(x), \quad M_{1,\infty}^{(3,1)}(x) = r_x^{(2)}(x) - r^{(2)}(x)I_0(x), \tag{A4b}$$

where

$$I_0(x) = \int_{-\infty}^{x} (\|\mathbf{q}(x')\|^2 - q_0^2)\mathrm{d}x', \quad I_1(x) = \int_{-\infty}^{x} \mathbf{q}^T(x')\mathbf{r}_{x'}(x')\mathrm{d}x'. \tag{A4c}$$

Furthermore, from Eq. (A3a) with $m=1$ it follows that

$$M_{1,\infty}^{(2,2)}(x) = ir^{(1)}(x)\Big[I_1(x) - \tfrac{1}{2}(I_0(x))^2\Big] - ir_{xx}^{(1)}(x) + i(r^{(1)}(x)I_0(x))_x, \tag{A5}$$

$$M_{1,\infty}^{(3,2)}(x) = ir^{(2)}(x)\Big[I_1(x) - \tfrac{1}{2}(I_0(x))^2\Big] - ir_{xx}^{(2)}(x) + i(r^{(2)}(x)I_0(x))_x \tag{A6}$$

which can be substituted into Eq. (A3b) for $m=2$ to get

$$M_{1,\infty}^{(1,2)}(x) = iI_0(x)I_1(x) - \frac{i}{6}(I_0(x))^3 - iI_2(x), \tag{A7}$$

where

$$I_2(x) = \int_{-\infty}^{x} [\mathbf{q}^T(x')\partial_{x'}^2\mathbf{r}(x') - \|\mathbf{q}(x')\|^2(\|\mathbf{q}(x')\|^2 - q_0^2)]\mathrm{d}x' \tag{A8}$$

and so on and so forth.

Similarly, one can write a Taylor series expansion of the eigenfunction $M_1(x,z)$ as $z\to 0$ in the form

$$M_1^{(1)}(x,z) = zM_{1,0}^{(1,1)}(x) + z^2M_{1,0}^{(1,2)}(x) + z^3M_{1,0}^{(1,3)}(x) + \cdots, \tag{A9a}$$

$$M_1^{(2)}(x,z) = M_{1,0}^{(2,0)}(x) + zM_{1,0}^{(2,1)}(x) + z^2M_{1,0}^{(2,2)}(x) + \cdots, \tag{A9b}$$

$$M_1^{(3)}(x,z) = M_{1,0}^{(3,0)}(x) + zM_{1,0}^{(3,1)}(x) + z^2M_{1,0}^{(3,2)}(x) + \cdots. \tag{A9c}$$

Substituting this into Eq. (2.1) and matching terms with the same powers of z^n yields $M_{1,0}^{(2,0)}(x)=\text{const}$ and $M_{1,0}^{(3,0)}(x)=\text{const}$. As before, the value of such constants is fixed by the integral equation (2.8a) to give

$$M_{1,0}^{(2,0)}(x) = ir_-^{(1)}, \quad M_{1,0}^{(3,0)}(x) = ir_-^{(3)}. \tag{A10a}$$

In turn, these allow one to get

$$q_0^2M_{1,0}^{(1,1)}(x) = \mathbf{q}^T(x)\mathbf{r}_-. \tag{A10b}$$

Proceeding to higher orders, one obtains the recurrence relations

$$\partial_x M_{1,0}^{(j,m)}(x) = iM_{1,0}^{(j,m-1)}(x) + r^{(j-1)}(x)M_{1,0}^{(1,m)}(x), \quad m=1,2,\ldots \tag{A11a}$$

for $j=2,3$, as well as

$$q_0^2M_{1,0}^{(1,m+1)}(x) = i\partial_x M_{1,0}^{(1,m)}(x) - iq^{(1)}(x)M_{1,0}^{(2,m)}(x) - iq^{(2)}(x)M_{1,0}^{(3,m)}(x), \quad m=0,1,\ldots. \tag{A11b}$$

For instance, the first terms are

$$q_0^2 M_{1,0}^{(2,1)}(x) = \int_{-\infty}^{x} [r^{(1)}(x')\mathbf{q}^T(x')\mathbf{r}_- - q_0^2 r_-^{(1)}]\mathrm{d}x',$$

$$q_0^2 M_{1,0}^{(3,1)}(x) = \int_{-\infty}^{x} [r^{(2)}(x')\mathbf{q}^T(x')\mathbf{r}_- - q_0^2 r_-^{(2)}]\mathrm{d}x',$$

which in turn give

$$q_0^4 M_{1,0}^{(1,2)}(x) = i\mathbf{r}_-^T\mathbf{q}_x(x) - i\mathbf{q}^T(x)\int_{-\infty}^{x} [\mathbf{r}(x')\mathbf{q}^T(x')\mathbf{r}_- - q_0^2\mathbf{r}_-]\mathrm{d}x'$$

and so on and so forth.

In a similar way one can obtain the asymptotic expansions for the remaining analytic eigenfunctions and adjoint eigenfunctions.

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