Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions

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The inverse scattering transform for the focusing nonlinear Schrödinger equation with non-zero boundary conditions at infinity is presented, including the determination of the analyticity of the scattering eigenfunctions, the introduction of the appropriate Riemann surface and uniformization variable, the symmetries, discrete spectrum, asymptotics, trace formulae and the so-called theta condition, and the formulation of the inverse problem in terms of a Riemann-Hilbert problem. In addition, the general behavior of the soliton solutions is discussed, as well as the reductions to all special cases previously discussed in the literature. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4868483]

I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation,

$$iq_t + q_{xx} - 2\nu(|q|^2 - q_o^2)q = 0 \tag{1.1}$$

(where $\nu = -1$ and $\nu = 1$ denote the focusing and defocusing cases, respectively), is a universal model for the evolution of the complex envelope of weakly nonlinear dispersive wave trains (see Ref. 14 for a derivation of the NLS equation from a generic wave equation). As such, it appears in many different physical contexts, such as deep water waves, optics, acoustics, Bose-Einstein condensation, etc. The equation has been studied extensively over the last 50 years (e.g., see the monographs in Refs. 4, 5, 20, 32, and 38 and references therein).

In this work we solve the initial value problem for the focusing NLS equation [namely, (1.1) with $\nu = -1$] with the following non-zero boundary conditions (NZBCs) as $x \to \pm\infty$:

$$\lim_{x\to\pm\infty} q(x,t) = q_\pm\,, \tag{1.2}$$

with $|q_\pm| = q_o \neq 0$. We do so by developing the appropriate inverse scattering transform (IST). The additional term $-2q_o^2 q$ in (1.1) can be removed by the simple rescaling $\tilde{q}(x,t) = \mathrm{e}^{2iq_o^2 t} q(x,t)$, and was added so that the boundary conditions (BCs) (1.2) are independent of time.

The IST for the focusing case with zero boundary conditions (ZBCs) was first developed in Ref. 44, while the IST for the defocusing case with non-zero boundary conditions (NZBC) was first developed in Ref. 45. Both cases have since been studied extensively. On the other hand, there are almost no known results on the IST for the focusing case with NZBCs. We believe there are two main reasons for this: (i) The technical difficulties resulting from the NZBCs; (ii) The presence of the modulational instability (MI) — known as the Benjamin-Feir instability[12,13] in the context of water waves. We discuss each of these issues in turn.

With regard to the technical difficulties, we note that, even for the defocusing case, the IST with NZBCs (which has been well studied) still presents several open questions.[15,19] For the focusing case with NZBCs, the only study of IST is Ref. 30, which only contains partial results. (In particular, no proof of analyticity was given, no Riemann surface and uniformization were used, the behavior at the branch points was not studied, no trace formulae or theta conditions were provided, symmetries were not studied in full, the inverse problem was not formulated in terms of a Riemann-Hilbert problem.) Moreover, and most importantly,[30] only considered the case in which $\lim_{x\to-\infty} q(x,t) = \lim_{x\to\infty} q(x,t)$ — i.e., the case in which the potential exhibits no asymptotic phase difference. Therefore, the IST in Ref. 30 only applies to a reduction of the full boundary conditions (1.2) considered here. As a result only purely imaginary discrete eigenvalues are included in the theory of Ref. 30. Partial results were also recently obtained in Ref. 24 to study the stability of the Peregrine soliton under perturbations.

With regard to the MI, we refer to the excellent article by Zakharov and Ostrovsky[43] for a historical perspective and an overview of the subject. Within the context of the NLS equation, the essence of the phenomenon is well-understood: the linearized stability analysis shows that a uniform background is unstable to long wavelength perturbations. The MI has received renewed interest in recent years, and has also been suggested as a possible mechanism for the generation of rogue waves.[33] In the framework of the NLS equation with periodic BCs, the underlying mechanism for the MI is known to be related to the existence of homoclinic solutions.[2,21] The MI is far from being an obstacle to the development of the IST, however; nor does it diminish the validity of the IST. In fact, the reverse is true: the IST provides a tool to study the nonlinear stage of modulational instability. Indeed, in recent studies[41,42] it was conjectured that the nonlinear stage of the modulational instability is mediated precisely by the soliton solutions. The IST is the perfect — indeed, the only — vehicle to test this hypothesis. We should also point out that, in the case of periodic BCs, the inverse problem in the IST is not as well characterized as that on the whole line case except for the class of finite-genus potentials.[11] As a result, the IST for periodic BCs is not an effective way to study initial-value problems for generic initial conditions except in an indirect way as a limit of finite-genus initial conditions. Such limitation does not apply to the present theory for the BCs (1.2).

II. DIRECT SCATTERING

A. Preliminaries: Lax pair, Riemann surface, and uniformization coordinate

Equation (1.1) with $\nu = -1$ is the compatibility condition of the Lax pair

$$\phi_x = X\,\phi\,, \quad \phi_t = T\,\phi\,, \tag{2.1}$$

with

$$X(x,t,z) = ik\sigma_3 + Q\,, \quad T(x,t,z) = -2ik^2\sigma_3 + i\sigma_3(Q_x - Q^2 - q_o^2) - 2kQ\,, \tag{2.2a}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}. \tag{2.2b}$$

It will be convenient to take $\phi(x, t, k)$ to be a 2×2 matrix throughout. Unlike the usual formulation of the IST for the defocusing NLS equation with NZBC, we will formulate the IST in a way that allows one to take the reduction $q_o \to 0$ explicitly.

The asymptotic scattering problem is $\phi_x = X_\pm\,\phi$, where $X_\pm = ik\sigma_3 + Q_\pm$. The eigenvalues of $X_\pm$ are $\pm i\sqrt{k^2+q_o^2}$. Since these eigenvalues are doubly branched, we introduce the two-sheeted Riemann surface defined by

$$\lambda^2 = k^2 + q_o^2\,, \tag{2.3}$$

so that $\lambda(k)$ is a single-valued function on this surface. The branch points are the values of k for which $\sqrt{q_o^2+k^2}=0$, i.e., $k = \pm iq_o$. (In the defocusing case, $\lambda^2 = k^2 - q_o^2$, and the branch points are $k = \pm q_o$.) Letting $k + iq_o = r_1\,e^{i\theta_1}$ and $k - iq_o = r_2\,e^{i\theta_2}$, one can write $\lambda(k) = \sqrt{r_1 r_2}\,e^{i\Theta}$, where $\Theta = (\theta_1+\theta_2)/2 + im\pi$, and $m = 0, 1$, respectively, on sheets I and II. We now take $-\pi/2 \le \theta_j < 3\pi/2$ for $j = 1, 2$. With these conventions, the discontinuity of λ (which defines the location of the branch cut) occurs on the segment $i[-q_o, q_o]$. The Riemann surface is then obtained by gluing the two copies of the complex plane along the cut. Along the real k axis we have $\lambda(k) = \pm\,\mathrm{sign}(k)\sqrt{q_o^2+k^2}$, where the plus/minus signs apply, respectively, on sheet I and sheet II of the Riemann surface, and where the square root sign denotes the principal branch of the real-valued square root function.

It is convenient to define the uniformization variable

$$z = k + \lambda \tag{2.4a}$$

(as in the defocusing case). The inverse transformation is

$$[-2ex]k = \tfrac{1}{2}\,(z - q_o^2/z)\,, \quad \lambda = \tfrac{1}{2}\,(z + q_o^2/z)\,. \tag{2.4b}$$

Finally, we let C_o be the circle of radius q_o in the complex z-plane. With these definitions: the branch cut on either sheet is mapped onto C_o. In particular, $z(\pm iq_o) = \pm iq_o$ from either sheet, $z(0_{\mathrm{I}}^{\pm}) = \pm q_o$ and $z(0_{\mathrm{II}}^{\pm}) = \mp q_o$; $\mathbb{C}_{\mathrm{I}}$ is mapped onto the exterior of C_o; $\mathbb{C}_{\mathrm{II}}$ is mapped onto the interior of C_o; in particular, $z(\infty_I) = \infty$ and $z(\infty_{\mathrm{II}}) = 0$; the first/second quadrants of $\mathbb{C}_{\mathrm{I}}$ are mapped into the first/second quadrants outside C_o, respectively; the first/second quadrants of $\mathbb{C}_{\mathrm{II}}$ are mapped into the second/first quadrants inside C_o, respectively. Note also $z_{\mathrm{I}}z_{\mathrm{II}} = q_o^2$.

Unlike the defocusing case, $\mathrm{Im}\,\lambda$ is not sign-definite in the upper-half plane (UHP) and lower-half plane (LHP). Instead, $\mathrm{Im}\ \lambda \gtrless 0$, respectively, in D^+ and D^-, where

$$D^+ = \{z \in \mathbb{C} : (|z|^2 - q_o^2)\,\mathrm{Im}\,z > 0\}\,, \quad D^- = \{z \in \mathbb{C} : (|z|^2 - q_o^2)\,\mathrm{Im}\,z < 0\}\,. \tag{2.5}$$

The two domains are shown in Fig. 1. As we show next, this property determines the analyticity regions of the Jost eigenfunctions. With some abuse of notation we will rewrite all the k dependence as dependence on z.

B. Jost solutions and analyticity

On either sheet of the Riemann surface, we can write the asymptotic eigenvector matrix as

$$Y_\pm(z) = I + i\sigma_3 Q_\pm/(k+\lambda) = I + (i/z)\sigma_3 Q_\pm\,, \tag{2.6}$$

where I denotes the 2×2 identity matrix, so that

$$X_\pm Y_\pm = Y_\pm\ i\lambda\sigma_3\,. \tag{2.7}$$

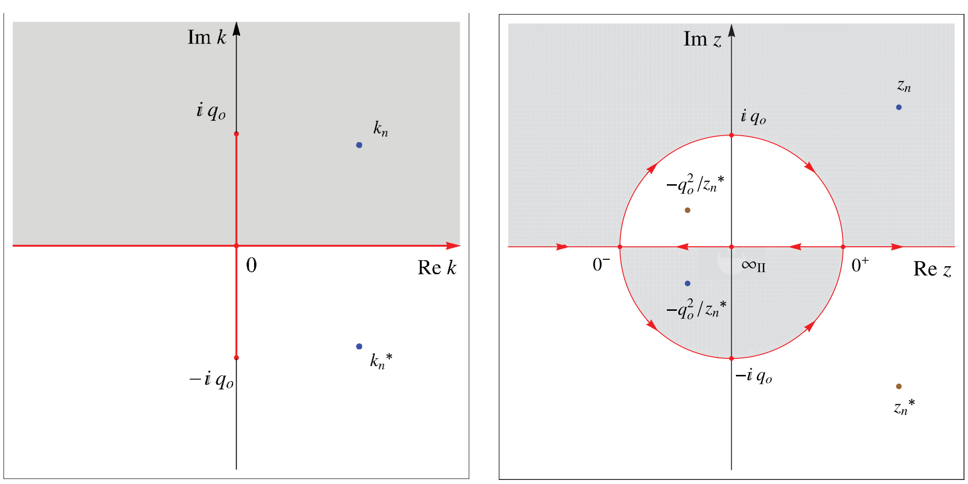

FIG. 1. Left: The first sheet of the Riemann surface, showing the branch cut (red) and the region where $\mathrm{Im}\,\lambda > 0$ (grey). Right: The complex z-plane, showing the regions $D^\pm$ where $\mathrm{Im}\,\lambda > 0$ (grey) and $\mathrm{Im}\,\lambda < 0$ (white), respectively. Also shown are the orientation of the contours for the Riemann-Hilbert problem and the symmetries of the discrete spectrum of the scattering problem [namely, the zeros of $s_{2,\,2}(z)$ (blue) and those of $s_{1,\,1}(z)$ (brown)]. See text for details.

For reference, note that

$$\det Y_\pm = 2\lambda/(\lambda + k) = 1 + q_o^2/z^2 =: \gamma(z)\,, \tag{2.8a}$$

$$Y_\pm^{-1} = (1/\gamma)\,[I - (i/z)\sigma_3 Q_\pm]\,. \tag{2.8b}$$

Let us now discuss the time dependence of the eigenfunctions. We expect that, as $x \to \pm\infty$, the time evolution of the solutions of (2.1) will be asymptotic to that of the problem $\phi_t = T_\pm\,\phi$, with $T_\pm = -2ik^2\sigma_3 - i\sigma_3(Q_\pm^2 + q_o^2) - 2kQ_\pm$. Note that $Q_\pm^2 = -q_o^2$, implying $T_\pm = -2kX_\pm$. Therefore, $T_\pm$ and $X_\pm$ share the same eigenvectors. (This is not a coincidence, of course, because the NLS equation (1.1) is the compatibility condition of the Lax pair (2.1), and the time independence of the BCs (1.2) is equivalent to the condition $[X_\pm, T_\pm] = 0$.) That is,

$$T_\pm Y_\pm = Y_\pm\,(-2ik\lambda\sigma_3)\,. \tag{2.9}$$

For reference, note that, in terms of the uniformization variable, $2k\lambda = \frac{1}{2}(z^2 - q_o^4/z^2)$.

As usual, the continuous spectrum Σ_k consists of all values of k (on either sheet) such that $\lambda(k) \in \mathbb{R}$, i.e., $\Sigma_k = \mathbb{R} \cup i[-q_o, q_o]$. The corresponding set in the complex z-plane is $\Sigma_z = \mathbb{R} \cup C_o$. Hereafter we will omit the subscripts on Σ, as the intended meaning will be clear from the context. For all $z \in \Sigma$, we can now define the Jost eigenfunctions $\phi_\pm(x, t, z)$ as the *simultaneous* solutions of both parts of the Lax pair such that

$$\phi_\pm(x,t,z) = Y_\pm(z)\,\mathrm{e}^{i\theta(x,t,z)\sigma_3} + o(1) \quad \text{as}\, x \to \pm\infty\,, \tag{2.10}$$

where

$$\theta(x,t,z) = \lambda(z)(x - 2k(z)t)\,. \tag{2.11}$$

We can subtract the asymptotic behavior of the potential and rewrite the first of (2.1) as $(\phi_\pm)_x = X_\pm\,\phi_\pm + \Delta Q_\pm\,\phi_\pm$, where $\Delta Q_\pm(x,t) = Q - Q_\pm$. As usual, we introduce modified eigenfunctions by factorizing the asymptotic exponential oscillations:

$$\mu_\pm(x,t,z) = \phi_\pm(x,t,z)\,\mathrm{e}^{-i\theta(x,t,z)\sigma_3}\,, \tag{2.12}$$

so that $\lim_{x\to\pm\infty}\mu_\pm(x, t, z) = Y_\pm$. The ODEs for $\mu_\pm$ can then be formally integrated to obtain linear integral equations for the modified eigenfunctions:

$$\mu_-(x,t,z) = Y_- + \int_{-\infty}^{x} Y_-\mathrm{e}^{i\lambda(x-y)\sigma_3}Y_-^{-1}\Delta Q_-(y,t)\mu_-(y,t,z)\,\mathrm{e}^{-i\lambda(x-y)\sigma_3}\,\mathrm{d}y\,, \tag{2.13a}$$

$$\mu_+(x,t,z) = Y_+ - \int_{x}^{\infty} Y_+\mathrm{e}^{i\lambda(x-y)\sigma_3}Y_+^{-1}\Delta Q_+(y,t)\mu_+(y,t,z)\,\mathrm{e}^{-i\lambda(x-y)\sigma_3}\,\mathrm{d}y\,. \tag{2.13b}$$

Using these integral equations, in Appendix A we show that, under mild integrability conditions on the potential, the eigenfunctions can be analytically extended in the complex z-plane into the following regions:

$$\mu_{+,1},\mu_{-,2}: \quad D^+, \quad \mu_{-,1},\mu_{+,2}: \quad D^-, \tag{2.14}$$

where the subscripts 1 and 2 identify matrix columns, i.e., $\mu_\pm = (\mu_{\pm,1}, \mu_{\pm,2})$. The analyticity properties of the columns of $\phi_\pm$ follow trivially from those of $\mu_\pm$. Hereafter, we will consistently use *subscripts* $\pm$ to denote limiting values as $x \to \pm\infty$, whereas *superscripts* $\pm$ will denote the regions $D^\pm$ of analyticity.

C. Scattering matrix

Abel's theorem implies that for any solution $\phi(x, t, z)$ of (2.1) one has $\partial_x(\det\phi) = \partial_t(\det\phi) = 0$. Thus, since for all $z \in \Sigma$ $\lim_{x\to\pm\infty}\phi_\pm(x, t, z)\,\mathrm{e}^{-i\theta\sigma_3} = Y_\pm$, we have

$$\det\phi_\pm(x, t, z) = \gamma(z) \quad x, t \in \mathbb{R}\,, \quad z \in \Sigma\,. \tag{2.15}$$

Letting $\Sigma_o = \Sigma\setminus\{\pm iq_o\}$ we then have that $\forall z \in \Sigma_o$ both ϕ_- and ϕ_+ are two fundamental matrix solutions of the scattering problem. Hence

$$\phi_+(x, t, z) = \phi_-(x, t, z)\,S(z)\,, \quad x, t \in \mathbb{R}, \quad z \in \Sigma_o\,. \tag{2.16}$$

(Of course one could equivalently write $\phi_-(x, t, z) = \phi_+(x, t, z)A(z)$, which is the traditional way to introduce the scattering matrix, e.g., see Refs. 5 and 32.) For the individual columns:

$$\phi_{+,1} = s_{1,1}\phi_{-,1} + s_{2,1}\phi_{-,2}\,, \quad \phi_{+,2} = s_{1,2}\phi_{-,1} + s_{2,2}\phi_{-,2}, \tag{2.17}$$

where $S(z) = (s_{i,j})$. Moreover, (2.15) also implies $\det S(z) = 1$. The reflection coefficients that will be needed in the inverse problem are

$$\rho(z) = s_{2,1}/s_{1,1}\,, \quad \tilde\rho(z) = s_{1,2}/s_{2,2}\,, \quad \forall z \in \Sigma\,. \tag{2.18}$$

The omission of the time dependence for the scattering matrix in the above equations is not a coincidence. Indeed, since $\phi_\pm$ are simultaneous solutions of both parts of the Lax pair, the entries of $S(z)$ are independent of time, as will be the norming constants. Also, using (2.16),

$$s_{1,1}(z) = \mathrm{Wr}(\phi_{+,1}, \phi_{-,2})/\gamma\,, \quad s_{1,2}(z) = \mathrm{Wr}(\phi_{+,2}, \phi_{-,2})/\gamma\,, \tag{2.19a}$$

$$s_{2,1}(z) = \mathrm{Wr}(\phi_{-,1}, \phi_{+,1})/\gamma\,, \quad s_{2,2}(z) = \mathrm{Wr}(\phi_{-,1}, \phi_{+,2})/\gamma\,. \tag{2.19b}$$

So $s_{1,1}$ is analytic in D^+, and $s_{2,2}$ is analytic in D^-. As usual, however, the off-diagonal scattering coefficients are nowhere analytic in general. One can also use the integral equations (2.13) to write an integral representation for the scattering matrix. Such a representation, however, is not as useful as in the case of ZBCs, since the exponential oscillations of the eigenfunctions cannot be easily factored out. If desired a different, more useful integral representation can be obtained using the methods of Ref. 19. Note also that $\det\phi_\pm(x, t, z) = 0$ at $z = \pm iq_o$. As a result, generically speaking the scattering coefficients have a pole at the branch points, as in the defocusing case.[20] The behavior of the eigenfunctions and scattering matrix at the branch points is discussed in Appendix B.

D. Symmetries

The symmetries for the IST with NZBCs are complicated by the fact that: (i) while with ZBCs one only needs to deal with the map $k \mapsto k^*$, here one must also deal with the sheets of the Riemann surface. (ii) Unlike the case of ZBCs, after removing the asymptotic oscillations, the Jost solutions do not tend to the identity matrix.

Recall $\lambda_{\mathrm{II}}(k) = -\lambda_{\mathrm{I}}(k)$, and consider the following transformations compatible with (2.3): (1) $z \mapsto z^*$ (UHP/LHP), implying $(k, \lambda) \mapsto (k^*, \lambda^*)$ (same sheet); (2) $z \mapsto -q_o^2/z$ (outside/inside C_o), implying $(k, \lambda) \mapsto (k, -\lambda)$ (opposite sheets). Both these transformations correspond to symmetries of the scattering problem. Indeed, in Appendix C we show that

$$\phi_\pm(x, t, z) = -\sigma_*\phi_\pm^*(x, t, z^*)\,\sigma_*\,, \quad z \in \Sigma\,, \tag{2.20}$$

where σ_* is defined in (C2). For the individual columns, this translates to

$$\phi_{\pm,1}(x, t, z) = \sigma_*\phi_{\pm,2}^*(x, t, z^*)\,, \quad \phi_{\pm,2}(x, t, z) = -\sigma_*\phi_{\pm,1}^*(x, t, z^*)\,. \tag{2.21}$$

Similarly,

$$\phi_\pm(x, t, z) = (i/z)\,\phi_\pm(x, t, -q_o^2/z)\,\sigma_3 Q_\pm\,, \quad z \in \Sigma\,. \tag{2.22}$$

For reference, note that

$$\sigma_3 Q_\pm = \begin{pmatrix} 0 & q_\pm \\ q_\pm^* & 0 \end{pmatrix}, \quad (\sigma_3 Q_\pm)^{-1} = \begin{pmatrix} 0 & 1/q_\pm^* \\ 1/q_\pm & 0 \end{pmatrix} = \frac{1}{q_o^2}\,\sigma_3 Q_\pm\,. \tag{2.23}$$

We then have, for the columns:

$$\phi_{\pm,1}(x,t,z) = (iq_\pm^*/z)\,\phi_{\pm,2}(x,t,-q_o^2/z)\,, \quad \phi_{\pm,2}(x,t,z) = (iq_\pm/z)\,\phi_{\pm,1}(x,t,-q_o^2/z)\,. \tag{2.24}$$

Of course one can also combine the results of the first two symmetries to obtain a relation between eigenfunctions at z and $-q_o^2/z^*$.

We now use these results to obtain the symmetries of the scattering coefficients. Recalling the scattering relation (2.16) and using (2.20) we have, for all $z \in \Sigma$,

$$S^*(z^*) = -\sigma_* S(z)\sigma_*\,. \tag{2.25}$$

We therefore have the following relations between the scattering coefficients:

$$s_{2,2}(z) = s_{1,1}^*(z^*)\,, \quad s_{1,2}(z) = -s_{2,1}^*(z^*)\,. \tag{2.26}$$

Similarly, from (2.16) and using (2.22) we have, for all $z \in \Sigma$,

$$S(-q_o^2/z) = \sigma_3 Q_-\, S(z)\,(\sigma_3 Q_+)^{-1}\,. \tag{2.27}$$

Recalling (2.23) we then have, elementwise,

$$s_{1,1}(z) = (q_+^*/q_-^*)\,s_{2,2}(-q_o^2/z)\,, \quad s_{1,2}(z) = (q_+/q_-^*)\,s_{2,1}(-q_o^2/z)\,, \tag{2.28}$$

Finally, combining (2.25) and (2.27) we have

$$S^*(z^*) = -\sigma_*(\sigma_3 Q_-)^{-1}\, S(-q_o^2/z)\,\sigma_3 Q_+ \sigma_*\,. \tag{2.29}$$

Elementwise, this is

$$s_{1,1}^*(z^*) = (q_+/q_-)\,s_{1,1}(-q_o^2/z)\,, \quad s_{1,2}^*(z^*) = -(q_+^*/q_-)\,s_{1,2}(-q_o^2/z)\,, \tag{2.30a}$$

$$s_{2,1}^*(z^*) = -(q_+/q_-^*)\,s_{2,1}(-q_o^2/z)\,, \quad s_{2,2}^*(z^*) = (q_+^*/q_-^*)\,s_{2,2}(-q_o^2/z)\,. \tag{2.30b}$$

The above symmetries yield immediately the symmetries for the reflection coefficients:

$$\rho(z) = -\tilde\rho^*(z^*) = (q_-/q_-^*)\,\tilde\rho(-q_o^2/z) = -(q_-^*/q_-)\,\rho^*(-q_o^2/z^*) \quad \forall z \in \Sigma\,. \tag{2.31}$$

Note that:

(i) Even though the above symmetries are only valid for $z \in \Sigma$, whenever the individual columns and scattering coefficients involved are analytic, they can be extended to the appropriate regions of the z-plane using the Schwartz reflection principle.
(ii) Unlike the case of ZBCs, and unlike the defocusing NLS equation with NZBCs, here even the symmetries of the non-analytic scattering coefficients involve the map $z \mapsto z^*$. This is because here the continuum spectrum is not just a subset of the real z-axis.
(iii) The first involution, $z \mapsto z^*$, is the same as for ZBCs. The second one, $z \mapsto -q_o^2/z$, simply expresses the switch from one sheet to the other. Since this transformation does not affect k, if $f(k)$ is any single-valued function of k, one has $f_{\rm I}(k) = f_{\rm II}(k)$. That is, f, when expressed as a function of z, satisfies the symmetry $f(z) = f(-q_o^2/z)$. That is because f depends not on z directly, but only through the combination $k = (z - q_o^2/z)/2$. More generally, (2.22) and (2.28) relate the values of the Jost eigenfunctions and scattering coefficients on opposite sheets of the Riemann surface.

E. Discrete spectrum and residue conditions

As usual, the discrete spectrum of the scattering problem is the set of all values $z \in \mathbb{C} \setminus \Sigma$ such that eigenfunctions exist in $L^2(\mathbb{R})$. We next show that these values are the zeros of $s_{1,1}(z)$ in D^+ and those of $s_{2,2}(z)$ in D^-. Note that, unlike what happens with the defocusing NLS equation, one cannot exclude the possible presence of zeros along Σ, which in the case of ZBCs give rise to the so-called real spectral singularities.[46] For now we restrict our consideration to potentials without spectral singularities. In Sec. IV, however, we will consider the limit of a soliton solution as the discrete eigenvalue tends to Σ, and we show that such limit is well defined and it gives rise to non-trivial solutions.

For all $z \in D^+$, $\phi_{+,1}(x, t, z) \to 0$ as $x \to \infty$, and $\phi_{-,2}(x, t, z) \to 0$ as $x \to -\infty$. Recalling the first of (2.19a), if $s_{1,1}(z) = 0$ at $z = z_n$ the eigenfunctions $\phi_{+,1}$ and $\phi_{-,2}$ at $z = z_n$ must be proportional:

$$\phi_{+,1}(x,t,z_n) = b_n\,\phi_{-,2}(x,t,z_n) \tag{2.32}$$

with $b_n \neq 0$ independent of x, t, and z. We therefore obtain an eigenfunction that is bounded $\forall x \in \mathbb{R}$. Suppose that $s_{1,1}(z)$ has a finite number N of simple zeros $z_1, \ldots, z_N$ in $D^+ \cap \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. That is, let $s_{1,1}(z_n) = 0$ and $s'_{1,1}(z_n) \neq 0$, with $|z_n| > q_o$ and $\operatorname{Im} z_n > 0$ for $n = 1, \ldots, N$, and where the prime denotes differentiation with respect to z. Owing to the symmetries (2.26) and (2.28) we have that

$$s_{1,1}(z_n) = 0 \Leftrightarrow s_{2,2}(z_n^*) = 0 \Leftrightarrow s_{2,2}(-q_o^2/z_n) = 0 \Leftrightarrow s_{1,1}(-q_o^2/z_n^*) = 0. \tag{2.33}$$

For each $n = 1, \ldots, N$ we therefore have a quartet of discrete eigenvalues. That is, the discrete spectrum is the set

$$Z = \{z_n, z_n^*, -q_o^2/z_n, -q_o^2/z_n^*\}_{n=1}^N\,. \tag{2.34}$$

This is similar to what happens for the vector defocusing NLS equation with NZBCs.[35] (Instead, in the focusing case with ZBCs and in the defocusing case with NZBCs one has symmetric pairs, respectively, in the k plane and in the z plane.)

Next we derive the residue conditions that will be needed for the inverse problem. We can write (2.32) equivalently as $\mu_{+,1}(x,t,z_n) = b_n \mathrm{e}^{-2i\theta(z_n)}\mu_{-,2}(x,t,z_n)$. Thus,

$$\operatorname{Res}_{z=z_n}\left[\mu_{+,1}(x,t,z)/s_{1,1}(z)\right] = C_n\,\mathrm{e}^{-2i\theta(z_n)}\mu_{-,2}(x,t,z_n)\,, \tag{2.35}$$

where $C_n = b_n/s'_{1,1}(z_n)$. Similarly, from the second of (2.19a), if $s_{2,2}(z_n^*) = 0$ we obtain

$$\phi_{+,2}(x,t,z_n^*) = \tilde{b}_n\,\phi_{-,1}(x,t,z_n^*)\,. \tag{2.36}$$

Equivalently, $\mu_{+,2}(x,t,z_n^*) = \tilde{b}_n \mathrm{e}^{2i\theta(z_n^*)}\mu_{-,1}(x,t,z_n^*)$, and as a result

$$\operatorname{Res}_{z=z_n^*}\left[\mu_{+,2}(x,t,z)/s_{2,2}(z)\right] = \tilde{C}_n\,\mathrm{e}^{2i\theta(z_n^*)}\mu_{-,1}(x,t,z_n^*)\,, \tag{2.37}$$

where $\tilde{C}_n = \tilde{c}_n/s'_{2,2}(z_n^*)$.

The above norming constants are related by the symmetries. Using (2.21) in (2.32) and comparing with (2.36) one easily obtains $\tilde{b}_n = -b_n^*$. It is also easy to see that (2.26) implies $s'_{1,1}(z_n) = (s'_{2,2}(z_n^*))^*$. Hence we have

$$\tilde{C}_n = -C_n^*\,. \tag{2.38}$$

Finally, we need to discuss the remaining two points of the eigenvalue quartet. Using (2.24) in (2.32) and (2.36) we have the relations

$$\begin{aligned}\phi_{+,2}(x,t,-q_o^2/z_n) &= (q_-/q_+^*)\,b_n\,\phi_{-,1}(x,t,-q_o^2/z_n)\,,\\ \phi_{+,1}(x,t,-q_o^2/z_n^*) &= (q_-^*/q_+)\,\tilde{b}_n\,\phi_{-,2}(x,t,-q_o^2/z_n^*).\end{aligned} \tag{2.39}$$

Moreover, differentiating (2.28), using (2.26), and evaluating at $z = z_n$ or $z = z_n^*$, we have

$$s'_{1,1}(-q_o^2/z_n^*) = (z_n^*/q_o)^2(q_-/q_+)\,(s'_{1,1}(z_n))^*\,, \quad s'_{2,2}(-q_o^2/z_n) = (z_n/q_o)^2(q_-/q_+)^*\,(s'_{2,2}(z_n^*))^*\,.$$

Combining these relations, we then have

$$\mathrm{Res}_{z=-q_o^2/z_n^*}\left[\mu_{+,1}(x,t,z)/s_{1,1}(z)\right] = C_{N+n}\,\mathrm{e}^{-2i\theta(-q_o^2/z_n^*)}\mu_{-,2}(x,t,-q_o^2/z_n^*)\,, \tag{2.40a}$$

$$\mathrm{Res}_{z=-q_o^2/z_n}\left[\mu_{+,2}(x,t,z)/s_{2,2}(z)\right] = \tilde{C}_{N+n}\,\mathrm{e}^{2i\theta(-q_o^2/z_n)}\mu_{-,1}(x,t,-q_o^2/z_n)\,, \tag{2.40b}$$

where for brevity we defined

$$C_{N+n} = (q_o/z_n^*)^2(q_-^*/q_-)\,\tilde{C}_n\,, \quad \tilde{C}_{N+n} = (q_o/z_n)^2(q_-/q_-^*)\,C_n\,. \tag{2.41}$$

Note that $\tilde{C}_{N+n} = -C_{N+n}^*$, consistently with (2.38).

F. Asymptotics as $z \to \infty$ and $z \to 0$

As usual, the asymptotic properties of the eigenfunctions and the scattering matrix are needed to properly define the inverse problem. Moreover, the next-to-leading-order behavior of the eigenfunctions will allow us to reconstruct the potential from the solution of the Riemann-Hilbert problem.

Again, the asymptotics with NZBCs is more complicated than with ZBCs, but the calculations are streamlined in the uniformization variable. Note that the limit $k \to \infty$ corresponds to $z \to \infty$ in $\mathbb{C}_{\mathrm{I}}$ and to $z \to 0$ in $\mathbb{C}_{\mathrm{II}}$, and we will need both limits. Consider the following formal expansion:

$$\mu_-(x,t,z) = \sum_{n=0}^{\infty} \mu^{(n)}(x,t,z)\,, \tag{2.42a}$$

$$[-1ex]\,\text{with}\,\mu^{(0)}(x,t,z) = Y_-\,, \tag{2.42b}$$

$$\mu^{(n+1)}(x,t,z) = \int_{-\infty}^{x} Y_-\mathrm{e}^{i\lambda(z)(x-y)\sigma_3}\big(Y_-^{-1}\Delta Q_-(y,t)\mu^{(n)}(y,t,z)\big)\mathrm{e}^{-i\lambda(z)(x-y)\sigma_3}\,\mathrm{d}y\,. \tag{2.42c}$$

Let A_d and A_o denote, respectively, the diagonal and off-diagonal parts of a matrix A. Equation (2.42) provides an asymptotic expansion for the columns of $\mu_-(x,t,z)$ as $z \to \infty$ in the appropriate region of the z-plane. More precisely, in Appendix D we show that, as long as the potential is smooth (i.e., it admits a continuous derivative),

$$\mu_d^{(2m)} = O(1/z^m), \quad \mu_o^{(2m)} = O(1/z^{m+1}), \quad \mu_d^{(2m+1)} = O(1/z^{m+1}), \quad \mu_o^{(2m+1)} = O(1/z^{m+1}) \tag{2.43}$$

for all $m \in \mathbb{N}$. Explicitly, the above expressions hold with $\mathrm{Im}\,z \leqslant 0$ for the first column and $\mathrm{Im}\,z \geqslant 0$ for the second column. Similar results hold for $\mu_+(x,t,z)$, and are proved in the same way.

Next we consider the asymptotics as $z \to 0$. In Appendix D we show that the same formal expansion (2.42) also provides an asymptotic expansion for the columns of $\mu_-(x,t,z)$ as $z \to 0$ in the appropriate region of the z-plane, with

$$\mu_o^{(2m)} = O(z^{m-1}), \quad \mu_d^{(2m)} = O(z^m), \quad \mu_o^{(2m+1)} = O(z^m), \quad \mu_d^{(2m+1)} = O(z^m), \tag{2.44}$$

for all $m \in \mathbb{N}$. Next, by computing explicitly the first five terms in (2.42) we have that, as $z \to \infty$,

$$\mu_-(x,t,z) = I + (i/z)\sigma_3 Q(x,t) + (i/z)\int_{-\infty}^{x}\big([\sigma_3 Q_-, \Delta Q_-(y,t)] + \Delta Q_-(y,t)\sigma_3\Delta Q_-(y,t)\big)\,\mathrm{d}y + O(1/z^2)\,. \tag{2.45}$$

Equation (2.45) will allow us to reconstruct the scattering potential $Q(x,t)$ from the solution of the inverse problem.

Finally, inserting the above asymptotic expansions for the Jost eigenfunctions into the Wronskian representations (2.19a) one shows that, as $z \to \infty$ in the appropriate regions of the complex z-plane,

$$S(z) = I + O(1/z)\,. \tag{2.46}$$

Explicitly, the above estimate holds with $\operatorname{Im} z \geqslant 0$ and $\operatorname{Im} z \leqslant 0$ for $s_{1,1}$ and $s_{2,2}$, respectively, and with $\operatorname{Im} z = 0$ for $s_{1,2}$ and $s_{2,1}$. Similarly, one shows that, as $z \to 0$,

$$S(z) = \operatorname{diag}(q_-/q_+,\, q_+/q_-) + O(z)\,, \tag{2.47}$$

again in the appropriate regions of the z-plane.

III. INVERSE PROBLEM

A. Riemann-Hilbert problem

As usual, the formulation of the inverse problem begins from (2.16), which we now regard as a relation between eigenfunctions analytic in D^+ and those analytic in D^-. Thus, we introduce the sectionally meromorphic matrices

$$M^+(x,t,z) = (\mu_{+,1}/s_{1,1},\ \mu_{-,2})\,, \quad M^-(x,t,z) = (\mu_{-,1},\ \mu_{+,2}/s_{2,2})\,. \tag{3.1}$$

(Recall that subscripts $\pm$ indicate normalization as $x \to \pm\infty$, while superscripts $\pm$ distinguish between analyticity in D^+ and D^-, respectively.) From (2.17) we then obtain the jump condition

$$M^-(x,t,z) = M^+(x,t,z)\,(I - G(x,t,z))\,, \quad z \in \Sigma\,, \tag{3.2}$$

where the jump matrix is

$$G(x,t,z) = \begin{pmatrix} 0 & -\mathrm{e}^{2i\theta(x,t,z)}\tilde\rho(z) \\ \mathrm{e}^{-2i\theta(x,t,z)}\rho(z) & \rho(z)\tilde\rho(z) \end{pmatrix}. \tag{3.3}$$

Equations (3.1)–(3.3) define a matrix, multiplicative, homogeneous Riemann-Hilbert problem (RHP). [Of course one can equivalently write the RHP as $M^+(x,t,z) = M^-(x,t,z)(I - \tilde G(x,t,z))$.] As usual, to complete the formulation of the RHP one needs a normalization condition, which in this case is the asymptotic behavior of $M^\pm$ as $z \to \infty$. Recalling the asymptotic behavior of the Jost eigenfunctions and scattering coefficients, it is easy to check that

$$M^\pm = I + O(1/z)\,, \quad z \to \infty\,. \tag{3.4}$$

On the other hand,

$$M^\pm = (i/z)\sigma_3 Q_- + O(1)\,, \quad z \to 0\,. \tag{3.5}$$

Thus, as with the defocusing NLS equation with NZBCs,[35] in addition to the behavior at $z = \infty$ and the poles from the discrete spectrum one also needs to subtract the pole at $z = 0$ in order to obtain a regular RHP.

To solve the RHP, one needs to regularize it by subtracting out the asymptotic behavior and the pole contributions. Recall that discrete eigenvalues come in symmetric quartets [cf. (2.33)]. It is then convenient to define $\zeta_n = z_n$ and $\zeta_{n+N} = -q_o^2/z_n^*$ for $n = 1, \ldots, N$ and rewrite (3.2) as

$$\begin{aligned} M^- - I - (i/z)\sigma_3 Q_- - \sum_{n=1}^{2N} \frac{\operatorname{Res}_{\zeta_n^*} M^-}{z - \zeta_n^*} - \sum_{n=1}^{2N} \frac{\operatorname{Res}_{\zeta_n} M^+}{z - \zeta_n} \\ = M^+ - I - (i/z)\sigma_3 Q_- - \sum_{n=1}^{2N} \frac{\operatorname{Res}_{\zeta_n} M^+}{z - \zeta_n} - \sum_{n=1}^{2N} \frac{\operatorname{Res}_{\zeta_n^*} M^-}{z - \zeta_n^*} - M^+ G\,, \end{aligned} \tag{3.6}$$

The left-hand side (LHS) of (3.6) is now analytic in D^- and is $O(1/z)$ as $z \to \infty$ there, while the sum of the first four terms of the right-hand side (RHS) is analytic in D^+ and is $O(1/z)$ as $z \to \infty$ there. Finally, the asymptotic behavior of the off-diagonal scattering coefficients implies that

$G(x, t, z)$ is $O(1/z)$ as $z \to \pm\infty$ and $O(z)$ as $z \to 0$ along the real axis. We then introduce the Cauchy projectors $P_\pm$ over Σ:

$$P_\pm[f](z) = \frac{1}{2\pi i}\int_\Sigma \frac{f(\zeta)}{\zeta - (z \pm i0)}\,\mathrm{d}\zeta\,,$$

where $\int_\Sigma$ denotes the integral along the oriented contour shown in Fig. 1, and the notation $z \pm i0$ indicates that when $z \in \Sigma$, the limit is taken from the left/right of it. Now recall Plemelj's formulae: if $f^\pm$ are analytic in $D^\pm$ and are $O(1/z)$ as $z \to \infty$, one has $P^\pm f^\pm = \pm f^\pm$ and $P^+f^- = P^-f^+ = 0$. Applying P^+ and P^- to (3.6) we then have

$$M(x,t,z) = I + (i/z)\sigma_3 Q_- + \sum_{n=1}^{2N}\frac{\mathrm{Res}_{\zeta_n} M^+}{z-\zeta_n} + \sum_{n=1}^{2N}\frac{\mathrm{Res}_{\zeta_n^*} M^-}{z-\zeta_n^*} + \frac{1}{2\pi i}\int_\Sigma \frac{M^+(x,t,\zeta)}{\zeta - z}\,G(x,t,\zeta)\,\mathrm{d}\zeta\,, \quad z \in \mathbb{C}\setminus\Sigma\,. \tag{3.7}$$

As usual, the expressions for M^+ and M^- are formally identical, except for the fact that the integral appearing in the right-hand side is a P^+ and a P^- projector, respectively.

B. Residue conditions and reconstruction formula

To close the system we need to obtain an expression for the residues appearing in (3.7). The residue relations (2.35) and (2.40) imply that only the first column of M^+ has a pole at $z = z_n$ and $z = -q_o^2/z_n^*$, and its residue is proportional to the second column of M^+ at that point. Explicitly,

$$\mathrm{Res}_{\zeta_n} M^+ = \left(C_n\,\mathrm{e}^{-2i\theta(x,t,\zeta_n)}\mu_{-,2}(x,t,\zeta_n)\,,0\right), \quad n = 1,\ldots,2N\,, \tag{3.8a}$$

$$\mathrm{Res}_{\zeta_n^*} M^- = \left(0\,,\tilde{C}_n\,\mathrm{e}^{2i\theta(x,t,\zeta_n^*)}\mu_{-,1}(x,t,\zeta_n^*)\right), \quad n = 1,\ldots,2N\,. \tag{3.8b}$$

We can therefore evaluate the second column of (3.7) at $z = z_n$ and at $z = -q_o^2/z_n^*$, obtaining

$$\mu_{-,2}(x,t,\zeta_n)=\begin{pmatrix}-iq_-/\zeta_n\\ 1\end{pmatrix}+\sum_{k=1}^{2N}\frac{\tilde{C}_k\,\mathrm{e}^{2i\theta(x,t,\zeta_k^*)}}{\zeta_n-\zeta_k^*}\,\mu_{-,1}(x,t,\zeta_k^*) + \frac{1}{2\pi i}\int_\Sigma\frac{M^+(x,t,\zeta)}{\zeta-\zeta_n}\,G(x,t,\zeta)\,\mathrm{d}\zeta\,, \tag{3.9a}$$

for $n = 1, \ldots, 2N$. Similarly, we can evaluate the first column of (3.7) at $z = z_n^*$ and at $z = -q_o^2/z_n$, obtaining

$$\mu_{-,1}(x,t,\zeta_n^*)=\begin{pmatrix}1\\ iq_-^*/\zeta_n^*\end{pmatrix}+\sum_{j=1}^{2N}\frac{C_j\,\mathrm{e}^{-2i\theta(x,t,\zeta_j)}}{\zeta_n^*-\zeta_j}\,\mu_{-,2}(x,t,\zeta_j)+\frac{1}{2\pi i}\int_\Sigma\frac{M^+(x,t,\zeta)}{\zeta-\zeta_n^*}\,G(x,t,\zeta)\,\mathrm{d}\zeta\,, \tag{3.9b}$$

again for $n = 1, \ldots, 2N$. Finally, evaluating $M^+(x, t, z)$ via (3.7) (thus with a P^+ projector) for $z \in \Sigma$ we obtain, together with Eqs. (3.9a) and (3.9b), a closed linear system of algebraic-integral equations for the solution of the RHP. We expect that solvability conditions for the RHP can be established using techniques similar to Ref. 9.

The last remaining task is to reconstruct the potential from the solution of the RHP. From (3.7), one obtains the asymptotic behavior of $M^\pm(x, t, z)$ as $z \to \infty$ as

$$M(x,t,z) = I + \frac{1}{z}\left\{i\sigma_3 Q_- + \sum_{n=1}^{2N}\left(\mathrm{Res}_{\zeta_n} M^+ + \mathrm{Res}_{\zeta_n^*} M^-\right) - \frac{1}{2\pi i}\int_\Sigma M^+(x,t,\zeta)\,G(x,t,\zeta)\,\mathrm{d}\zeta\right\} + O(1/z^2)\,, \tag{3.10}$$

where the residues are given by (3.8). Taking $M = M^+$ and comparing the 1,2 element of (3.10) of this expression with (2.45) we then obtain the reconstruction formula for the potential:

$$q(x,t) = q_- + i\sum_{n=1}^{2N} \tilde{C}_n\, \mathrm{e}^{2i\theta(x,t,\zeta_n^*)}\,\mu_{-,1,1}(x,t,\zeta_n^*) + \frac{1}{2\pi}\int_\Sigma (M^+G)_{1,2}(x,t,\zeta)\,\mathrm{d}\zeta\,. \qquad (3.11)$$

Recall that the time dependence of the solution is automatically taken into account by the fact that the Jost eigenfunctions are simultaneous solutions of both parts of the Lax pair.

C. Trace formulae and "theta" condition

Recall that $s_{1,1}$ and $s_{2,2}$ are analytic in D^+ and in D^-, respectively. Also recall that the discrete spectrum is composed of quartets: $z_n, z_n^*, -q_o^2/z_n, -q_o^2/z_n^*$ $\forall n = 1,\ldots,N$. Then the functions

$$\beta^+(z) = s_{1,1}(z)\prod_{n=1}^N \frac{(z-z_n^*)(z+q_o^2/z_n)}{(z-z_n)(z+q_o^2/z_n^*)}\,,\qquad \beta^-(z) = s_{2,2}(z)\prod_{n=1}^N \frac{(z-z_n)(z+q_o^2/z_n^*)}{(z-z_n^*)(z+q_o^2/z_n)}\,,$$

are analytic in $D^\pm$ like $s_{1,1}(z)$ and $s_{2,2}(z)$, respectively. But, unlike $s_{1,1}(z)$ and $s_{2,2}(z)$, they have no zeros. Moreover, $\beta^\pm(z) \to 1$ as $z \to \infty$ in the appropriate domains. Finally, for all $z \in \Sigma$ we have $\beta^+(z)\beta^-(z) = s_{1,1}(z)s_{2,2}(z)$, and the relation $\det S(z) = 1$ yields $1/[s_{1,1}(z)s_{2,2}(z)] = 1 - \rho(z)\tilde{\rho}(z) = 1 + \rho(z)\rho^*(z^*)$, implying

$$\beta^+(z)\beta^-(z) = 1/(1+\rho(z)\rho^*(z^*))\,,\qquad z\in\Sigma\,. \qquad (3.12)$$

(Note that, since Σ is not just the real z-axis, the above expression does not reduce to $1 + |\rho(z)|^2$, unlike the defocusing case and the focusing case with ZBCs.) Equation (3.12) amounts to a jump condition for a scalar, multiplicative, homogeneous RHP. Taking logarithms and applying the Cauchy projectors (as in Sec. III A) we have

$$\log\beta^\pm(z) = \mp\frac{1}{2\pi i}\int_\Sigma \frac{\log[1+\rho(\zeta)\rho^*(\zeta^*)]}{\zeta - z}\mathrm{d}\zeta\,,\qquad z\in D^\pm\,.$$

Substituting $\beta^+(z)$ for $s_{1,1}(z)$, we then obtain the so-called "trace" formula

$$s_{1,1}(z) = \exp\left[-\frac{1}{2\pi i}\int_\Sigma \frac{\log[1+\rho(\zeta)\rho^*(\zeta^*)]}{\zeta-z}\mathrm{d}\zeta\right]\prod_{n=1}^N\frac{(z-z_n)(z+q_o^2/z_n^*)}{(z-z_n^*)(z+q_o^2/z_n)}\,,\qquad z\in D^+\,, \qquad (3.13)$$

which expresses the analytic scattering coefficient in terms of the discrete eigenvalues and the reflection coefficient. A similar formula is obtained for $s_{2,2}(z)$. In the special case of reflectionless solutions, $s_{1,2}(z) = s_{2,1}(z) \equiv 0$ $\forall z \in \Sigma$, and the integral in (3.13) is identically zero.

Taking the limit $z \to 0$ of (3.13) from the LHP and recalling (2.47) we also obtain the so-called "theta" condition,

$$\arg(q_-/q_+) = 4\sum_{n=1}^N \arg z_n + \frac{1}{2\pi i}\int_\Sigma \frac{\log[1+\rho(\zeta)\rho^*(\zeta^*)]}{\zeta}\mathrm{d}\zeta\,, \qquad (3.14)$$

which relates the phase difference between the asymptotic values of the potential to the discrete spectrum and reflection coefficient. Note that

$$\int_{q_o}^{\infty} \log[1+|\rho(\zeta)|^2]\,\mathrm{d}\zeta/\zeta = -\int_{-q_o}^{0}\log[1+|\rho(\zeta)|^2]\,\mathrm{d}\zeta/\zeta\,,$$

because $|\rho(\zeta)| = |\rho(-q_o^2/\zeta)|$ thanks to the symmetry (2.31). A similar relation holds between the integral from $-\infty$ to $-q_o$ and that from 0 to q_o. Finally,

$$\int_{C_o^+} \log[1+\rho(\zeta)\rho^*(\zeta^*)]\,\mathrm{d}\zeta/\zeta = -\int_{C_o^-}\log[1+\rho(\zeta)\rho^*(\zeta^*)]\,\mathrm{d}\zeta/\zeta\,,$$

where $C_0^{\pm}$ denote, respectively, the upper-half and lower-half semicircles of radius q_o. Due to the orientation of Σ, however, these individual contributions do not cancel each other out, but rather add together (cf. Fig. 1), implying that the reflection coefficient could in principle contribute to the asymptotic phase difference, similarly to the defocusing case.[15]

D. Reflectionless potentials

We now look at potentials $q(x, t)$ for which the reflection coefficient $\rho(z)$ vanishes identically. As usual, in this case there is no jump from M^+ to M^- across the continuous spectrum, and the inverse problem therefore reduces to an algebraic system, whose solution yields the soliton solutions of the integrable nonlinear equation.

Recall $\zeta_{N+j} = -q_o^2/z_j$ and $C_{N+j} = -(q_o/z_j^*)^2(q_-^*/q_-)\,C_j^*$ for all $j = 1, \ldots, N$, and that $\theta(x, t, z^*) = \theta^*(x, t, z)$. Recall also $\tilde{C}_j = -C_j^*$ for all $j = 1, \ldots, 2N$. It is convenient to introduce the quantities

$$c_j(x,t,z) = \frac{C_j}{z-\zeta_j}\,\mathrm{e}^{-2i\theta(x,t,\zeta_j)}, \qquad j = 1,\ldots,2N\,. \tag{3.15}$$

Note from (3.11) that only the first component of the eigenfunction is needed in the reconstruction formula. The algebraic system obtained from the inverse problem is then expressed as

$$\mu_{-,1,2}(\zeta_j) = -iq_-/\zeta_j - \sum_{k=1}^{2N} c_k^*(\zeta_j^*)\,\mu_{-,1,1}(\zeta_k^*)\,, \quad j = 1,\ldots,2N\,, \tag{3.16a}$$

$$\mu_{-,1,1}(\zeta_n^*) = 1 + \sum_{j=1}^{2N} c_j(\zeta_n^*)\,\mu_{-,1,2}(\zeta_j)\,, \quad n = 1,\ldots,2N\,, \tag{3.16b}$$

where for brevity we omitted the x and t dependence. Substituting (3.16a) into (3.16b) yields

$$\mu_{-,1,1}(x,t,\zeta_n^*) = 1 - iq_-\sum_{j=1}^{2N} c_j(\zeta_n^*)/\zeta_j - \sum_{j=1}^{2N}\sum_{k=1}^{2N} c_j(\zeta_n^*)c_k^*(\zeta_j^*)\,\mu_{-,1,1}(x,t,\zeta_k^*)\,, \quad n = 1,\ldots,2N\,. \tag{3.17}$$

We now write this system in matrix form. Introducing $\mathbf{X} = (X_1, \ldots, X_{2N})^t$ and $\mathbf{B} = (B_1, \ldots, B_{2N})^t$, where

$$X_n = \mu_{-,1,1}(x,t,\zeta_n)\,, \quad B_n = 1 - iq_-\sum\nolimits_{j=1}^{2N} c_j(\zeta_n^*)/\zeta_j\,, \quad n = 1,\ldots,2N\,,$$

and defining the $2N \times 2N$ matrix $A = (A_{n,k})$, where

$$A_{n,k} = \sum_{j=1}^{2N} c_j(\zeta_n^*)c_k^*(\zeta`_j^*)\,, \quad n,k = 1,\ldots,2N,$$

the system (3.17) becomes simply $M\,\mathbf{X} = \mathbf{B}$, where $M = I + A = (\mathbf{M}_1, \ldots, \mathbf{M}_{2N})$. The solution of the system is simply $X_n = \det\, M_n^{\mathrm{ext}}/\det M$ for $n = 1, \ldots, 2N$, where

$$M_n^{\mathrm{ext}} = (\mathbf{M}_1,\ldots,\mathbf{M}_{n-1},\mathbf{B},\mathbf{M}_{n+1},\ldots,\mathbf{M}_{2N})\,.$$

Finally, upon substituting $X_1, \ldots, X_{2N}$ into the reconstruction formula, the resulting expression for the potential can be written compactly as

$$q(x,t) = q_- - i\,\frac{\det M^{\mathrm{aug}}}{\det M}\,, \tag{3.18}$$

where the augmented $(2N + 1) \times (2N + 1)$ matrix M^{aug} is given by

$$M^{\mathrm{aug}} = \begin{pmatrix} 0 & \mathbf{Y}^t \\ \mathbf{B} & M \end{pmatrix}, \quad \mathbf{Y} = (Y_1,\ldots,Y_{2N})^t\,,$$

and $Y_n = \tilde{C}_n \, e^{2i\theta(x,t,\zeta_n^*)}$ for $n = 1, \ldots, 2N$. Note that, even though the discrete eigenvalues appear in quartets in the NZBC case as opposed to pairs in the ZBC case, the number of unknowns in the inverse problem is still the same as that of the ZBC case. This is because the symmetry (2.24) implies

$$\phi_{-,1}(\zeta_{N+j}^*) = i(\zeta_j/q_-)\,\phi_{-,2}(\zeta_j)\,, \quad \phi_{-,2}(\zeta_{N+j}) = i(\zeta_j/q_-)^*\,\phi_{-,1}(\zeta_j^*)\,,$$

for all $j = 1, \ldots, N$. Therefore, one can equivalently write the linear algebraic system (3.16) in terms of just $2N$ unknowns, as in the case of ZBCs.

IV. SOLITON SOLUTIONS

The focusing NLS equation with NZBC possesses a rich family of soliton solutions[8,29,30,34,40] (some of which have been re-discovered several times[10,31,41,42]). It should be noted that some of these solutions have recently been observed experimentally.[26] It should also be noted that, in light of the relationship between these soliton solutions, the MI and rogue waves, it is likely that these soliton solutions are relevant to rogue waves[6] in water waves[7] as well as optics.[39]

A. Stationary solitons

The simplest non-trivial solutions are of course the one-soliton solutions: $N = 1$. Recall that the NLS equation possesses a scaling symmetry. That is, if $q(x, t)$ is a solution, so is $aq(ax, a^2t)$, for any $a \in \mathbb{R}$. Therefore, without loss of generality we can set $q_o = 1$ in what follows.

We first discuss the case of a purely imaginary eigenvalue. Let $z_1 = iZ$, with $Z > 1$, and $C_1 = e^{\xi + i\varphi}$, with $\xi, \varphi \in \mathbb{R}$. The theta condition (3.14) implies that the corresponding asymptotic phase difference is 2π, that is, no phase difference in this case. From the general N-solution formula (3.18) we then have in this case

$$q(x,t) = \frac{\cosh\chi + \frac{1}{2}\,c_+(1 + c_-^2/c_+^2)\sin s - i c_- \cos s}{\cosh\chi + A\sin s}\,, \tag{4.1}$$

with

$$\chi(x,t) = c_- x + c_0 + \xi\,, \quad s(x,t) = c_+ c_- t + \varphi\,, \tag{4.2}$$

and where

$$c_\pm = Z \pm 1/Z\,, \quad c_0 = \operatorname{arc\,tanh}\frac{c_+^2 - 4Z^2c_-^2}{c_+^2 + 4Z^2c_-^2}\,, \quad A = 2/c_+ < 1\,. \tag{4.3}$$

This solution was first found by Kuznetsov in 1977 using direct methods,[29] and was rediscovered by Ma in 1979[30] as well as others later on. An example of this solution is shown in Fig. 2 (left). Note that this solution is homoclinic in x and periodic in t, which is the opposite situation compared to the solutions of the focusing NLS equation with periodic BCs.

It is easy to see that, for any fixed value of t, the maximum of the solution occurs at $\chi = 0$, i.e., at $x_{\max} = -c_0/c_-$. Moreover, simple calculus shows that the maximum and the minimum values of the modulus of $|q(x, t)|$ at $\xi = 0$ occur for $t_{\max} = -\pi/(2c_+c_-)$ and $t_{\min} = \pi/2(c_+c_-)$, and that these maximum and minimum values of $|q(x, t)|$ are, respectively, $c_+ \pm 1 = Z + 1/Z \pm 1$. The whole solution is also periodic in t with a period of $2\pi/c_+c_- = 2\pi/(Z^2 - 1/Z^2)$.

The width of the solution can be quantified by noting that at $t = t_{\max}$, there are always two cavitation point, i.e., values $x = \pm x_0$ such that $q(\pm x_o, t_{\max}) = 0$. These values are easily found to be $\pm x(\xi_o)$ with $\xi_o = \operatorname{arc\,cosh}((Z^2 + 1/Z^2)/(Z + 1/Z))$.

Performing a translation of coordinates so that the origin of the soliton is located at the origin and taking the limit $Z \to 1$ one obtains Peregrine's rational solution of focusing NLS:[34]

$$q(x,t) = \frac{4x^2 - 16it + 16t^2 - 3}{4x^2 + 16t^2 + 1}\,. \tag{4.4}$$

Such a solution is shown in Fig. 2 (right). Note that this solution corresponds to a zero of $s_{1,1}(z)$ at $z = iq_o$, i.e., along the continuous spectrum. As a result, it does not give rise to a bound state

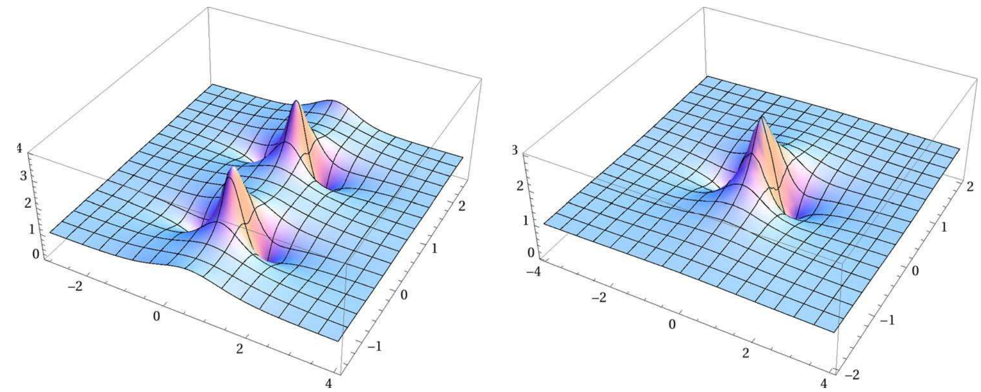

FIG. 2. Left: A one-soliton solution of the focusing NLS equation with NZBC at infinity, obtained with and $q_o = 1$ and $z_1 = 2i$, yielding no asymptotic phase difference. Right: The Peregrine soliton, obtained by taking the limit $z_1 \to i$. Here and in all subsequent three-dimensional plots, the horizontal axes are x and t and the vertical axis is $|q(x, t)|$.

(i.e., the corresponding eigenfunctions do not decay as $x \to \pm\infty$). Conversely, restoring the background amplitude parameter q_o, it is straightforward to see that, in the limit $q_o \to 0$, one recovers the usual 1-soliton solution of the focusing NLS equation with ZBC. Explicitly, keeping the location of the discrete eigenvalue fixed at $z_1 = iZ$, as $q_o \to 0$ one obtains

$$q(x,t) = -i\,\mathrm{e}^{iZ^2t+\phi}\,\mathrm{sech}[Zx + \log(2Z) + \xi] + O(q_o)\,.$$

B. Non-stationary solitons

We now discuss the one-soliton solutions obtained for a generic position of the discrete eigenvalue. Again, we use the scaling invariance of the NLS equation to set $q_o = 1$ without loss of generality. It is convenient to parametrize such eigenvalue as $z_1 = iZ\mathrm{e}^{i\alpha}$, with $Z > 1$ and $\alpha \in (-\pi/2, \pi/2)$. Tedious but straightforward algebra shows that from the general N-solution formula (3.18) we then have

$$q(x,t) = \frac{\cosh(\chi + 2i\alpha) + \frac{1}{2}A\,[c_{+2}\,(Z^2\sin(s+2\alpha) - \sin s) - i\,c_{-2}\,(Z^2\cos(s+2\alpha) - \cos s)]}{\cosh\chi + A\,[Z^2\sin(s+2\alpha) - \sin s]}\,, \tag{4.5}$$

with

$$\chi(x,t) = c_- x\cos\alpha - c_{+2}t\sin(2\alpha) + c_0' + \xi\,, \tag{4.6a}$$

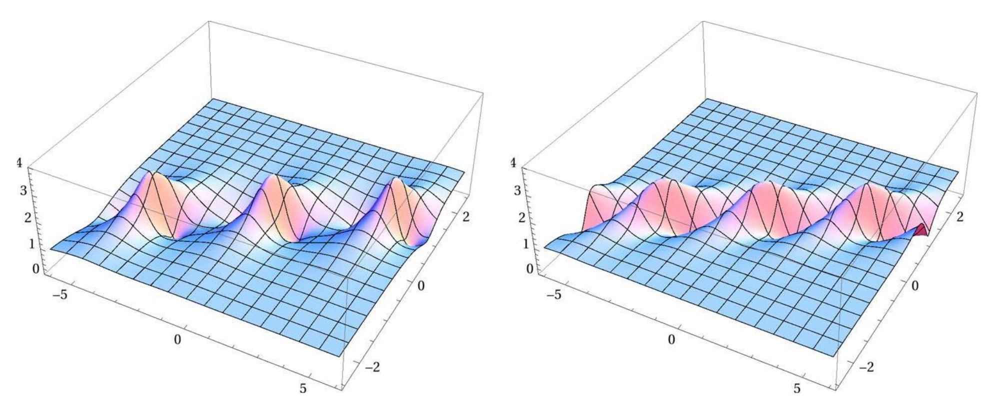

FIG. 3. Non-stationary 1-soliton solutions of the focusing NLS with NZBC, obtained with a discrete eigenvalue $z_1 = \sqrt{2}\,\mathrm{e}^{i\pi/4}$ (left) and with $z_1 = 2\,\mathrm{e}^{i\pi/6}$ (right), resulting in an asymptotic phase difference of π and $2\pi/3$, respectively.

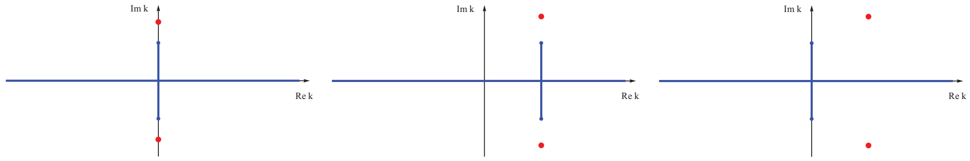

FIG. 4. Discrete spectrum (red dots) and continuous spectrum (blue lines) in the k-plane for a stationary soliton (left), a Galilean-boosted stationary soliton (center), and the non-stationary soliton (center) described in the text.

$$s(x,t) = c_+ x \sin\alpha + c_{-2} t \cos(2\alpha) + \varphi\,, \tag{4.6b}$$

and where

$$c_{+2} = Z^2 + 1/Z^2\,, \quad c_{-2} = Z^2 - 1/Z^2 = c_+ c_-\,, \tag{4.7a}$$

$$A = 1/(c'_+ c'_-)\,, \quad c'_0 = \log(c'_+/c'_-)\,, \tag{4.7b}$$

$$c'_+ = |1 - Z^2\,\mathrm{e}^{-2i\alpha}|\,, \quad c'_-(Z + 1/Z)/(2\cos\alpha)\,, \tag{4.7c}$$

with $c_\pm$ as before and where the addition formula for the hyperbolic cosine was used: $\cosh(a + ib) = \cosh a \cos b + i \sinh a \sin b$. This solution was first derived by Tajiri and Watanabe in 1998[40] and rediscovered recently by Zakharov and Gelash using the dressing method.[41] Figure 3 (left) shows a solution with $z_1 = 1 + i$, yielding an asymptotic phase difference of π, while Fig. 3 (right) shows a solution with $z_1 = 2\,\mathrm{e}^{i\pi/6}$, yielding an asymptotic phase difference of $2\pi/3$.

Like the solutions (4.1), the solutions (4.5) are homoclinic in x. Now however the peak of the solution does not remain localized at a fixed value in x, unlike the solutions generated from purely imaginary eigenvalues. The motion of the center of mass can be easily obtained by noting that, like with the stationary solitons, the maximum of $|q(x,t)|$ is still found for $\xi = 0$. The equation $\chi(x,t) = 0$ then yields the straight line in the xt-plane along which the peak is located, resulting in a soliton velocity of $v_{Z,\alpha} = \sin\alpha\,(Z^2 + 1/Z^2)/(Z - 1/Z)$.

It is important to realize, unlike what happens with the solitons of the focusing NLS equation with ZBC, the traveling solution (4.5) is *not* a simply Galilean transformation of the stationary solution (4.1). This difference can be understood in two ways. First, by noting that both solutions satisfy the *same* constant BCs $q(x,t) \to q_\pm$ as $x \to \pm\infty$, whereas a Galilean-boosted stationary solutions would have an oscillating phase with respect to x as $x \to \pm\infty$. The difference between the two solutions can also be understood from a spectral point of view, as shown in Fig. 4. For the Galilean-boosted stationary solution, the real part of the discrete eigenvalue coincides with the location of the branch cut. In contrast, for the traveling soliton solution (4.5), the discrete eigenvalue does not lie directly above the branch cut. Of course, in the limit $q_o \to 0$ both kinds of solutions

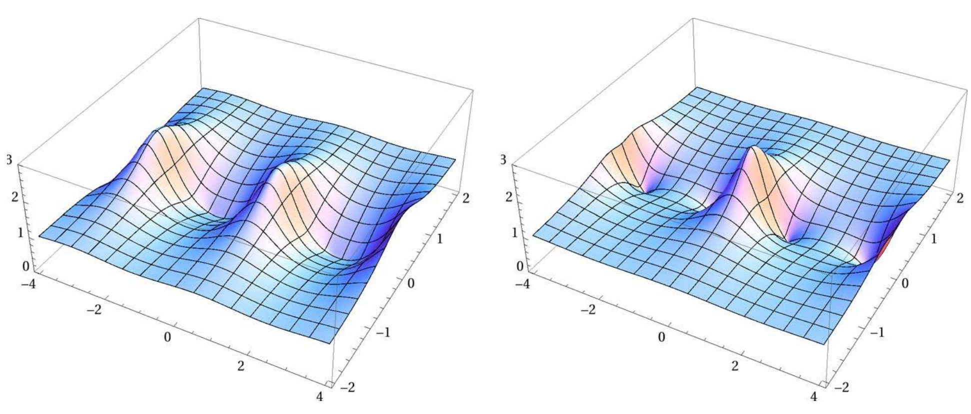

FIG. 5. Two so-called Akhmediev solitons, obtained for $\alpha = \pi/3$ (left) and $\alpha = \pi/4$ (right).

— namely, the traveling soliton (4.5) and the Galilean-boosted stationary soliton — reduce to the same, traveling soliton solution of the focusing NLS equation with ZBC.

As with the stationary solitons, one can take the limit $Z \to 1$ of the solution (4.5). In this case, however, the additional parameter $\alpha = \arg z_1 - \pi/2$ is present. As a result, performing again a translation of coordinates so that the maximum of the solution is at the origin, one obtains the so-called Akhmediev solitons:[8]

$$q(x,t) = \frac{\cosh[2\sin(2\alpha)t - 2i\alpha] - \cos\alpha\,\cos[2\sin(\alpha)x]}{\cosh[2\sin(2\alpha)t] - \cos\alpha\,\cos[2\sin(\alpha)x]}\,, \tag{4.8}$$

where again the complex addition formula for the hyperbolic cosine was used. Two examples are described in Fig. 5, corresponding to $\alpha = \pi/3$ (left) and $\alpha = \pi/4$ (right). Note that these solutions are periodic in x and homoclinic in t, which is the opposite kind of behavior to that of the solution (4.5). While this may appear counter-intuitive, it is easily understood by noting that the slope of the characteristic line that governs the center of mass motion tends to infinity as $Z \to 1$. In other words, $\lim_{Z\to 1} |v_{z,\alpha}| = \infty$.

Of course the solution (4.8) reduces to the Peregrine soliton for $\alpha = 0$. And, like the Peregrine soliton, this solution originates from a zero of $s_{1,1}(z)$ along the branch cut, which is part of the continuous spectrum, and as a result it does not correspond to a bound state for the eigenfunctions. For generic non-zero values of α, the spatial period of the solution (4.8) is easily found to be $\pi/\sin\alpha$, while the maximum is $[\cos\alpha - \cos(2\alpha)]/(1 - \cos\alpha)$. Both are decreasing functions of $|\alpha|$, and in particular the maximum tends to 1 (resulting in the uniform solution) as $\alpha \to \pm\pi/2$. Like the soliton solution (4.1), the Akhmediev solitons (4.8) also possess cavitation points. More precisely, for $|\alpha| \le \pi/3$, $q(x, 0) = 0$ at $x = \pm\frac{1}{2}\csc\alpha\ \operatorname{arc\,cos}[\sec\alpha\cos(2\alpha)]$.

C. Multi-soliton solutions

Of course the expressions derived in Sec. IV are not limited to one-soliton solutions, and it allows one to obtain explicit solutions with an arbitrary number of solitons. As an example, Figure 6 shows two different two-soliton solutions, obtained with different sets of discrete eigenvalues. Explicitly, Fig. 6 (left) shows a bound state, obtained for $z_1 = 2i$ and $z_2 = 3i$, while Fig. 6 (right) shows a soliton interaction, obtained for $z_1 = -1 + 2i$ and $z_2 = 2 + i$. Solutions with larger numbers of solitons can be obtained just as easily.

V. CONCLUDING REMARKS

The results of this work provide a framework to address a number of interesting issues. Among them: (i) the characterization of the soliton interactions (e.g., see Refs. 5 and 32 for the case of ZBCs); (ii) the use of the IST to solve the direct problem and study the time evolution of various classes of ICs

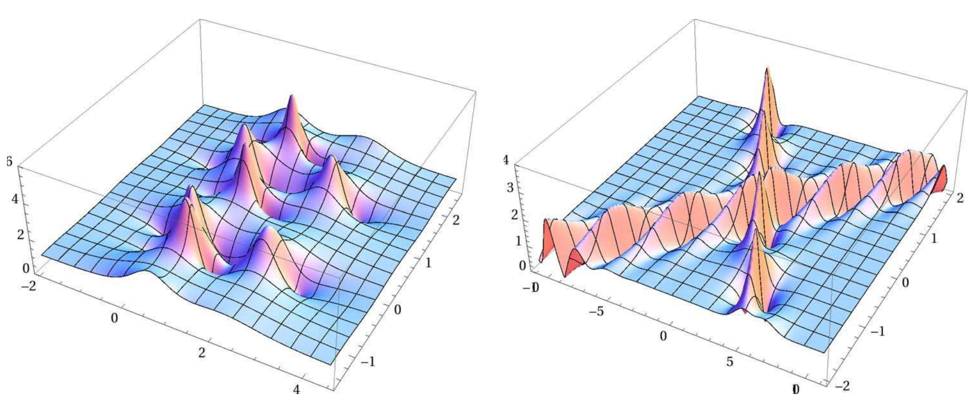

FIG. 6. Two-soliton solutions of the NLZ equation with NZBCs at infinity and $q_o = 1$. Left: $z_1 = 2i$ and $z_2 = 3i$, resulting in a bound state. Right: $z_1 = -1 + 2i$ and $z_2 = 2 + i$, resulting in a soliton interaction.

(e.g., as in Ref. 37 for the focusing case with ZBCs and Ref. 22 for the defocusing case with NZBCs); (iii) the characterization of the nonlinear stage of the Benjamin-Feir instability; (iv) the study of the long-time asymptotic using the nonlinear steepest descent method[17,18] or other techniques;[5,32] (v) a characterization of the scattering problem with regard to the existence of discrete eigenvalues (similar Refs. 27 and 28 for the focusing case with ZBCs and to Ref. 15 for the defocusing case with NZBCs). Also, a number of related research problems exist that can now be studied using a similar approach. Among them: (vi) the solution of the initial-value problem (IVP) for the vector focusing NLS equation with NZBCs, which, remarkably, is still completely open (indeed, even the IVP for the defocusing case with NZBCs was only solved recently[35]); (vii) the solution of the IVP for the Maxwell-Bloch equations with NZBCs (see Refs. 3 and 23 for the IST with ZBCs). We plan to address some of the above issues and investigate some of these problems in the near future.

ACKNOWLEDGMENTS

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APPENDIX A: ANALYTICITY OF THE EIGENFUNCTIONS

In this and Appendices B–D we provide the explicit proofs of various results presented in the text. For brevity, we omit the time dependence when doing so does not cause ambiguity.

We start by rewriting the first of the integral equations (2.13) that define the Jost eigenfunctions:

$$\mu_-(x,z) = Y_-\left[I + \int_{-\infty}^{x} \mathrm{e}^{i\lambda(x-y)\sigma_3} Y_-^{-1}\Delta Q_-(y)\mu_-(y,z)\,\mathrm{e}^{-i\lambda(x-y)\sigma_3}\,\mathrm{d}y\right]. \tag{A1}$$

The limits of integration imply that $x - y$ is always positive for μ_- and always negative for μ_+. Note the special structure of the product $\mathrm{e}^{i\xi\sigma_3} M\,\mathrm{e}^{-i\xi\sigma_3}$, namely,

$$\mathrm{e}^{i\xi\sigma_3} M\,\mathrm{e}^{-i\xi\sigma_3} = \begin{pmatrix} m_{1,1} & \mathrm{e}^{2i\xi} m_{1,2} \\ \mathrm{e}^{-2i\xi} m_{2,1} & m_{2,2} \end{pmatrix}. \tag{A2}$$

Also note that the matrix products in the RHS of (A1) operate column-wise. In particular, letting $W(x,z) = Y_-^{-1}\mu_-$, for the first column w of W one has

$$w(x,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} G(x-y,z)\Delta Q_-(y)Y_-(z)\,w(y,z)\,\mathrm{d}y\,, \tag{A3}$$

where

$$G(\xi,z) = \mathrm{diag}\left(1, \mathrm{e}^{-2i\lambda(z)\xi}\right) Y_-^{-1}(z) = \frac{1}{\gamma}\begin{pmatrix} 1 & -i(q_-/z) \\ i(r_-/z)\,\mathrm{e}^{-2i\lambda\xi} & \mathrm{e}^{-2i\lambda\xi} \end{pmatrix}. \tag{A4}$$

Now we introduce a Neumann series representation for w:

$$w(x,z) = \sum_{n=0}^{\infty} w^{(n)}\,, \tag{A5a}$$

$$\text{with } w^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w^{(n+1)}(x,z) = \int_{-\infty}^{x} C(x,y,z)\,w^{(n)}(y,z)\,\mathrm{d}y\,, \tag{A5b}$$

and where $C(x, y, z) = G(x - y, z)\Delta Q(y)Y_-(z)$. Introducing the L^1 vector norm $\|w\| = |w_1| + |w_2|$ and the corresponding subordinate matrix norm $\|C\|$, we then have

$$\|w^{(n+1)}(x,z)\| \leqslant \int_{-\infty}^{x} \|C(x,y,z)\|\|w^{(n)}(y,z)\|\,\mathrm{d}y\,. \tag{A6}$$

Note $\|Y_\pm\| = 1 + q_o/|z|$ and $\|Y_\pm^{-1}\| = (1 + q_o/|z|)/|1 + q_o^2/z^2|$. The properties of the matrix norm imply

$$\|C(x,y,z)\| \leqslant \|\,\mathrm{diag}(1, \mathrm{e}^{-2i\lambda(z)(x-y)})\|\|Y_-\|\|\Delta Q(y)\|\|Y_-^{-1}\| \\ = c(z)\,(1 + \mathrm{e}^{2\lambda_{\mathrm{im}}(z)(x-y)})\,|q(y) - q_-|\,, \tag{A7}$$

where $\lambda_{\mathrm{im}}(z) = \mathrm{Im}\;\lambda(z)$ and $c(z) = \|Y_-\|\|Y_-^{-1}\| = (1 + q_o/|z|)^2/|1 + q_o^2/z^2|$ is the condition number of Y_-. Now recall that $\mathrm{Im}\;\lambda(z) < 0$ for z in D^-. On the other hand, $c(z) \to \infty$ as $z \to \pm iq_o$. Thus, given $\epsilon > 0$, we restrict our attention to the domain $D_\epsilon^- = D^- \setminus (B_\epsilon(iq_o) \cup B_\epsilon(-iq_o))$, where $B_\epsilon(z_o) = \{z \in \mathbb{C} : |z - z_o| < \epsilon q_o\}$. It is straightforward to show that $c_\epsilon = \max_{z\in D_\epsilon^-} c(z) = 2 + 2/\epsilon$. Next we prove that, for all $z \in D_\epsilon^-$ and for all $n \in \mathbb{N}$,

$$\|w^{(n)}(x,z)\| \leqslant \frac{M^n(x)}{n!}\,, \tag{A8a}$$

where

$$M(x) = 2c_\epsilon \int_{-\infty}^{x} |q(y) - q_-|\,\mathrm{d}y\,. \tag{A8b}$$

We will prove the result by induction, following Ref. 4. The claim is trivially true for $n = 0$. Also, note that, for all $z \in \overline{D^-}$ and for all $y \leq x$ one has $1 + \mathrm{e}^{2\lambda_{\mathrm{im}}(x-y)} \leqslant 2$. Then, if (A8a) holds for $n = j$, (A6) implies

$$\|w^{(j+1)}(x,z)\| \leqslant \frac{2c_\epsilon}{j!}\int_{-\infty}^{x} |q(y) - q_-|\,M^j(y)\,\mathrm{d}y = \frac{1}{j!(j+1)}M^{j+1}(x)\,, \tag{A9}$$

proving the induction step, namely, that the validity of (A8a) for $n = j$ implies its validity for $n = j + 1$. Thus, for all $\epsilon > 0$, if $q(x) - q_- \in L^1(-\infty, a]$ for some $a \in \mathbb{R}$ the Neumann series converges absolutely and uniformly with respect to $x \in (-\infty, a)$ and $z \in D_\epsilon^-$. Since a uniformly convergent series of analytic functions converges to an analytic function,[1,25] this demonstrates that the corresponding column of the Jost solution is analytic in this domain. It is important to note that, since $q_+ \neq q_-$ in general, $q(x) - q_- \notin L_1(\mathbb{R})$, and therefore one cannot take $a = \infty$. This non-uniformity with respect to $x \in \mathbb{R}$ is analogous to that in the defocusing case. This problem can be resolved using an alternative approach, similar to that in Ref. 36. Note also that, as in the defocusing case, additional conditions need to be imposed on the potential to establish convergence of the Neumann series at the branch points.[19]

APPENDIX B: BEHAVIOR AT THE BRANCH POINTS

We now discuss the behavior of the Jost eigenfunctions and the scattering matrix at the branch points $k = z = \pm iq_o$. The complication there is due to the fact that the $\lambda(\pm iq_o) = 0$, and therefore at $z = \pm iq_o$ the two exponentials $\mathrm{e}^{\pm i\lambda x}$ reduce to the identity. Correspondingly, at $z = \pm iq_o$ the matrices $Y_\pm(z)$ are degenerate. Note however that, even though $\det Y_\pm(\pm iq_o) = 0$ and $Y_\pm^{-1}(\pm iq_o)$ do not exist, the term $Y_\pm(z)\,\mathrm{e}^{-i\lambda(x-y)\sigma_3}Y_\pm^{-1}(z)$ appearing in (2.13) remains finite as $z \to \pm iq_o$:

$$\lim_{z\to\pm iq_o} Y_\pm(z)\,\mathrm{e}^{-i\lambda(x-y)\sigma_3}Y_\pm^{-1}(z) = I + i(x-y)(Q_\pm \mp q_o\sigma_3)\,.$$

Thus, both columns of $\phi_\pm(x,t,z)$ remain well-defined at $z = \pm iq_o$ as long as the potential satisfies appropriate regularity conditions (see Appendix A).

Let us therefore investigate the integral representation of the Jost solutions at the branch points. Since $\det\phi_\pm(x,t,\pm iq_o)=0$, the two columns of $\phi_-(x,t,iq_o)$ are proportional to each other, and similarly for the columns of $\phi_-(x,t,-iq_o)$, $\phi_+(x,t,iq_o)$, and $\phi_+(x,t,-iq_o)$. Comparing the boundary conditions of $\phi_\pm(x,t,\pm iq_o)$ as $x\to\pm\infty$, one then obtains

$$\phi_{\pm,2}(x,t,iq_o)=-\mathrm{e}^{i\theta_\pm}\phi_{\pm,1}(x,t,iq_o)\,,\quad \phi_{\pm,2}(x,t,-iq_o)=\mathrm{e}^{i\theta_\pm}\phi_{\pm,1}(x,t,-iq_o)\,, \tag{B1}$$

where $\theta_\pm=\arg q_\pm$. Owing to the well-defined limit of the Jost solutions at the branch points and to the Wronskian representations (2.19a), all entries of the scattering matrix have a well-defined limiting behavior near the branch points as well. That is, we can write

$$S(z)=\frac{1}{z\mp iq_o}A^\pm+B^\pm+O(z\mp iq_o). \tag{B2a}$$

Or, elementwise,

$$s_{i,j}(z)=\frac{1}{z\mp iq_o}a_{i,j}^\pm+b_{i,j}^\pm+O(z\mp iq_o) \tag{B2b}$$

in the appropriate regions of the complex z-plane. In (B2) and throughout this section, we use the superscripts $\pm$ for the scattering matrix and the scattering coefficients to denote the behavior in a neighborhood of $z=\pm iq_o$, respectively. In particular,

$$a_{1,1}^\pm=\pm\frac{1}{2}iq_o\mathrm{Wr}(\phi_{-,1}(x,t,\pm iq_o),\phi_{+,2}(x,t,\pm iq_o))\,, \tag{B3a}$$

$$b_{1,1}^\pm=\pm\frac{3}{2iq_o}a_{1,1}^\pm\pm\frac{1}{2}iq_o\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Wr}(\phi_{-,1}(x,t,z),\phi_{+,2}(x,t,z))|_{z=\pm iq_o}\,. \tag{B3b}$$

From (B3a) it is clear that there are two possibilities: (i) If $\phi_{-,1}(x,t,z)$ and $\phi_{+,2}(x,t,z)$ are linearly independent at $z=\pm iq_o$, then $a_{1,1}^\pm\neq 0$, and $s_{1,1}(z)$ has a simple pole singularity at both points, which is also what happens generically in the defocusing case.[20] (ii) If $\phi_{-,1}(x,t,z)$ and $\phi_{+,2}(x,t,z)$ are linearly dependent at $z=iq_o$ or at $z=-iq_o$, then either $a_{11,+}$ or $a_{11,-}$ or both vanish, and $s_{1,1}(z)$ is non-singular at iq_o or $-iq_o$. In this case the points $z=iq_o$ or $z=-iq_o$ are called *virtual levels*.[20] We next characterize the constants $a_{i,j}^\pm$ and $b_{i,j}^\pm$ in both of these cases.

Comparing the Wronskian representations of the scattering coefficients and using the proportionality relations (B1), one obtains

$$a_{1,2}^\pm=\mp\mathrm{e}^{i\theta_+}a_{1,1}^\pm\,,\quad a_{2,1}^\pm=\pm\mathrm{e}^{-i\theta_-}a_{1,1}^\pm\,,\quad a_{2,2}^\pm=-\mathrm{e}^{i\Delta\theta}a_{1,1}^\pm\,, \tag{B4}$$

where $\Delta\theta=\theta_+-\theta_-$. Thus, the reflection coefficients $\rho(z)$ and $\tilde\rho(z)$ defined in Sec. II C satisfy

$$\lim_{z\to\pm iq_o}\rho(z)=\pm\mathrm{e}^{-i\theta_-}\,,\quad \lim_{z\to\pm iq_o}\tilde\rho(z)=\pm\mathrm{e}^{i\theta_-}\,. \tag{B5}$$

Relations between the $a_{i,j}^\pm$ and the $b_{i,j}^\pm$ can be obtained by recalling that $\det S(z)=1$ for all $z\in\Sigma\backslash\{\pm iq_o\}$ and taking the limit of the determinant as $z\to\pm iq_o$.

For reflectionless potentials, $s_{2,1}(z)\equiv 0$ for all $z\in\Sigma\backslash\{\pm iq_o\}$ implies $a_{2,1}^\pm=0$. Then, by the first of (B4), we have $a_{1,1}^\pm=0$ as well, so we conclude that both iq_o and $-iq_o$ are always virtual levels for reflectionless potentials. In the generic case in which $z=\pm iq_o$ are not virtual levels, using the symmetries in Sec. II D, one has

$$a_{1,1}^\pm=(a_{2,2}^\mp)^*=\mathrm{e}^{-i\Delta\theta}a_{2,2}^\pm\,,\quad a_{1,2}^\pm=-(a_{2,1}^\mp)^*=\mathrm{e}^{i(\theta_++\theta_-)}a_{2,1}^\pm\,, \tag{B6}$$

implying a relation between the behavior at the two branch points.

$$(a_{1,1}^\pm)^*=\mathrm{e}^{i\Delta\theta}a_{1,1}^\mp\,,\quad (a_{1,2}^\pm)^*=-\mathrm{e}^{-i(\theta_++\theta_-)}a_{1,2}^\mp\,. \tag{B7}$$

Note that $z=iq_o$ is a virtual level if and only if $z=-iq_o$ is.

APPENDIX C: SYMMETRIES OF THE EIGENFUNCTIONS

We first prove (2.20). If $\phi(x, t, z)$ is a solution of the scattering problem [the first of (2.1)], so is

$$w(x,t,z) = \sigma_* \phi^*(x,t,z^*)\,, \tag{C1}$$

where

$$\sigma_* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{C2}$$

[because $\sigma_*^2 = -I\sigma_*\sigma_3\sigma_*, = \sigma_3$, and $\sigma_* Q^* \sigma_* = Q^\dagger = -Q$]. By the same token, so is $w\,C$, where C is any constant 2×2 matrix. Now let $z \in \Sigma$, take $\phi \equiv \phi_\pm$, and look the asymptotic behavior of $w_\pm$ as $x \to \pm\infty$. Note that $\theta^*(x, t, z^*) = \theta(x, t, z)$ and $\sigma_* Y_\pm^*(z^*) = \sigma_*(I - i\sigma_3 Q_\pm^*/z) = Y(z)\sigma_*$. Also, a little algebra shows that $\sigma_* \mathrm{e}^{ia\sigma_3}\sigma_* = -\mathrm{e}^{-ia\sigma_3}$. Therefore, $w_\pm(x,t,z)\,\sigma_* = Y_\pm(z)\,\mathrm{e}^{i\theta(x,t,z)\sigma_3} + o(1)$ as $x \to \pm\infty$. The uniqueness of the solution of the scattering problem with given BCs then implies $w_\pm\,\sigma_* = \phi_\pm$, which in turn yields (2.20).

Next we prove (2.22). If $\phi(x, t, z)$ is a solution of the scattering problem, so is

$$w(x,t,z) = \phi(x,t,-q_o^2/z) \tag{C3}$$

[since $k(-q_o^2/z) = k(z)$]. As before, this implies that $w\,C$ is also a solution, for any matrix C independent of t. If $\phi = \phi_\pm$, one has $w_\pm(x,t,z)C = Y_\pm(-q_o^2/z)\,\mathrm{e}^{-i\lambda(z)x\sigma_3}C + o(1)$ as $x \to \pm\infty$, since $\theta(x, t, -q_o^2/z) = -\theta(x, t, z)$. Now note that $Y_\pm(-q_o^2/z)\,\mathrm{e}^{-i\theta\,\sigma_3}\,\sigma_3 Q_\pm = -izY_\pm(z)\,\mathrm{e}^{i\theta\,\sigma_3}$ [because $\mathrm{e}^{-ia\sigma_3}Q = Q\mathrm{e}^{ia\sigma_3}$ and $(\sigma_3 Q)^2 = q_o^2 I$]. Thus, taking $C = (i/z)\,\sigma_3 Q$ we conclude, using the same arguments as before, we obtain (2.22).

APPENDIX D: ASYMPTOTIC BEHAVIOR AS $k \to \infty$

We first prove Eqs. (2.43). The proof proceeds by induction. The statement is trivially true for $\mu_d^{(0)}$ and $\mu_o^{(0)}$. Moreover, using (2.6) and separating the diagonal and off-diagonal parts of (2.42c), we have

$$\mu_d^{(n+1)}(x,z) = \frac{1}{1+(q_o/z)^2}\bigg[\int_{-\infty}^{x}\bigg(\Delta Q_-(y)\mu_o^{(n)}(y,z) - \frac{i\sigma_3 Q_-}{z}\,\Delta Q_-(y)\,\mu_d^{(n)}(y,z)\bigg)\,\mathrm{d}y + \frac{i\sigma_3 Q_-}{z}\int_{-\infty}^{x}\mathrm{e}^{i\lambda(z)(x-y)\hat\sigma_3}\bigg(\Delta Q_-(y)\mu_d^{(n)}(y,z) - \frac{i\sigma_3 Q_-}{z}\Delta Q_-(y)\,\mu_o^{(n)}(y,z)\bigg)\,\mathrm{d}y\bigg], \tag{D1a}$$

$$\mu_o^{(n+1)}(x,z) = \frac{1}{(1+(q_o/z)^2)}\bigg[\frac{i\sigma_3 Q_-}{z}\int_{-\infty}^{x}\bigg(\Delta Q_-(y)\mu_o^{(n)}(y,z) - \frac{i\sigma_3 Q_-}{z}\Delta Q_-(y)\,\mu_d^{(n)}(y,z)\bigg)\,\mathrm{d}y + \int_{-\infty}^{x}\mathrm{e}^{i\lambda(z)(x-y)\hat\sigma_3}\bigg(\Delta Q_-(y)\mu_d^{(n)}(y,z) - \frac{i\sigma_3 Q_-}{z}\Delta Q_-(y)\,\mu_o^{(n)}(y,z)\bigg)\,\mathrm{d}y\bigg]. \tag{D1b}$$

where we have temporarily suppressed the time dependence for brevity. As $z \to \infty$, the four terms in the RHS of (D1a) are, respectively,

$$O(\mu_o^{(n)}),\quad O(\mu_d^{(n)}/z),\quad O(\mu_d^{(n)}/z^2),\quad O(\mu_o^{(n)}/z^3),$$

where the last two estimates are obtained using integration by parts (e.g., see Ref. 16), taking advantage of the differentiability of $q(y, t)$ and the fact that $\lambda(z) = z/2 + O(1/z)$ as $z \to \infty$. Similarly, as $z \to \infty$, the four terms in the RHS of (D1b) are, respectively,

$$O(\mu_o^{(n)}/z),\quad O(\mu_d^{(n)}/z^2),\quad O(\mu_d^{(n)}/z),\quad O(\mu_o^{(n)}/z^2),$$

(where again the last two estimates are obtained using integration by parts). Looking for the dominant contribution when n is either odd or even one can then complete the induction step and therefore the proof of (2.43).

We next prove (2.44). Again, the result is proved by induction. The claim is trivially true for $\mu_d^{(0)}$ and $\mu_o^{(0)}$. And decomposing (2.42c) into its diagonal and off-diagonal parts yields (D1a) and (D1b) as before. Finally, one shows that the four terms in the RHS of (D1a) and (D1b) are, respectively

$$O(z^2\mu_o^{(n)}), \quad O(z\,\mu_d^{(n)}), \quad O(z^2\mu_d^{(n)}), \quad O(z\,\mu_o^{(n)})$$

for (D1a) and

$$O(z\mu_o^{(n)}), \quad O(\mu_d^{(n)}), \quad O(z^3\mu_d^{(n)}), \quad O(z\,\mu_o^{(n)})$$

for (D1b). Again, the last two estimates in each row are obtained using integration by parts and $\lambda(z) = O(1/z)$ as $z \to 0$. And again, looking for the dominant terms one completes the induction step and thereby the proof of (2.44).

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