Line Soliton Interactions of the Kadomtsev-Petviashvili Equation

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We study soliton solutions of the Kadomtsev-Petviashvili II equation $(-4u_t + 6uu_x + 3u_{xxx})_x + u_{yy} = 0$ in terms of the amplitudes and directions of the interacting solitons. In particular, we classify elastic N-soliton solutions, namely, solutions for which the number, directions, and amplitudes of the N asymptotic line solitons as $y \to \infty$ coincide with those of the N asymptotic line solitons as $y \to -\infty$. We also show that the $(2N-1)!!$ types of solutions are uniquely characterized in terms of the individual soliton parameters, and we calculate the soliton position shifts arising from the interactions.

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The Kadomtsev-Petviashvili (KP) equation,

$$(-4u_t + uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0 \qquad (1)$$

(subscripts x, y, and t denote partial derivatives) is perhaps the prototypical $(2+1)$-dimensional integrable system and is a universal model for weakly two-dimensional small-amplitude waves in the long wavelength regime [1–3]. As such it appears in many physical settings. The cases $\sigma = i$ and $\sigma = 1$ are referred to as the KPI and KPII equation. One of the hallmarks of integrability is the existence of exact N-soliton solutions for any positive integer N. One-soliton solutions of KP are one-dimensional structures called line solitons. Since KP is an integrable system, it was believed that the soliton parameter space of ordinary N-soliton solutions is simply the N-fold Cartesian product of the parameter space of one-soliton solutions, apart from degenerate cases. It has been known, however, that for KPII this is not the case [3]. At the same time, recent studies have shown that the solitonic sector of KPII is richer, and more general soliton solutions exist [4–7]. These solutions describe a rich and highly nontrivial phenomenology of line-soliton interactions. The purpose of this work is to characterize a special family of solutions and the interactions they describe in terms of the physical parameters of the individual solitons. Namely, we discuss *elastic* N-soliton interactions, defined as those N-soliton solutions for which the number, directions, and amplitudes of the N outgoing solitons (asymptotic line solitons as $y \to \infty$) coincide with those of the N incoming solitons (asymptotic line solitons as $y \to -\infty$).

Wronskian solutions of KP.—Equation (1) admits the well-known line-soliton solution [2,3]:

$$u(x, y, t) = \tfrac{1}{2}a^2 \operatorname{sech}^2[\tfrac{1}{2}a(x - cy - \omega t/a - x_0)], \qquad (2)$$

which is in the form of a traveling-wave solution, $u(x, y, t) = U(\mathbf{k} \cdot \mathbf{x} - \omega t)$, with $\mathbf{x} = (x, y)$ and $\mathbf{k} = (k_x, k_y) = (a, -ac)$. Apart from an overall translation, the solution depends on two parameters: the *soliton amplitude*, a (taken to be positive throughout), and the *soliton direction*, c (that is, the soliton inclination in the xy plane: $c = \tan\alpha$, where α is the angle from the positive y axis, measured counterclockwise). Note that $\mathbf{k}$ and ω satisfy the soliton dispersion relation $D(\mathbf{k}, \omega) = 4\omega k_x + k_x^4 - 3\sigma^2 k_y^2 = 0$.

It is also well known that many solutions of KP can be written in Wronskian form [8]:

$$u(x, y, t) = 2[\log \tau(x, y, t)]_{xx}, \qquad (3a)$$

where the tau function is

$$\tau(x, y, t) = \mathrm{Wr}(f_1, \ldots, f_N), \qquad (3b)$$

and where the functions $f_1, \ldots, f_N$ are linearly independent solutions of the Lax pair of KP with zero potential, namely, $f_y = f_{xx}$ and $f_t = f_{xxx}$. One-soliton solutions simply correspond to the "scalar" case $N = 1$ with $f = e^{\theta_1} + e^{\theta_2}$, where the exponential "phases" are

$$\theta_m(x, y, t) = k_m x + \sigma k_m^2 y + k_m^3 t + \theta_{0m}. \qquad (4)$$

Then $u = \frac{1}{2}(k_2 - k_1)^2 \operatorname{sech}^2[\frac{1}{2}(\theta_2 - \theta_1)]$, implying $\mathbf{k} = (k_2 - k_1, \sigma(k_1^2 - k_2^2))$ and $\omega = k_1^3 - k_2^3$. For KPII the *phase parameters* k_{i_n} and k_{j_n} are real, and they determine the soliton parameters a_n and c_n as

$$a_n = k_{j_n} - k_{i_n}, \qquad c_n = k_{i_n} + k_{j_n}. \qquad (5a)$$

(Here, $n = 1$, $i_n = 1$, and $j_n = 2$; the indices n, i_n, and j_n, which are for now redundant, will be useful later.) Equivalently, if the soliton parameters are given, the phase parameters are

$$k_{i_n} = \tfrac{1}{2}(c_n - a_n), \qquad k_{j_n} = \tfrac{1}{2}(c_n + a_n). \qquad (5b)$$

Equation (5b) implies that real, nonsingular one-soliton solutions exist for any value of amplitude and direction.

The generalization of the above to N-soliton solutions is obtained taking $f_n = e^{\theta_{2n-1}} + e^{\theta_{2n}}$. Neglecting the spatial shifts arising from interactions, these solutions simply produce a pattern of N straight lines in the xy plane [cf. Fig. 1(a)]. We refer to these as *ordinary* N-soliton solutions. Similarly to one-soliton solutions, these are parametrized by the soliton amplitudes $a_1, \ldots, a_N$ and directions $c_1 \ldots, c_N$. Surprisingly, however, while for KPI such solutions exist for any choice of amplitudes and directions, the same is not true for KPII [3].

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General soliton solutions are obtained by taking arbitrary linear combinations of exponentials for the functions $f_1, \ldots, f_N$ [5,6]. Namely, $f_n = \sum_{m=1}^{M} a_{nm} e^{\theta_m}$, where $k_1 < \cdots < k_M$ without loss of generality. Then

$$\tau(x, y, t) = \det(K e^{\Theta} A) = \sum_{1 \le m_1 < m_2 < \cdots < m_N \le M} V_{m_1,\ldots,m_N} A_{m_1,\ldots,m_N} e^{\theta_{m_1} + \cdots + \theta_{m_N}}, \quad (6)$$

where $\Theta(x, y, t) = \mathrm{diag}(\theta_1, \ldots, \theta_M)$, $K = (k_m^{n-1})$, $A = (a_{nm})$ is the $N \times M$ coefficient matrix, $A_{m_1,\ldots,m_N}$ is its $N \times N$ minor obtained from columns $m_1, \ldots, m_N$, and $V_{m_1,\ldots,m_N} > 0$ is a Van der Monde determinant. The spatiotemporal dependence of the tau function is confined to the exponential phases. Also, each term in Eq. (6) contains combinations of N distinct phases $\theta_{m_1}, \ldots, \theta_{m_N}$ chosen out of $\theta_1, \ldots, \theta_M$. Finally, note that transformations $A \to A' = GA$ with $G \in \mathrm{GL}_N(\mathbb{R})$ amount to a rescaling $\tau(x, y, t) \to \det(G)\tau(x, y, t)$, which leaves $u(x, y, t)$ invariant [9]. Thus, $\tau(x, y, t)$ describes an orbit in the Grassmannian $\mathrm{Gr}_{N,M}(\mathbb{R})$ [5]. The $\mathrm{GL}_N(\mathbb{R})$ invariance can be exploited to write A in reduced row-echelon form (RREF).

The tau function in Eq. (6) is parametrized by the M phase parameters $k_1, \ldots, k_M$ and by the coefficient matrix A [10]. Nonsingular solutions are obtained when all $N \times N$ minors of A are non-negative. Asymptotically as $y \to \pm\infty$, the solution becomes a linear superposition of one-soliton solutions, with the amplitude and direction of each soliton given by Eq. (5a) for a specific pair of indices i_n and j_n out of $1, \ldots, M$ [6]. We say that A is *irreducibile* if it is rank N and, when put in RREF, each column contains at least one nonzero element, and each row contains at least another nonzero element in addition to the pivot. Any irreducible coefficient matrix generates a solution of KPII in which $N_- = M - N$ *incoming* solitons (asymptotic solitons as $y \to -\infty$) interact to become $N_+ = N$ *outgoing* solitons (asymptotic solitons as $y \to \infty$) [6]. Each outgoing soliton is identified by an *index pair* $[i_n^+, j_n^+]$, with $i_n^+ < j_n^+$, where $i_1^+, \ldots, i_N^+$ label the N pivot columns of A. Similarly, each incoming soliton is identified by an index pair $[i_n^-, j_n^-]$, with $i_n^- < j_n^-$, where $j_1^-, \ldots, j_N^-$ label the $M - N$ nonpivot columns of A. It is then clear that N-soliton solutions are produced when $M = 2N$. Not all N-soliton solutions are elastic, however: elastic solutions are those for which $i_n^- = i_n^+$ and $j_n^- = j_n^+$ for all $n = 1, \ldots, N$, implying that $i_1^-, \ldots, i_N^-$ and $j_1^+, \ldots, j_N^+$ form a disjoint partition of $1, \ldots, 2N$.

Two-soliton solutions.—The simplest interactions of course arise in 2-soliton solutions. It was shown [5] that there are three types of elastic 2-soliton solutions, whose coefficient matrices A_{ord}, A_{res}, and A_{asymm} are, respectively,

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & a_{2,3} & a_{2,4} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

with $a_{2,3} > a_{2,4} > 0$. We refer to the two additional types, respectively, as *resonant* and *asymmetric* (the reason for these names will be clear shortly). From the asymptotics of $\tau(x, y, t)$ as $y \to \pm\infty$ one obtains that the individual line solitons are identified, respectively, by the index pairs $[1, 2]$ and $[3, 4]$ for ordinary 2-soliton solutions, $[1, 3]$ and $[2, 4]$ for resonant solutions, and $[1, 4]$ and $[2, 3]$ for asymmetric solutions. The requirement that $\mathbf{k}_1, \ldots, \mathbf{k}_M$ are well ordered, however, yields relations among the soliton parameters [11]. Denote (a_n, c_n) and $(a_{n'}, c_{n'})$ the individual soliton parameters, with $n = 1, 2$ and $n' = 2, 1$. (Again, n and n' are for later convenience.) One can show that (i) an ordinary two-soliton solution exists if and only if

$$|c_n - c_{n'}| > a_n + a_{n'}; \quad (7a)$$

(ii) a resonant two-soliton solution exists if and only if

$$|a_n - a_{n'}| < |c_n - c_{n'}| < a_n + a_{n'}; \quad (7b)$$

(iii) an asymmetric two-soliton solution exists if and only if

$$|c_n - c_{n'}| < |a_n - a_{n'}|. \quad (7c)$$

Since these inequalities are mutually exclusive, the three types of solutions divide the soliton parameter space of amplitudes and directions into three disjoint sectors. That is, exactly one type of elastic 2-soliton solution exists for any given choice of soliton amplitudes and directions. Note however that, unless $c_1 = c_2$, solutions of each type exist for any choice of soliton directions [12]. For example, Fig. 2 shows a resonant and an asymmetric 2-soliton solution with the same soliton directions as the ordinary solution in Fig. 1(a) [13].

Position shifts and degenerate sector.—The difference between resonant interactions versus ordinary or asymmetric ones is apparent: resonant interactions are mediated by a collection of Y unctions [such as the one in Fig. 1(b)], each of which satisfies Miles' resonance conditions $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ and $\omega_1 + \omega_2 = \omega_3$ [14]; in contrast, both ordinary and asymmetric interactions manifest as X junctions [15]. Ordinary and asymmetric interactions also differ, however: Denote, respectively, Δx_n and $\delta x_n = \Delta x_n / a_n$ the absolute and the reduced position shift of the nth soliton as $y \to \pm\infty$ as a result of the interaction. It is

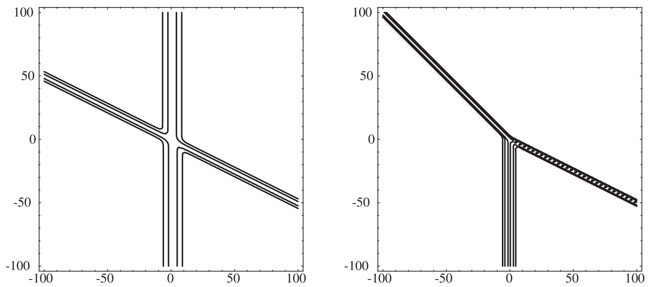

FIG. 1. Left: an ordinary 2-soliton solution of KPII with $(k_1, \ldots, k_4) = (-\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4})$, resulting in soliton amplitudes $(a_1, a_2) = (\frac{1}{2}, \frac{1}{2})$ and directions $(c_1, c_2) = (0, 2)$. Right: A Miles resonance with $(k_1, k_2, k_3) = (-\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$. The soliton directions are the same, but now $(a_1, a_2) = (1, 1)$. In all figures, the horizontal and vertical axes are, respectively, x and y, and the graph shows contour lines of $\log u(x, y, t)$ at fixed time.

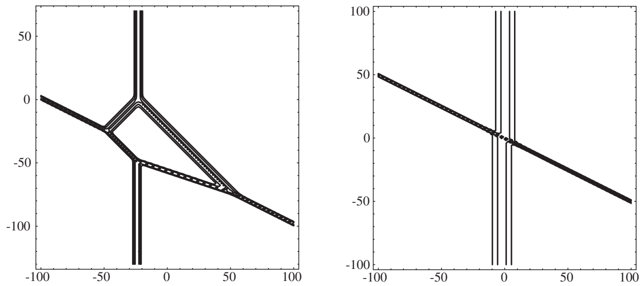

FIG. 2. A resonant (left) and an asymmetric (right) 2-soliton solution with the same soliton directions in the xy plane as the ordinary solution in Fig. 1(a). The phase parameters are $(k_1, \ldots, k_4) = (-1, 0, 1, 2)$ and $(k_1, \ldots, k_4) = (-\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{5}{4})$, respectively, resulting in soliton amplitudes of $(a_1, a_2) = (2, 2)$ and $(a_1, a_2) = (\frac{1}{2}, 3)$.

$$\delta x_n = \mathrm{sgn}(c_n - c_{n'}) \log S_{nn'} + \log S_{\mathrm{coeff}}, \qquad \text{(8a)}$$

with

$$S_{nn'} = \left| \frac{(c_n - c_{n'})^2 - (a_n - a_{n'})^2}{(c_n - c_{n'})^2 - (a_n + a_{n'})2} \right|, \qquad \text{(8b)}$$

$$S_{\mathrm{coeff}} = |A_{j_n j_{n'}} A_{i_n i_{n'}} / (A_{i_n j_{n'}} A_{i_{n'} j_n})|^{\mathrm{sgn}(c_n - c_{n'})}. \qquad \text{(8c)}$$

Explicitly, $S_{\mathrm{coeff}} = 1$ for ordinary and asymmetric solutions, while $S_{\mathrm{coeff}} = (a_{2,3}/a_{2,4} - 1)^{\mathrm{sgn}(c_n - c_{n'})}$ for resonant solutions. In all cases the constants θ_{0m} cancel (implying that individual soliton translations do not affect the position shift), and the "center of mass" invariance $a_1 \Delta x_1 + a_2 \Delta x_2 = 0$ holds. From Eqs. (8) it follows that (i) for ordinary solutions $\log S_{nn'} > 0$ [cf. Fig. 3(a)] and it can take any positive value depending on the soliton parameters; (ii) for asymmetric solutions $\log S_{nn'} < 0$ [cf. Fig. 3(b)] and it can take any negative value; (iii) for resonant solutions both $\log S_{nn'}$ and $\log S_{\mathrm{coeff}}$ can take any real value, and the sign of $\log S_{nn'}$ coincides with that of $(c_2 - c_1)^2 - (a_1^2 + a_2^2)$. Figure 4 shows a "bow-tie" solution, namely, a resonant solution with a large position shift arising from the coefficient matrix via S_{coeff}.

The boundaries of the three sectors of the soliton parameter space are given the hyperplanes $|c_2 - c_1| = a_1 + a_2$ and $|c_2 - c_1| = |a_1 - a_2|$. As the soliton parameters approach these boundaries, all position shifts tend to infinity, and two of the four phase parameters coalesce. In this limit, Miles' resonance conditions are satisfied, and each type of solution reduces to a Y junction, such as the one shown in Fig. 1(b). Figure 3 shows an ordinary and an asymmetric solution that are both almost degenerate, resulting in a large position shift.

Multisoliton solutions.—The generalization of these results to N-soliton solution is naturally formulated in terms of a direct and an inverse problem. The *inverse problem* consists in characterizing the soliton solution arising from given phase parameters and coefficient matrix. This problem was studied in Refs. [5–7]. In particular, the $N \times 2N$ coefficient matrices that lead to disjoint index pairs (and thus to elastic solutions) were identified [5] as those whose

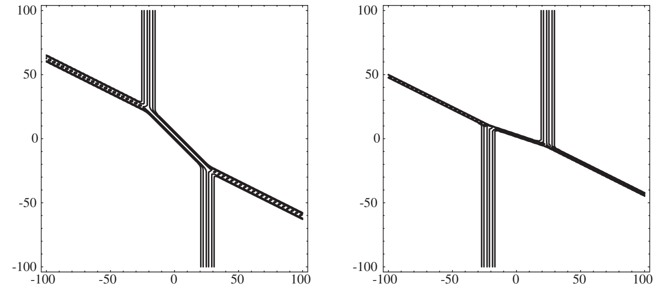

FIG. 3. Position shifts of almost-degenerate 2-soliton solutions with same soliton directions as in Fig. 1(a). Left: An ordinary solution with $(k_1, \ldots, k_4) = (-\frac{1}{2}(1-\varepsilon), \frac{1}{2}(1-\varepsilon), \frac{1}{2}, \frac{3}{2})$. Right: An asymmetric solution with $(k_1, \ldots, k_4) = (-\frac{1}{2} \times (1+\varepsilon), -\frac{1}{2}, \frac{1}{2}, \frac{5}{2}(1+\varepsilon))$. In both cases $\varepsilon = 10^{-20}$.

zero minors satisfy a duality relation: $A_{m_1,\ldots,m_N} = 0$ if and only if $A_{m'_1,\ldots,m'_N} = 0$, where $m'_1, \ldots, m'_N$ are the complement of $m_1, \ldots, m_N$ in $1, \ldots, 2N$. Also, a constructive method was given [6] to find the index pairs (and thus the soliton amplitudes and directions) from the coefficient matrix.

Conversely, the *direct problem* consists in characterizing the soliton interactions in terms of the soliton parameters $a_1, \ldots, a_N$ and $c_1, \ldots, c_N$. To do so, note that each choice of soliton parameters defines a set of $2N$ phase parameters via Eq. (5b). Sorting these into an ordered list yields an index pair $[i_n, j_n]$ for each soliton, corresponding to the position of its phase parameters in the list. The issue of the existence of an elastic N-soliton solution with the given soliton parameters is then translated into that of the existence of a coefficient matrix that generates the given set of index pairs. This is a highly nontrivial problem, however, which is closely related to the classification of $\mathrm{Gr}^{\mathrm{tnn}}_{N,M}$, the totally nonnegative part of $\mathrm{Gr}_{N,M}(\mathbb{R})$. Fortunately, the latter problem was recently solved [16] by introducing a refinement of the Schubert cell decomposition of $\mathrm{Gr}_{N,M}(\mathbb{R})$. One can now use the finer decomposition of $\mathrm{Gr}^{\mathrm{tnn}}_{N,2N}$ to show that a coefficient matrix exists for *any* choice of disjoint index pairs. Many combinatorial properties of the solutions were

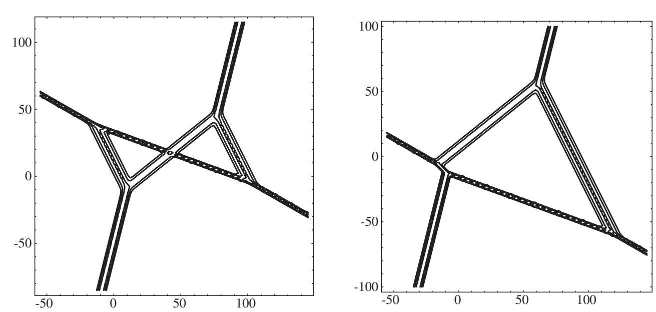

FIG. 4. A "bow-tie" resonant solution, obtained with $(k_1, \ldots, k_4) = (-1, -\frac{1}{4}, \frac{3}{4}, 2)$, $a_{2,3} = 1 + \varepsilon$, $a_{2,4} = 1$, and $\varepsilon = 10^{-40}$. Left: $t = -15$. Right: $t = 7$. Here the intermediate solitons intersect each other during a finite range of values of time. The interaction among the asymptotic solitons, however, is always resonant (i.e., mediated by Y junctions).

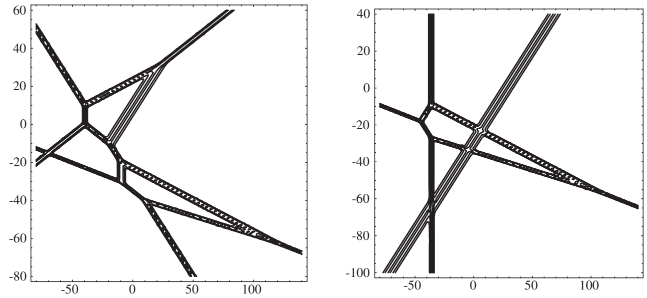

FIG. 5. Two elastic 3-soliton solutions of KPII. Left: Pairwise interactions 1R2, 2R3, and 1O3, generated by index pairs $[1, 3]$, $[2, 5]$, $[4, 6]$. Right: 1A2, 2O3 and 1R3, generated by $[1, 5]$, $[2, 3]$, $[4, 6]$. In both cases $(k_1, \ldots, k_6) = (-2, -1, 0, 1, 2, 3)$.

derived in [5,7] assuming existence. In particular, one finds that there exist $(2N-1)!!$ types of elastic N-soliton solutions [17].

The types of solutions are uniquely identified by the soliton parameters. Define the *degree of overlap* between two pairs $[i_n, j_n]$ and $[i_{n'}, j_{n'}]$ with $i_n < i_{n'}$ as follows: no overlap if $j_n < i_{n'}$, partial overlap if $i_{n'} < j_n < j_{n'}$, and total overlap if $j_n > j_{n'}$ [5]. The degree of overlap determines the type of *pairwise* interaction among solitons n and n' via certain zero minor conditions. More precisely, the pairwise interaction is ordinary (denoted by $n\mathrm{O}n'$) if there is no overlap, resonant (denoted by $n\mathrm{R}n'$) if there is partial overlap, and asymmetric (denoted by $n\mathrm{A}n'$) if there is total overlap. In turn, however, the degree of overlap is determined by which one of Eqs. (7) holds. Thus, the choice of pairs is in 1-to-1 correspondence with the specific combination of $2^{N(N-1)/2}$ pairwise inequalities [18]. Therefore, the soliton parameter space of the $2N$ amplitudes and directions is also divided into $(2N-1)!!$ disjoint open sectors via Eqs. (7), and each type of elastic N-soliton solution is uniquely characterized in terms of the soliton amplitudes and directions. As an example, Fig. 5 shows two elastic 3-soliton solutions, corresponding to different combinations of pairwise interactions.

The position shift of the nth soliton in an elastic N-soliton solution can be obtained from the asymptotics as $y \to \pm\infty$ in a similar way as for 2-soliton solutions:

$$\delta x_n = \sum_{1\le n' \ne n \le N} \mathrm{sgn}(c_n - c_{n'}) \log S_{nn'} + \log S_{\mathrm{coeff}}, \quad (9)$$

with $S_{nn'}$ as in Eq. (8b), and where S_{coeff} is now the ratio of four appropriate $N \times N$ minors of A. One can also identify the degenerate sector of the N-soliton parameter space. Indeed, when the soliton parameters are such that one or more of the inequalities in Eqs. (7) is replaced by an equal sign, inelastic N-soliton solutions are obtained. In turn, this condition implies that two or more of the $2N$ phase parameters obtained from the soliton parameters via Eq. (5b) coincide. Of course, like the elastic sector (and unlike the 2-soliton case) many different types of inelastic N-soliton solutions exist. Their classification is still an open problem.

Conclusion.—Since KP arises in many different settings, these results should have a wide range of applicability. Moreover, since aspects of Miles' resonance have been experimentally verified [19,20], we expect that it will also be possible to observe the more general interactions described here.

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[9] $\mathrm{GL}_N(\mathbb{R})$ transformations correspond to independent linear combinations of $f_1, \ldots, f_N$.
[10] The choice of θ_{0m} amounts to rescaling the corresponding column of A by an arbitrary positive constant.
[11] One could always use Eq. (5b) to define a set of phase parameters for any choice of soliton parameters. If the appropriate inequality is not satisfied, however, the resulting phase parameters are not sorted, which makes $\tau(x, y, t)$ sign indefinite and in turn results in a singular solution.
[12] Only asymmetric solutions are possible when the soliton directions coincide. Instead, in ordinary and resonant solutions the directions are sorted: $i_n < i_{n'}$ implies $c_n < c_{n'}$. Note also that asymmetric solutions only exist for unequal amplitudes.
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