

Prof. Cowen

Chapter 5.5

5.5.1. For which p does the improper integral $\int_4^\infty \frac{1}{x(\ln x)^p} dx$ converge/diverge?

5.5.2. For which p does the improper integral $\int_4^\infty \frac{1}{x(\ln x)(\ln \ln x)^p} dx$ converge/diverge?

5.5.3. Does $\int_{10}^\infty \frac{1}{\sqrt{x^3 - 5x}} dx$ converge or diverge? Why?

5.5.4. Does $\int_0^1 \frac{1}{\sqrt{3x - x^5}} dx$ converge or diverge? Why?

5.5.5. Does $\int_1^\infty \frac{x}{\sqrt{x^4 - 1}} dx$ converge or diverge? Why?

Chapter 6.1

6.1.1. Let $a_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$. Show that a_n is decreasing.

Hint: $\ln(n+1) - \ln n = \int_n^{n+1} \frac{1}{x} dx$. Why is this greater than or equal to $\frac{1}{n+1}$?

6.1.2. Show that a_n is bounded below. Hint: $1 + \frac{1}{2} + \dots + \frac{1}{n-1} \geq \int_1^n \frac{1}{x} dx$.

Note: The limit of a_n as $n \rightarrow \infty$ is the Euler constant $\gamma = .5772156649\dots$

6.1.3. Let $f(x)$ and $g(x)$ be positive functions which are continuous on the interval $(a, b]$. Let $L = \lim_{x \rightarrow a^+} \frac{f(x)}{g(x)}$. State and prove a theorem relating the improper integrals $\int_a^b f(x) dx$ and $\int_a^b g(x) dx$ when $L = 0$. *Hint: This is similar to the case where $L \neq 0, \infty$, but $L = 0$ means that $f(x)$ is much smaller than $g(x)$ near a .*

6.1.4. Same question as problem 6.1.3 but for $L = \infty$. *Hint: $L = \infty$ means that $f(x)$ is much larger than $g(x)$ near a .*

6.1.5. We showed in class that an increasing sequence bounded above has a limit. Use this result to show that a decreasing sequence bounded below has a limit.

6.1.6. We showed in class that a continuous function on a closed finite interval is bounded above. Use this result to prove that continuous function on a closed finite interval is bounded below.

6.1.7. Let $f(x)$ be a continuous function defined on a closed finite interval $[a, b]$. Prove that the range of f is a closed finite interval $[c, d]$. *Hint: draw the graph of a general continuous function. What are c and d ? Now prove the result.*

6.1.8. Draw the graph of a continuous function defined on an open finite interval (a, b) such that the range is a closed finite interval $[c, d]$.

Chapter 6.4

6.4.1. Use the Taylor approximation to find e^{-5} to eight decimal places. Include an analysis of how large n has to be. You may use a calculator or computer to do the arithmetic. Check your answer against the actual value for e^{-5} given by your calculator.

6.4.2. Same question for $\sin 0.2$.

6.4.3. We showed that $1/\ln \frac{1}{x}$ goes to 0 slower than x^p for any real number $p > 0$, in the sense that

$$\lim_{x \rightarrow 0^+} \frac{x^p}{1/\ln \frac{1}{x}} = 0.$$

Give an example of a function $f(x)$ defined for $x > 0$ such that $f(x)$ goes to 0 as $x \rightarrow 0^+$, but $f(x)$ goes to 0 slower than $\left(1/\ln \frac{1}{x}\right)^p$ for any real number $p > 0$.

Prove that your example actually does what it is supposed to do.

6.4.4. (Continuation of 6.4.3) Give an example of a function $g(x)$ defined for $x > 0$ such that $g(x)$ goes to 0 as $x \rightarrow 0^+$, but $g(x)$ goes to 0 slower than $(f(x))^p$ for any real number $p > 0$.

Prove that your example actually does what it is supposed to do.

6.4.5. (Continuation of 6.4.4) Give an example of a sequence of functions $f_1(x), f_2(x), \dots$ defined for $x > 0$ such that $f_1(x) = x$; for each n , $f_n(x)$ goes to 0 as $x \rightarrow 0^+$; but $f_{n+1}(x)$ goes to 0 slower than $(f_n(x))^p$ for any real number $p > 0$.

Do not give a proof.

6.4.6. Prove that e is irrational as follows:

If e is rational, then $e = m/n$ where m and n are positive integers with no common factors. Note that $n \geq 2$, since otherwise $n = 1$ and $e = m$; this would imply that e is an integer, which we know is not true by calculation.

Use Taylor's Theorem to show that

$$0 < e - \{1 + 1/1! + \dots + 1/n!\} < 3/(n+1)!.$$

Now let $r = n!(m/n - \{1 + 1/1! + \dots + 1/n!\})$. Show that r is an integer and that $0 < r < 3/(n+1)$. Why is this a contradiction? *Note: to prove that π is irrational is infinitely harder.*

6.4.7. We showed in class that: if $f(x)$ is differentiable at a ; if $L(x) = f(a) + f'(a)(x - a)$ is the linearization; and if $R(x) = f(x) - L(x)$, then

$$\lim_{x \rightarrow a} \frac{R(x)}{x - a} = 0.$$

Let $\tilde{L}(x)$ be any linear function of x , and let $\tilde{R}(x) = f(x) - \tilde{L}(x)$. If

$$\lim_{x \rightarrow a} \frac{\tilde{R}(x)}{x - a} = 0,$$

show that $\tilde{L}(x) = L(x)$. Hint: write $\tilde{L}(x)$ in the form $L(x) = b_0 + b_1(x - a)$.

Chapter 6.5

6.5.1. Let f and g be continuous on $[a, b]$ and differentiable on (a, b) , with $g'(x) \neq 0$ for all x in (a, b) .

i. Prove that $g(b) \neq g(a)$.

ii. Let

$$h(x) = (g(x) - g(a))(f(b) - f(a)) - (f(x) - f(a))(g(b) - g(a)).$$

Show that $h(a) = h(b) = 0$.

iii. Use $h(x)$ from ii. to prove there is a c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Hint: prove that $h'(c) = 0$ for some c in (a, b) .

6.5.2. $\lim_{x \rightarrow 1} \frac{\ln x}{1 - x}$

6.5.3. $\lim_{x \rightarrow 0} \frac{e^x - 1}{\ln(1 + x)}$

6.5.4. $\lim_{x \rightarrow 1} \frac{\ln(x + 2)}{1 + x}$

6.5.5. $\lim_{x \rightarrow 0} (e^x + x)^{1/x}$

6.5.6. $\lim_{x \rightarrow \infty} x^2 \sin 1/x$

6.5.7. $\lim_{n \rightarrow \infty} (n^2 + n)^{1/n}$

Hint: if $f(x) \rightarrow L$ as $x \rightarrow \infty$ then $f(n) \rightarrow L$ as $n \rightarrow \infty$. So switch the variable to x , find the limit, then switch back to n (note: it doesn't make sense to take derivative with respect to n , but it does make sense to take derivative with respect to x .)

$$6.5.8. \lim_{x \rightarrow 0^+} \frac{x^p}{1/\ln x} \quad \text{for } p > 0.$$

Hint: let $y = 1/x$ and express the limit in terms of y

6.5.9. Prove the following version of L'Hospital's rule: if $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = 0$ and $\lim_{x \rightarrow \infty} f'(x)/g'(x)$ exists, then $\lim_{x \rightarrow \infty} f(x)/g(x) = \lim_{x \rightarrow \infty} f'(x)/g'(x)$.

Hint: let $y = 1/x$, so $y \rightarrow 0^+$ as $x \rightarrow \infty$. Define $F(y) = f(1/y)$ and $G(y) = g(1/y)$ for $y > 0$. Apply L'Hospital's rule to $F(y)/G(y)$. What is the relationship between $F'(y)/G'(y)$ and $f'(1/y)/g'(1/y)$?

6.5.10. Here is another version of L'Hospital's rule, with a more complicated proof (an expanded version of the proof in Helson, Problem 2 p. 149): If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \infty$ and $\lim_{x \rightarrow a} f'(x)/g'(x)$ exists, then $\lim_{x \rightarrow a} f(x)/g(x) = \lim_{x \rightarrow a} f'(x)/g'(x)$.

Note: you may use a part of the problem, even if you cannot prove it, in subsequent parts of the problem.

i. We show the rule is true for $\lim_{x \rightarrow a^+}$. Assume that $f(x), g(x), f'(x)$, and $g'(x)$ are defined on $(a, b]$, for some $b > a$. (At this point it really doesn't matter what b is, as long as $b > a$, but we will see below that we have to control how close b is to a in order to make everything work.) If $a < x < b$, then

$$\frac{f(b) - f(x)}{g(b) - g(x)} = \frac{f'(c_x)}{g'(c_x)}$$

for some c_x , $x < c_x < b$. Now divide the numerator and denominator of the left side of the equation above by $g(x)$ and show that

$$\frac{f(x)}{g(x)} = \frac{f(b)}{g(x)} - \frac{g(b)}{g(x)} \frac{f'(c_x)}{g'(c_x)} + \frac{f'(c_x)}{g'(c_x)}.$$

Why can we assume that $f(x) > 0$ and $g(x) > 0$ for $a < x \leq b$?

ii. Let $\lim_{x \rightarrow a^+} f'(x)/g'(x) = L$. We want to show that $\lim_{x \rightarrow a^+} f(x)/g(x) = L$. That is, given $\varepsilon > 0$, we need to show there exists $\delta > 0$ such that $\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$ whenever $0 < x - a < \delta$.

Using part i), show that

$$\left| \frac{f(x)}{g(x)} - L \right| \leq \frac{f(b)}{g(x)} + \frac{g(b)}{g(x)} \left| \frac{f'(c_x)}{g'(c_x)} \right| + \left| \frac{f'(c_x)}{g'(c_x)} - L \right|,$$

if $a < x < b$.

iii. Show that if a function has a limit as $x \rightarrow a^+$, then the function is bounded for x near a and greater than a .

iv. Use parts ii and iii to show that if b is close enough to a , then

$$\left| \frac{f(x)}{g(x)} - L \right| \leq \frac{f(b)}{g(x)} + \frac{g(b)}{g(x)} K + \left| \frac{f'(c_x)}{g'(c_x)} - L \right|,$$

for some constant $K > 0$, whenever $a < x < b$.

v. By the definition of limit, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\varepsilon}{2}$ whenever $0 < x - a < \delta$. Show there exists δ_1 , $0 < \delta_1 \leq \delta$, such that if $b - a < \delta_1$, then

$$\left| \frac{f(x)}{g(x)} - L \right| < \frac{f(b) + Kg(b)}{g(x)} + \frac{\varepsilon}{2},$$

whenever $a < x < b$.

vi. Use v and the fact that $g(x) \rightarrow \infty$ as $x \rightarrow a^+$ to prove there exists δ_2 , $0 < \delta_2 \leq \delta_1$, such that if $0 < x - a < \delta_2$, then $\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$. This shows that $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$.

Chapter 7.1

In the next three problems, assume that you have never heard of \log or \ln —so these functions are ineligible for any use whatsoever.

7.1.1. Let $f(x)$ be a function defined for all $x > 0$ such that $f'(x) = 1/x$. Let a be a positive constant and let $g(x) := f(ax)$. Find $g'(x)$ and simplify your answer.

7.1.2. Find a positive integer n such that when you chop the interval from 1 to 2.8 into n equal pieces and approximate $\int_1^{2.8} \frac{dx}{x}$ by the Riemann sum using right endpoints (i.e. by the lower sum), then you get a sufficiently accurate approximation to show $\int_1^{2.8} \frac{dx}{x} > 1$. You may use a spreadsheet or a computer algebra system (such as matlab/maple/mathematica). Include a printout verifying your answer.

7.1.3. Find a positive integer n such that when you chop the interval from 1 to 2.7 into n equal pieces and approximate $\int_1^{2.7} \frac{dx}{x}$ by the Riemann sum using left endpoints (i.e. by the upper sum), then you get a sufficiently accurate approximation to show $\int_1^{2.7} \frac{dx}{x} < 1$. You may use a spreadsheet or a computer algebra system (such as matlab/maple/mathematica). Include a printout verifying your answer.

7.1.4. Let $b > 0$ be a real number. Use the definition of b^x to show:

- 1) $b^{s+t} = b^s b^t$ for all s and t real.
- 2) $b^{s-t} = b^s / b^t$ for all s and t real.
- 3) $\log b^s = s \log b$ for all s real.
- 4) $b^{st} = (b^s)^t$ for all s and t real.

7.1.5. Let $b > 0$ be a real number. Use the definition of b^x to show:

- 1) $\frac{db^x}{dx} = b^x \log b$.
- 2) $\int b^x dx = \frac{b^x}{\log b} + C$.

Use logarithmic differentiation to find the derivative with respect to x of:

$$7.1.6. \frac{(x+1)^3(x^2+2)^4(x^3+3)^5}{(x^4+4)^6(x^5+5)^7(x^6+6)^8}$$

$$7.1.7. \sqrt{(x^2+x+2)^7(x^5+9x^3+8)^{11}}$$

In problems 7.1.8-7.1.12, find the derivative with respect to x . Note:

$$(1) \quad \frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2}$$

$$(2) \quad \frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$7.1.8. (\sqrt{2})^x$$

$$7.1.9. (\tan x)^{\ln x}$$

$$7.1.10. \pi^x \tan^{-1} x$$

$$7.1.11. e^{x^2 \sin^{-1} x}$$

$$7.1.12. \log_{10}(xe^x + 1)$$

Find the following integrals:

$$7.1.13. \int (\sin 4x) 5^{\cos 4x} dx$$

$$7.1.14. \int \frac{x \log_2(x^2+1)}{x^2+1} dx$$

$$7.1.15. \int \frac{\pi^{\log_{10} x}}{x} dx$$

Chapter 8.1

8.1.1. Let $1, 1, 2, 3, 5, 8, \dots$ be the Fibonacci sequence. Find the first six terms of a series $a_1 + a_2 + \dots$ such that the first six partial sums S_1, \dots, S_6 for the series match the first six terms of the Fibonacci sequence. That is, $S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 3, S_5 = 5, S_6 = 8$.

8.1.2. Let s_1, s_2, \dots be any sequence. Find a series $a_1 + a_2 + \dots$ such that the partial sum S_n for the series satisfies $S_n = s_n$ for all n . *This shows that partial sums can exhibit the behavior of arbitrary sequences.*

8.1.3. A series $a_1 + a_2 + \dots$ is called a *telescoping series* if $a_1 = b_1 - b_2, a_2 = b_2 - b_3, \dots$. For example, $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ is a telescoping series. Show that a telescoping series converges if and only if $b_n \rightarrow L$ as $n \rightarrow \infty$. What is the sum of the series if it does converge?

8.1.4. Find $\sum_{n=2}^{\infty} 1/(n^2-1)$. *Hint: use partial fractions to show the series is something like a telescoping series.*

8.1.5. Does $\sum_{n=5}^{\infty} \frac{n}{n^3 - 7n^2 + 4}$ converge or diverge? Give complete reasons.

8.1.6. Does $\sum_{n=5}^{\infty} \frac{5^n - 2n}{8^n + 4n^2}$ converge or diverge? Give complete reasons.

8.1.7. $1 + 3/2 + 9/4 + 27/8 + \dots = ?$

Chapter 9.2

In problems 9.2.1 - 9.2.4, find the radius of convergence and check convergence at the endpoints for each of the following:

9.2.1. $\sum_{n=1}^{\infty} \frac{x^n}{n}$

$$9.2.2. \sum_{n=1}^{\infty} \frac{(3x-5)^n}{n^2}$$

$$9.2.3. \sum_{n=1}^{\infty} \frac{8^n(x-1)^{3n}}{n}$$

$$9.2.4. \sum_{n=1}^{\infty} \frac{8^n(4x+1)^{2n+1}}{\sqrt{n}}$$

9.2.5. Estimate $\int_0^2 e^{x^2} dx$ correct to 6 decimal places. How do you know your estimate is correct?

Chapter 10.2

$$10.2.1. \int \csc x \, dx$$

$$10.2.2. \int \cos^5 x \, dx$$

$$10.2.3. \int \tan^2 x \, dx$$

$$10.2.4. \int \sec^4 x \, dx$$

$$10.2.5. \int \cos^4 x \, dx$$

$$10.2.6. \int \frac{dx}{\sqrt{1-4x^2}}$$

$$10.2.7. \int \frac{dx}{\sqrt{9x^2-1}}$$

$$10.2.8. \int \frac{x^2 \, dx}{\sqrt{9-x^2}}$$

$$10.2.9. \int \frac{x^3 \, dx}{\sqrt{x^2+1}}$$

$$10.2.10. \int \frac{dx}{\sqrt{2x - x^2}}$$

$$10.2.11. \int \frac{dx}{4x^2 + 4x + 2}$$

$$10.2.12. \int \frac{dx}{\sqrt{2 - 5x^2}}$$

$$10.2.13. \int \frac{dx}{1 + \sin x} \text{ Hint: multiply numerator and denominator by } 1 - \sin x$$

10.2.14. Show that

$$(3) \quad \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

$$(4) \quad \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$(5) \quad d\theta = \frac{d \tan \theta}{1 + \tan^2 \theta}$$

10.2.15. Let

$$z = \tan \frac{x}{2}$$

Show that

$$(6) \quad \sin x = \frac{2z}{1 + z^2}$$

$$(7) \quad \cos x = \frac{1 - z^2}{1 + z^2}$$

$$(8) \quad dx = \frac{2dz}{1 + z^2}$$

Hint: let $\theta = x/2$ and apply problem 10.2.14.

$$10.2.16. \text{ Use 10.2.15 to find } \int \frac{dx}{2 + \sin x}$$

Chapter 10.3

$$10.3.1. \int \frac{x+7}{x^2-6x+5} dx$$

$$10.3.2. \int \frac{3x+1}{x^2-10x+25} dx$$

$$10.3.3. \int \frac{x+2}{x^3-3x^2} dx$$

$$10.3.4. \int \frac{1}{(x+1)(x^2+4)} dx$$

$$10.3.5. \int \frac{x^5}{x^2-4} dx$$

$$10.3.6. \int \frac{e^x}{e^{2x}+4e^x+3} dx$$