Prof. Cowen

1. Numerical Integration

The goal of this section is to give a numerical approximation to $\int_a^b f(x)\,dx$, where $f(x)$ is a continuous function. We chop the interval $[a, b]$ into n equal pieces and use the following notation:

$h = \dfrac{b-a}{n}$	Width of the intervals
$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h, \ldots,\ x_n = b$	Endpoints
$\bar{x}_0 = a + \dfrac{h}{2},\ \bar{x}_1 = \bar{x}_0 + h,\ \bar{x}_2 = \bar{x}_0 + 2h, \ldots,\ \bar{x}_n = \bar{x}_0 + nh$	Midpoints
$y_0 = f(x_0), \ldots, y_n = f(x_n)$	y values endpts
$\bar{y}_0 = f(\bar{x}_0), \ldots, \bar{y}_n = f(\bar{x}_n)$	y values midpts
$\text{Left}_n = h \cdot (y_0 + \ldots y_{n-1})$	Left endpoint rule
$\text{Right}_n = h \cdot (y_1 + \ldots y_n)$	Right endpoint rule
$\text{Mid}_n = h \cdot (\bar{y}_1 + \ldots \bar{y}_n)$	Midpoint rule
$\text{Trap}_n = \dfrac{h}{2} \cdot (y_0 + 2y_1 + \ldots 2y_{n-1} + y_n)$	Trapezoidal rule
$\text{Simp}_n = \dfrac{h}{3} \cdot (y_0 + 4y_1 + 2y_2 + \ldots 2y_{n-2} + 4y_{n-1} + y_n)$	Simpson's rule; n is even

2. Error Bounds

In this section we give estimates on the error given by approximating the integral $\int_a^b f(x)\,dx$ by the various numerical methods. The errors are defined as follows:

$$\text{LeftError}_n = \left| \int_a^b f(x)\,dx - \text{Left}_n \right| \quad \text{Left error}$$

$$\text{RightError}_n = \left| \int_a^b f(x)\,dx - \text{Right}_n \right| \quad \text{Right error}$$

$$\text{MidError}_n = \left| \int_a^b f(x)\,dx - \text{Mid}_n \right| \quad \text{Midpoint error}$$

$$\text{TrapError}_n = \left| \int_a^b f(x)\,dx - \text{Trap}_n \right| \quad \text{Trapezoidal error}$$

$$\text{SimpError}_n = \left| \int_a^b f(x)\,dx - \text{Simp}_n \right| \quad \text{Simpson error}$$

Note that by definition the errors are always positive; we do not distinguish between overestimates and underestimates in the error.

Assume that we can find constants $M_1, M_2,$ and M_4 so that $|f'(x)| \leq M_1, |f''(x)| \leq M_2,$ and $\left|f^{(4)}(x)\right| \leq M_4$ on $[a, b]$. Then we have the following error bounds:

$$\text{LeftError}_n \leq \frac{M_1\,(b-a)\,h}{2} \quad \text{Left error bound}$$

$$\text{RightError}_n \leq \frac{M_1\,(b-a)\,h}{2} \quad \text{Right error bound}$$

$$\text{MidError}_n \leq \frac{M_2\,(b-a)\,h^2}{24} \quad \text{Midpoint error bound}$$

$$\text{TrapError}_n \leq \frac{M_2\,(b-a)\,h^2}{12} \quad \text{Trapezoid error bound}$$

$$\text{SimpError}_n \leq \frac{M_4\,(b-a)\,h^4}{180} \quad \text{Simpson error bound}$$

3. Problems

Problems from Calculus, 2nd edition, by Hunt; Calculus with Early Transcendentals, 4th edition, by Stewart; and Calculus with Analytic Geometry, Early Transcendentals, 5th edition, by Edwards & Penney.

1. Use the Trapezoidal Rule and the Midpoint Rule to approximate $\int_0^2 e^{-x^2}\,dx$ using $n = 10$. Estimate (that is, find bounds on) the errors $\mathrm{TrapError}_{10}$ and $\mathrm{MidError}_{10}$. How large do we have to take n so that we could guarantee that the errors in using the Trapezoidal Rule and the Midpoint Rule would be less than 0.000001?

2. Use Simpson's Rule to approximate $\int_1^2 \frac{dx}{x}$ using $n = 10$. Estimate the error $\mathrm{SimpError}_{10}$. How large do we have to take n so that we could guarantee the error in using Simpson's Rule would be less than 0.000001? (This would be a practical way to calculate $\ln 2$.)

3. Because the number e is the base for natural logarithms, it follows that

$$\int_1^e \frac{1}{x}\,dx = 1.$$

Use numerical integration to approximate the integrals $\int_1^{2.7} \frac{1}{x}\,dx$ and $\int_1^{2.8} \frac{1}{x}\,dx$ with sufficient accuracy to show that $2.7 < e < 2.8$.

4. Use the Trapezoidal Rule to approximate $\int_0^{20} \cos(\pi x)\,dx$ using $n = 10$. Find the exact error $\mathrm{TrapError}_{10}$. Explain why the error is so big.

5. Let $f(x)$ be a continuous function on $[a, b]$. The *average value* of f is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Show that if we approximate $\int_a^b f(x)\,dx$ by the Trapezoidal Rule, then the average value of f is approximated by the average of $y_1, y_2, \ldots, y_{n-1}, \frac{y_0+y_n}{2}$.

6. Prove that if $L(x) = Ax + B$ is a linear function, then both the Trapezoidal Rule and the Midpoint Rule give the exact answer for $\int_a^b L(x)\,dx$. (Hint: what is the bound on the error?)

7. Prove that if $P(x)$ is a polynomial of degree 3, then Simpson's rule gives the exact answer for $\int_a^b P(x)\,dx$. (Hint: what is the bound on the error?)

8. Let $f(x)$ be continuous and positive on $[a, b]$. If $f''(x) > 0$ for all x in $[a, b]$, prove that $\text{Mid}(n) \leq \int_a^b f(x)\,dx$. (Hint: it suffices to prove this result for $n = 1$. Why is that?)