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SPECHT FILTRATIONS FOR HECKE ALGEBRAS OF TYPE A

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Abstract

Let $\mathcal{H}_q(d)$ be the Iwahori–Hecke algebra of the symmetric group, where q is a primitive lth root of unity. Using results from the cohomology of quantum groups and recent results about the Schur functor and adjoint Schur functor, it is proved that, contrary to expectations, for $l \geqslant 4$ the multiplicities in a Specht or dual Specht module filtration of an $\mathcal{H}_q(d)$-module are well defined. A cohomological criterion is given for when an $\mathcal{H}_q(d)$-module has such a filtration. Finally, these results are used to give a new construction of Young modules that is analogous to the Donkin–Ringel construction of tilting modules. As a corollary, certain decomposition numbers can be equated with extensions between Specht modules. Setting $q = 1$, results are obtained for the symmetric group in characteristic $p \geqslant 5$. These results are false in general for $p = 2$ or 3.

1. *Introduction*

1.1

Let k be an algebraically closed field of characteristic $p \geqslant 0$ and let G be a reductive algebraic group over k. Twenty years ago, Donkin [**7**] first defined the notion of when a rational G-module admits a good filtration, that is, a filtration with successive quotients isomorphic to induced modules $\nabla(\lambda)$. Furthermore, he proved a cohomological criterion which gives both a necessary and a sufficient condition for when a rational G-module admits such a filtration. The filtration multiplicities in a good filtration are easily shown to be well defined. The collection of indecomposable modules with both a good filtration and a Weyl filtration are called tilting modules, and there is exactly one for each dominant weight.

Now let $G = \mathrm{GL}_n(k)$ be the general linear group over k, and let Σ_d be the symmetric group on d letters. It is well known that, for $n \geqslant d$, the representation theory for G and Σ_d are related via the Schur and inverse Schur functors. Under the Schur functor, injective (polynomial) representations for G are sent to Young modules, induced modules are sent to Specht modules, and Weyl modules are sent to dual Specht modules. By using the aforementioned results for reductive groups, Donkin [**8**] showed that the Young modules admit both Specht and dual Specht filtrations.

However, for $p = 2$ and $p = 3$, there are examples (see [**21**, p. 126] or Subsection 3.2) which demonstrate that in general Specht filtration multiplicities are not well defined. Even for Young modules, which are known to have Specht filtrations, it was not known if the multiplicities were well defined. At first glance, it appears that developing a theory of Specht filtrations for the symmetric groups similar to the existing theory for reductive groups is a hopeless undertaking.

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1.2

In this paper we prove that these small examples are exceptional, and that for $p \geqslant 5$ there is a theory of Specht and dual Specht filtrations with well defined multiplicities. In fact, we will state our results in a more general framework using the Hecke algebra $\mathcal{H}_q(d)$ of type A and the q-Schur algebra $S_q(n,d)$. For $n \geqslant d$, one has a Schur functor from $\mathrm{Mod}(S_q(n,d))$ to $\mathrm{Mod}(\mathcal{H}_q(d))$. Tilting modules, Young modules, Specht modules and dual Specht modules all have analogues in the quantum setting. In Section 3 we prove that if q is an lth root of unity with $l \geqslant 4$, then there is a notion of Specht and dual Specht filtrations with well defined multiplicities for $\mathcal{H}_q(d)$. Using the (quantum) Schur functor and its adjoint, we give a cohomological condition which provides necessary and sufficient conditions for an $\mathcal{H}_q(d)$-module to admit a Specht (respectively dual Specht) module filtration.

Finally we apply this theory to give a new construction of the Young modules via extensions of Specht modules. The motivation for this construction is a similar construction of the tilting modules. This construction allows us to equate dimensions of certain extension groups between Specht modules to decomposition numbers for the Hecke algebra.

2. *Notation and preliminaries*

2.1

Let q be a unit in k. The Hecke algebra $\mathcal{H}_q(d) = \mathcal{H}_{k,q}(\Sigma_d)$ is the free k-module with basis $\{T_w : w \in \Sigma_d\}$. The multiplication in the algebra is defined by the rule

$$T_w T_s = \begin{cases} T_{ws} & \ell(ws) > \ell(w) \\ qT_{ws} + (q-1)T_w & \text{otherwise,} \end{cases}$$

where $s = (i, i+1) \in \Sigma_d$ is a basic transposition and $w \in \Sigma_d$. The function $\ell : \Sigma_d \longrightarrow \mathbb{N}$ is the usual length function.

Let l be the smallest integer such that $1 + q + \ldots + q^{l-1} = 0$. Notice that when $q = 1$, then $l = p$ is the characteristic of k. In this (classical) case we will use the more traditional notation p for the characteristic of k when $q = 1$. When q is a primitive jth root of unity, then $l = j$. We only consider the case where q is a root of unity (so $l < \infty$), since otherwise $\mathcal{H}_q(d)$ is semisimple. There are an automorphism $\#$ and an antiautomorphism * of $\mathcal{H}_q(d)$ defined by

$$\begin{aligned} T_w^{\#} &= (-q)^{l(w)}(T_{w^{-1}})^{-1}, \\ T_w^{*} &= T_{w^{-1}}. \end{aligned}$$

The maps $\#$ and * are both involutions, and they will be used in the next subsection.

2.2

The representation theory of $\mathcal{H}_q(d)$ largely parallels that of $k\Sigma_d$. It was first described by Dipper and James [**5**] and is thoroughly surveyed in [**22**]. We just sketch it here.

Let Λ be the set of partitions of d. For each $\lambda \in \Lambda$ there is a q-Specht module of the Hecke algebra $\mathcal{H}_q(d)$, denoted by S^λ. A partition $(\lambda_1, \lambda_2, \ldots)$ is called l-*restricted* if $\lambda_i - \lambda_{i+1} \leqslant l - 1$ for all i. The set of the l-restricted partitions of d will be denoted by Λ_{res}. A partition λ is called *l-regular* if its *transpose* λ' is l-restricted. We denote the set of all l-regular partitions of d by Λ_{reg}. If $\lambda \in \Lambda_{\text{reg}}$, then S^λ has a unique simple quotient, which we denote by D^λ. The collection of modules D^λ for $\lambda \in \Lambda_{\text{reg}}$ is a complete set of non-isomorphic simple $\mathcal{H}_q(d)$-modules.

For any $\mathcal{H}_q(d)$-module M, we can define a dual module $M^* := \operatorname{Hom}_k(M, k)$, where the action of $\mathcal{H}_q(d)$ is given by $(hf)(m) := f(h^* m)$. The simple modules D^λ are self-dual and absolutely irreducible.

Let $\Lambda(n, d)$ be the set of compositions of d into at most n parts. For $\lambda \in \Lambda(n, d)$, let Σ_λ be the corresponding Young subgroup of Σ_d, so that $\Sigma_\lambda \cong \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \dots$. There is also a corresponding parabolic subalgebra of $\mathcal{H}_q(d)$

$$\mathcal{H}_q(\lambda) := \mathcal{H}_q(\lambda_1) \times \mathcal{H}_q(\lambda_2) \times \dots$$

generated by $\{T_w \mid w \in \Sigma_\lambda\}$. Set

$$x_\lambda = \sum_{w \in \Sigma_\lambda} T_w.$$

Define $M^\lambda := \mathcal{H}_q(d) x_\lambda$. For a partition $\lambda \in \Lambda$, the module M^λ has a unique submodule isomorphic to the Specht module S^λ, and there is a unique indecomposable direct summand of M^λ containing S^λ; this is the Young module Y^λ. We have $Y^\lambda \cong Y^\mu$ if and only if $\lambda = \mu$. For any $\alpha \in \Lambda(n, d)$, the module M^α is a direct sum of such Young modules (see [**21**, 4.6]).

We remark that $\mathcal{H}_q(d)$ has two one-dimensional representations [**22**, 1.14], which we denote k and sgn, given by

$$k(T_w) = q^{l(w)} \quad \text{and} \quad \operatorname{sgn}(T_w) = (-1)^{l(w)}. \tag{2.2.1}$$

Of course, when $q = 1$ these specialize to the usual trivial and sign representations of $k\Sigma_d$.

In general, the tensor product of two $\mathcal{H}_q(d)$-modules is not an $\mathcal{H}_q(d)$-module, since $\mathcal{H}_q(d)$ is not a Hopf algebra. However, the automorphism $\#$ lets us define, for each $\mathcal{H}_q(d)$-module M, a new module $M^\#$ with the same underlying vector space and with action given by $h \circ m := h^\# m$. Since this specializes for $q = 1$ to tensoring with the sign representation, we henceforth denote it by

$$M \otimes \operatorname{sgn} := M^\#.$$

The simple $\mathcal{H}_q(d)$-modules can also be indexed by Λ_{res}. For $\lambda \in \Lambda_{\text{res}}$ we denote the corresponding simple module by D_λ. Then it is known that $D_\lambda = \operatorname{soc}(S^\lambda)$. The relationship between these two parameterizations is given by

$$D^\lambda \cong D_{\lambda'} \otimes \operatorname{sgn} \quad \text{for any } \lambda \in \Lambda_{\text{reg}}. \tag{2.2.2}$$

We also make extensive use of the following [**15**, 6.7; **22**, Exercise 3.14].

$$S^\lambda \otimes \operatorname{sgn} \cong (S^{\lambda'})^* := S_{\lambda'}. \tag{2.2.3}$$

In particular, (2.2.3) shows that tensoring with the sign representation turns Specht modules into dual Specht modules, and vice versa.

2.3

Henceforth, let $n \geqslant d$. The q-Schur algebra was introduced by Dipper and James as follows:

$$A := S_q(n,d) = \mathrm{End}_{\mathcal{H}_q(d)}\left(\bigoplus_{\lambda\in\Lambda(n,d)} M^\lambda\right).$$

When $q=1$, this is the classical Schur algebra $S(n,d)$ studied in [**14**].

The representation theory of A was described in [**6**], and it parallels the $S(n,d)$-theory. The simple A-modules are indexed by Λ, and we denote the simple A-module with highest weight λ by $L(\lambda)$. We also write $\Delta(\lambda)$ and $\nabla(\lambda)$ for the standard and costandard A-modules with highest weight λ, respectively. Moreover, if $\lambda \in \Lambda$, let $P(\lambda)$ be the projective cover of $L(\lambda)$ and let $T(\lambda)$ be the corresponding tilting module. These modules are described in detail in [**21**, Chapters 3, 4, 7].

Just as the classical Schur algebra has an involutory antiautomorphism corresponding to the taking of matrix transposes, so does A [**6**]. Thus A-modules have a contravariant dual, fixing simple modules, which we denote by M^τ.

Let e be the idempotent in A called $\phi^1_{w,w}$ in [**6**]. Then $eAe \cong \mathcal{H}_q(d)$. The Schur functor $\mathcal{F}$ is the covariant exact functor from $\mathrm{Mod}(A)$ to $\mathrm{Mod}(\mathcal{H}_q(d))$ defined on objects by $\mathcal{F}(M)=eM$. The duality τ and the usual duality '$*$' in $\mathrm{Mod}(\mathcal{H}_q(d))$ are compatible in the sense that $e(M^\tau)\cong(eM)^*$ for any finite-dimensional $M\in\mathrm{Mod}(A)$. It is well known that $\mathcal{F}(L(\lambda))$ is non-zero if and only if $\lambda\in\Lambda_{\mathrm{res}}$, in which case $\mathcal{F}(L(\lambda))\cong D_\lambda$.

One has the following correspondences between A-modules and $\mathcal{H}_q(d)$-modules under $\mathcal{F}$ (see [**9**]).

$$\mathcal{F}(\nabla(\lambda))=S^\lambda,\quad \mathcal{F}(\Delta(\lambda))=S_\lambda\cong S^{\lambda'}\otimes\mathrm{sgn},\quad \mathcal{F}(P(\lambda))=Y^\lambda,\quad \mathcal{F}(T(\lambda))=Y^{\lambda'}\otimes\mathrm{sgn}.$$

2.4. *The inverse Schur functor $\mathcal{G}$*

Recall that $A=S_q(n,d)$. The Schur functor $\mathcal{F}$ can be represented as a tensor functor $\mathcal{F}(M)\cong eA\otimes_A M$. The functor $\mathcal{F}$ admits a right adjoint functor $\mathcal{G}$ defined by

$$\begin{aligned}\mathcal{G}(N) &:= \mathrm{Hom}_{eAe}(eA,N)\\ &= \mathrm{Hom}_{\mathcal{H}_q(d)}(V^{\otimes d},N),\end{aligned}$$

where $V^{\otimes d}$ is the q-tensor space. The q-tensor space $V^{\otimes d}$ is *defined* in [**6**] as being eA, but for an explicit construction, see [**22**, Exercise 3.19]. In particular,

$$V^{\otimes d}\cong\bigoplus_{\lambda\in\Lambda(n,d)} M^\lambda.$$

The functor $\mathcal{G}$ is a right inverse to $\mathcal{F}$, that is, $\mathcal{F}\circ\mathcal{G}(N)\cong N$, but not a two-sided inverse. However, we do have the following.

Proposition 2.4.1 [**12**, 3.3]. *Suppose that $M\in\mathrm{Mod}(A)$ has l-restricted socle. Then there is an exact sequence of A-modules*

$$0\longrightarrow M\longrightarrow\mathcal{G}\circ\mathcal{F}(M)\longrightarrow D\longrightarrow 0,$$

where D has no l-restricted composition factors.

The functor $\mathcal{G}$ is only left exact, and so it has higher right derived functors

$$R^j\mathcal{G}(N) = \mathrm{Ext}^i_{\mathcal{H}_q(d)}(V^{\otimes d}, N).$$

These functors seem to control a great deal of the relationship between the cohomology theory of A and $\mathcal{H}_q(d)$. There is a nice spectral sequence relating these two functors. Under suitable assumptions, these higher right derived functors allow one to extend the adjointness to Ext^1.

THEOREM 2.4.2 [**12**, 2.2]. *Let $M \in \mathrm{Mod}(A)$ and let $N \in \mathrm{Mod}(\mathcal{H}_q(d))$.*
(i) *There exists a first quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}^i_A(M, R^j\mathcal{G}(N)) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{H}_q(d)}(\mathcal{F}(M), N).$$

(ii) *If either $M/\operatorname{rad} M$ has only l-restricted composition factors, or if $R^1\mathcal{G}(N) = 0$, then*

$$\mathrm{Ext}^1_A(M, \mathcal{G}(N)) \cong \mathrm{Ext}^1_{\mathcal{H}_q(d)}(\mathcal{F}(M), N).$$

2.5. *Good and Weyl filtrations*

In this subsection we collect some necessary results from the cohomology of quantum groups. The standard reference here is [**19**] for the classical case. For the quantum setting, the results are proved in [**10**].

An ascending chain

$$0 = M_0 \subset M_1 \subset M_2 \dots$$

of submodules of an A-module $M = \bigcup M_i$ is called a *good filtration* if each M_i/M_{i-1} is isomorphic to some $\nabla(\lambda_i)$. Similarly, M has a *Weyl filtration* if the quotients are isomorphic to $\Delta(\lambda_i)$. Since $\Delta(\lambda)^\tau \cong \nabla(\lambda)$, a module M has a good filtration if and only if M^τ has a Weyl filtration.

The first three parts of the following proposition are due to Donkin [**7**, 1.3], and the last is due to Ringel [**23**, Corollary 4].

PROPOSITION 2.5.1. *Let $V \in \mathrm{Mod}(A)$. The following are equivalent.*
(i) *V admits a Weyl filtration.*
(ii) $\mathrm{Ext}^1_A(V, \nabla(\lambda)) = 0$ *for all* $\lambda \in \Lambda$.
(iii) $\mathrm{Ext}^i_A(V, \nabla(\lambda)) = 0$ *for all* $\lambda \in \Lambda$ *and for all* $i > 0$.
(iv) $\mathrm{Ext}^i_A(V, T(\lambda)) = 0$ *for all* $\lambda \in \Lambda$ *and for all* $i > 0$.

Since V has a Weyl filtration exactly when V^τ has a good filtration, Proposition 2.5.1 also gives equivalent conditions for good filtrations, and we will use this without comment. As far as we know, part (iv) cannot be strengthened to require vanishing only of Ext^1.

A crucial result in the cohomology theory for A is the following.

$$\mathrm{Ext}^i_A(\Delta(\lambda), \nabla(\lambda)) \cong \begin{cases} k & i = 0, \lambda = \mu \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.2}$$

Equation 2.5.2 allows one to conclude easily (see [**19**, II.4.16]) that the filtration multiplicities in an A-module with a good filtration are well defined, and similarly for a Weyl filtration. In particular, if V has a Weyl filtration, and if $[V : \Delta(\lambda)]$ denotes the number of factors isomorphic to $\Delta(\lambda)$, then

$$[V : \Delta(\lambda)] = \dim_k \mathrm{Hom}_A(V, \nabla(\lambda)). \tag{2.5.3}$$

3. *Specht and dual Specht filtrations*

3.1

We define a Specht and dual Specht filtration analogously to that of a good or Weyl filtration, namely, an $\mathcal{H}_q(d)$-module N has a *Specht filtration* if we have

$$0 = N_0 \subset N_1 \subset N_2 \ldots,$$

where $N = \bigcup N_i$ and N_i/N_{i-1} is isomorphic to some S^{λ_i}. Similarly, N has a *dual Specht filtration* if the quotients are isomorphic to some S_{λ_i}. We remark that (2.2.3) shows that N has a Specht filtration if and only if $N \otimes \mathrm{sgn}$ has a dual Specht filtration.

3.2

We would like a theory of Specht filtrations that is like that of good filtrations. Unfortunately, the result corresponding to (2.5.2) for the symmetric group is not true; indeed $\mathrm{Ext}^1_{k\Sigma_d}(S_\lambda, S^\mu)$ can be nonzero. It has long been assumed that the filtration multiplicities for Specht or dual Specht filtrations are not well defined, since they are not well defined for $p = 2$ and $p = 3$. For example, when $p = d = 2$, the two Specht modules $S^{(2)}$ and $S^{(1^2)}$ are both one-dimensional and are isomorphic. Thus the two-dimensional regular representation has a filtration with two copies of $S^{(2)}$, a filtration with one with a copy of $S^{(2)}$ and one copy of $S^{(1^2)}$, and a filtration with two copies of $S^{(1^2)}$, so the multiplicities are not well defined. This module is isomorphic to the Young module $Y^{(1^2)}$, providing an example of where the multiplicities are not well defined, even for a Young module.

For $p = 3$ and $d = 7$, the Specht module $S^{(3^2,1)}$ has the obvious Specht filtration (it is already a Specht module!). However, it also has a filtration

$$0 \longrightarrow S^{(5,1^2)} \longrightarrow S^{(3^2,1)} \longrightarrow S^{(2,1^5)} \longrightarrow 0.$$

3.3

We first look at an important calculation involving the higher right derived functors. The corresponding result for symmetric groups is in [**11**, 4.1], but the proof does not generalize to the Hecke algebra setting. Let k and sgn be the one-dimensional representations of $\mathcal{H}_q(d)$ from (2.2.1). Then we have the following proposition.

PROPOSITION 3.3.1. *Let $l \geqslant 4$. Then the following hold.*
(i) $\mathrm{Ext}^1_{\mathcal{H}_q(d)}(k, \mathrm{sgn}) = 0$.
(ii) $R^1\mathcal{G}(\mathrm{sgn}) = 0$.

Proof. (i) Without loss of generality, we may assume that $l \mid d$, because otherwise the Nakayama rule [**22**, 5.38] implies that k and sgn are in different blocks. Now $S^{(d-1)} \cong k$, so the usual branching rule [**22**, 6.2] shows that

$$M := \mathrm{Ind}_{\mathcal{H}_q(d-1)}^{\mathcal{H}_q(d)} k$$

has a filtration with quotients being Specht modules $S^{(d)}$ and $S^{(d-1,1)}$. Since M is self-dual, we have

$$M := \operatorname{Ind}_{\mathcal{H}_q(d-1)}^{\mathcal{H}_q(d)} k \cong \begin{array}{c} k \\ D^{(d-1,1)}. \\ k \end{array}$$

Now let

$$N := \begin{array}{c} k \\ D^{(d-1,1)} \end{array}$$

and consider the short exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow N \longrightarrow 0.$$

This yields a long exact sequence of the form

$$\ldots \longrightarrow \operatorname{Hom}_{\mathcal{H}_q(d)}(k, \mathrm{sgn}) \longrightarrow \operatorname{Ext}^1_{\mathcal{H}_q(d)}(N, \mathrm{sgn}) \longrightarrow \operatorname{Ext}^1_{\mathcal{H}_q(d)}(M, \mathrm{sgn}) \longrightarrow \ldots . \tag{3.3.2}$$

Since $l \neq 2$, we know that $\operatorname{Hom}_{\mathcal{H}_q(d)}(k, \mathrm{sgn}) = 0$. Therefore Frobenius reciprocity and the fact that $l \not| d-1$ imply that

$$\operatorname{Ext}^1_{\mathcal{H}_q(d)}(M, \mathrm{sgn}) \cong \operatorname{Ext}^1_{\mathcal{H}_q(d-1)}(k, \mathrm{sgn}) = 0.$$

It follows from (3.3.2) that $\operatorname{Ext}^1_{\mathcal{H}_q(d)}(N, \mathrm{sgn}) = 0$. Now consider the short exact sequence

$$0 \longrightarrow D^{(d-1,1)} \longrightarrow N \longrightarrow k \longrightarrow 0.$$

From this we obtain the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{H}_q(d)}\big(D^{(d-1,1)}, \mathrm{sgn}\big) \longrightarrow \operatorname{Ext}^1_{\mathcal{H}_q(d)}(k, \mathrm{sgn}) \longrightarrow 0.$$

However, $\operatorname{Hom}_{\mathcal{H}_q(d)}(D^{(d-1,1)}, \mathrm{sgn}) = 0$ since $l \neq 3$; thus $\operatorname{Ext}^1_{\mathcal{H}_q(d)}(k, \mathrm{sgn}) = 0$.

(ii) From part (i), we can conclude that

$$\begin{aligned} R^1\mathcal{G}(\mathrm{sgn}) &\cong \operatorname{Ext}^1_{\mathcal{H}_q(d)}(V^{\otimes d}, \mathrm{sgn}) \\ &\cong \bigoplus_{\sigma \in \Lambda(n,d)} \operatorname{Ext}^1_{\mathcal{H}_q(d)}(M^\sigma, \mathrm{sgn}) \\ &\cong \bigoplus_{\sigma \in \Lambda(n,d)} \operatorname{Ext}^1_{\mathcal{H}_q(\sigma)}(k, \mathrm{sgn}). \end{aligned} \tag{3.3.3}$$

Now we apply the Kunneth formula to each $\operatorname{Ext}^1_{\mathcal{H}_q(\sigma)}(k, \mathrm{sgn})$. Since every term in the expansion will include an $\operatorname{Ext}^1_{\mathcal{H}_q(t)}(k, \mathrm{sgn})$ for some $t \leqslant d$, part (i) implies that $R^1\mathcal{G}(\mathrm{sgn}) = 0$. $\square$

Henceforth, assume that $n = d$. Recall that $S^\lambda \otimes \mathrm{sgn} \cong S_{\lambda'}$. Part (i) of the next result generalizes [**2**, 2.4], although the proof is closer to that given in [**11**, 4.2].

THEOREM 3.3.4. *Let $l \geqslant 4$ and $\lambda \in \Lambda$.*

(i) $\operatorname{Ext}^1_{\mathcal{H}_q(d)}(k, S_{\lambda'}) = 0$.
(ii) $R^1\mathcal{G}(S^\lambda \otimes \mathrm{sgn}) = 0$.
(iii) $R^1\mathcal{G}(Y^\lambda \otimes \mathrm{sgn}) = 0$.

Proof. (i) Putting $M=\Delta(\lambda)$ and $N=\mathrm{sgn}$ into the spectral sequence from Theorem 2.4.2(i) gives

$$E_2^{i,j}=\mathrm{Ext}^i_{S_q(n,d)}(\Delta(\lambda),R^j\mathcal{G}(\mathrm{sgn}))\Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{H}_q(d)}(S^{\lambda'}\otimes \mathrm{sgn},\mathrm{sgn}).$$

From [**12**, Theorem 5.2(i)], $\mathcal{G}(\mathrm{sgn})=\delta$, where δ is the determinant representation. Thus the spectral sequence yields an exact sequence [**12**, 2.2]

$$0\longrightarrow \mathrm{Ext}^1_{S_q(n,d)}(\Delta(\lambda),\delta)\longrightarrow \mathrm{Ext}^1_{\mathcal{H}_q(d)}(k,S_{\lambda'})\longrightarrow \mathrm{Hom}_{S_q(n,d)}(\Delta(\lambda),R^1\mathcal{G}(\mathrm{sgn})). \tag{3.3.5}$$

However, $\mathrm{Ext}^i_{S_q(n,d)}(\Delta(\lambda),\delta)=0$ for $i>0$ since $\delta\cong\nabla(1^d)$, and $R^1\mathcal{G}(\mathrm{sgn})=0$ by Proposition 3.3.1(ii), so (3.3.5) proves (i).

(ii) The proof of (ii) is similar to (3.3.3), and it is modelled on [**20**, 6.4]:

$$\begin{aligned}R^1\mathcal{G}(S^\lambda\otimes\mathrm{sgn})&\cong \mathrm{Ext}^1_{\mathcal{H}_q(d)}(V^{\otimes d},S_{\lambda'})\\ &\cong \bigoplus_{\sigma\in\Lambda(n,d)}\mathrm{Ext}^1_{\mathcal{H}_q(d)}(M^\sigma,S_{\lambda'})\\ &\cong \bigoplus_{\sigma\in\Lambda(n,d)}\mathrm{Ext}^1_{\mathcal{H}_q(\sigma)}(k,S_{\lambda'}).\end{aligned} \tag{3.3.6}$$

Now we use the fact [**18**] that $S_{\lambda'}$ has a filtration with factors of the form $S_{\mu_1}\otimes\dots\otimes S_{\mu_t}$ when restricted to $\mathcal{H}_q(\sigma)$. Thus the Kunneth formula and part (i), when applied to (3.3.6), complete the proof.

(iii) The proof of (iii) follows from (ii), because $Y^\lambda\otimes\mathrm{sgn}$ admits a dual Specht module filtration. □

The restrictions on l in Theorem 3.3.4 cannot be improved. Indeed, for $l=3$ and for d small, $R^1\mathcal{G}(S^{(d)}\otimes\mathrm{sgn})$ may be nonzero in the classical symmetric group case [**11**, 4.1].

3.4

Part (i) of the following lemma is due to James [**16**, 2.8] in the classical case, and it follows from the same argument in the quantum case [**9**, Section 4.3]. Part (ii) is immediate from the adjointness of $\mathcal{F}$ and $\mathcal{G}$.

LEMMA 3.4.1. *Let $\lambda\in\Lambda$ and let $N\in\mathrm{Mod}(\mathcal{H}_q(d))$. Then the following hold.*
(i) *$\Delta(\lambda)$ has l-restricted socle.*
(ii) *$\mathcal{G}(N)$ has l-restricted socle.*
(iii) *$T(\lambda)$ has l-restricted socle.*

Proof. Only (iii) needs to be proved, and it follows since $T(\lambda)$ is well known to have a filtration by Weyl modules. Any module with a Weyl filtration will have l-restricted socle. This follows from (i) because, given a short exact sequence

$$0\longrightarrow V\longrightarrow T\longrightarrow U\longrightarrow 0,$$

$\mathrm{soc}(T)$ embeds in $\mathrm{soc}(U)\oplus\mathrm{soc}(V)$. □

The following theorem (the motivation for which is [**20**, 3.1, 3.2]) shows that the functor $\mathcal{G}$ is well behaved on dual Specht and twisted Young modules. We remark that Theorem 3.4.2 and Corollary 3.8.2 first appeared in [**13**, Theorem 7.6, Proposition 7.5; **4**, Theorem 5.2.4] with fewer restrictions on l. Their approach uses base changes. We provide alternative proofs of these results to make the exposition more self-contained.

THEOREM 3.4.2. *Suppose that $l \geqslant 4$. Then the following hold.*
(i) $\mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn}) \cong T(\lambda)$.
(ii) $\mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn}) \cong \Delta(\lambda)$.

Proof. (i) First we will show that $\mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn})$ has a good filtration. A similar exact sequence to (3.3.5) gives an injection.

$$0 \longrightarrow \mathrm{Ext}^1_{S_q(n,d)}(\Delta(\mu), \mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn})) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}_q(d)}(S_\mu, Y^{\lambda'} \otimes \mathrm{sgn}).$$

However, $Y^{\lambda'}$ is a summand of $V^{\otimes d}$; therefore,

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{H}_q(d)}(S_\mu, Y^{\lambda'} \otimes \mathrm{sgn}) &\hookrightarrow \mathrm{Ext}^1_{\mathcal{H}_q(d)}(S_\mu, V^{\otimes d} \otimes \mathrm{sgn}) \\ &\cong \mathrm{Ext}^1_{\mathcal{H}_q(d)}(V^{\otimes d}, S^\mu \otimes \mathrm{sgn}) \\ &= R^1\mathcal{G}(S^\mu \otimes \mathrm{sgn}) \\ &= 0. \end{aligned}$$

Therefore, by Proposition 2.5.1 , $\mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn})$ has a good filtration.

By Lemma 3.4.1 , $T(\lambda)$ has l-restricted socle. Thus, by Proposition 2.4.1, there exists an exact sequence of the form

$$0 \longrightarrow T(\lambda) \longrightarrow \mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn}) \longrightarrow X \longrightarrow 0,$$

where X has no l-restricted composition factors. Because the modules $T(\lambda)$ and $\mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn})$ have good filtrations, it follows that X must have a good filtration. Consequently, $X = 0$ because each $\nabla(\mu)$ has an l-restricted head.

(ii) Since $\Delta(\lambda)$ has an l-restricted socle, Proposition 2.4.1 yields a short exact sequence of the form

$$0 \longrightarrow \Delta(\lambda) \longrightarrow \mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn}) \longrightarrow X \longrightarrow 0.$$

We will prove that $\Delta(\lambda)$ and $\mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn})$ have the same composition factors with the same multiplicities to show that $X = 0$. We have

$$\begin{aligned} [\mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn}) : L(\mu)] &= \dim_k \mathrm{Hom}_{\mathcal{H}_q(d)}(Y^\mu, S^{\lambda'} \otimes \mathrm{sgn}) \quad \text{by [12, 2.6]} \\ &= \dim_k \mathrm{Hom}_{\mathcal{H}_q(d)}(S_{\lambda'}, Y^\mu \otimes \mathrm{sgn}) \\ &= \dim_k \mathrm{Hom}_{S_q(n,d)}(\Delta(\lambda'), \mathcal{G}(Y^\mu \otimes \mathrm{sgn})) \quad \text{by adjointness} \\ &= \dim_k \mathrm{Hom}_{S_q(n,d)}(\Delta(\lambda'), T(\mu')) \\ &= [T(\mu') : \nabla(\lambda')] \quad \text{by (2.5.3)} \\ &= [\Delta(\lambda) : L(\mu)] \quad \text{by reciprocity.} \end{aligned}$$

□

3.5

Despite the failure of (2.5.2) for symmetric groups, we will show that Specht filtration multiplicities are well defined, as long as $l \geqslant 4$. The key observation is that $\mathcal{G}$ is a two-sided inverse to $\mathcal{F}$ on modules with Weyl filtrations. We begin with the following theorem.

THEOREM 3.5.1. *Let $l \geqslant 4$ and let $N \in \mathrm{Mod}(\mathcal{H}_q(d))$. Then there is a one-to-one correspondence between dual Specht module filtrations of N and Weyl filtrations of $\mathcal{G}(N)$.*

Proof. The Schur functor $\mathcal{F}$ is exact and $\mathcal{F}(\Delta(\lambda)) = S_\lambda$. Since $\mathcal{F}(\mathcal{G}(N)) = N$, it is clear that a Weyl filtration of $\mathcal{G}(N)$ induces a dual Specht filtration of N via $\mathcal{F}$. We prove the converse by induction on the length of the filtration. Thus suppose that N has a dual Specht filtration; we show that $\mathcal{G}(N)$ has a Weyl filtration.

Theorem 3.4.2(ii) proves the case where $N \cong S_\mu$. Now suppose that we have

$$0 \longrightarrow S_\mu \longrightarrow N \longrightarrow U \longrightarrow 0,$$

and assume by induction that $\mathcal{G}(U)$ has a Weyl filtration. Applying $\mathcal{G}$ gives

$$0 \longrightarrow \mathcal{G}(S_\mu) \longrightarrow \mathcal{G}(N) \longrightarrow \mathcal{G}(U) \longrightarrow R^1\mathcal{G}(S_\mu).$$

However, $R^1\mathcal{G}(S_\mu) = 0$ by Theorem 3.3.4(ii), so $\mathcal{G}(N)$ has a Weyl filtration. □

3.6

Theorem 3.5.1 allows us to translate Proposition 2.5.1 into a criterion for an $\mathcal{H}_q(d)$-module to have a dual Specht filtration in terms of A-cohomology.

THEOREM 3.6.1. *Let $l \geqslant 4$ and let $N \in \mathrm{Mod}(\mathcal{H}_q(d))$. The following are equivalent.*

(i) *N has a dual Specht module filtration.*
(ii) $\mathrm{Ext}^1_A(\mathcal{G}(N), \nabla(\lambda)) = 0$ *for all $\lambda \in \Lambda$.*
(iii) $\mathrm{Ext}^i_A(\mathcal{G}(N), T(\lambda)) = 0$ *for all $\lambda \in \Lambda$ and for all $i > 0$.*

Of course N has a dual Specht filtration if and only if $N \otimes \mathrm{sgn}$ has a Specht filtration, so Theorem 3.6.1 also yields a criterion for Specht filtrations. That is, we have the following theorem.

THEOREM 3.6.2. *Let $l \geqslant 4$ and let $N \in \mathrm{Mod}(\mathcal{H}_q(d))$. The following are equivalent.*

(i) *N has a Specht module filtration.*
(ii) $\mathrm{Ext}^1_A(\mathcal{G}(N \otimes \mathrm{sgn}), \nabla(\lambda)) = 0$ *for all $\lambda \in \Lambda$.*
(iii) $\mathrm{Ext}^i_A(\mathcal{G}(N \otimes \mathrm{sgn}), T(\lambda)) = 0$ *for all $\lambda \in \Lambda$ and for all $i > 0$.*

3.7

It is well known that a Young module for $k\Sigma_d$ is a p-modular reduction of a unique characteristic zero module, and so it has a well defined ordinary character. For $\lambda, \mu \in \Lambda$, let $(Y^\lambda : S^\mu)$ denote the multiplicity of the complex character corresponding to S^μ in the character of Y^λ. Then Y^λ is known to have a Specht filtration with each S^μ appearing $(Y^\lambda : S^\mu)$ times. This is the filtration induced by $\mathcal{F}$ from the good filtration of $I(\lambda)$, the injective hull of $L(\lambda)$. Although this is often denoted by $[Y^\lambda : S^\mu]$, this notation is ambiguous, since we saw in Subsection 3.2 that Young modules can have Specht filtrations with different multiplicities. The following theorem, which is immediate from (2.5.3) and Theorem 3.5.1, proves that for $l \geqslant 4$, filtration multiplicities in Specht and dual Specht filtrations *are* well defined for any $\mathcal{H}_q(d)$-module. In particular, for $k\Sigma_d$ there is no ambiguity when

$p \geqslant 5$, that is, $[Y^\lambda : S^\mu] = (Y^\lambda : S^\mu)$ is well defined. Thus we can define $[N : S_\lambda]$ as the multiplicity of S_λ in any dual Specht filtration of N.

THEOREM 3.7.1. *Let $l \geqslant 4$ and suppose that $N \in \mathrm{Mod}(\mathcal{H}_q(d))$ has a dual Specht filtration and that $M \in \mathrm{Mod}(\mathcal{H}_q(d))$ has a Specht filtration. Then the filtration multiplicities are well defined. Specifically,*

$$[N : S_\lambda] = \dim_k \mathrm{Hom}_A(\mathcal{G}(N), \nabla(\lambda)).$$

Similarly, for Specht filtrations,

$$[M : S^\lambda] = \dim_k \mathrm{Hom}_A(\mathcal{G}(N^*), \nabla(\lambda)).$$

3.8

Theorem 3.5.1 and Proposition 2.4.1 show that on the category of modules with Weyl filtrations, $\mathcal{G}$ is a two-sided inverse to $\mathcal{F}$. Namely we have the following theorem.

THEOREM 3.8.1. *Let $l \geqslant 4$. Suppose that $M \in \mathrm{Mod}(A)$ has a Weyl filtration. Then $M \cong \mathcal{G}(\mathcal{F}(M))$.*

Proof. Since M has a Weyl filtration, we can apply Proposition 2.4.1 to M to obtain

$$0 \longrightarrow M \longrightarrow \mathcal{G}(\mathcal{F}(M)).$$

However, M and $\mathcal{G}(\mathcal{F}(M)))$ both have Weyl filtrations with the same multiplicities, so they must be isomorphic. □

This result lets us determine $\mathcal{G}$ on Young modules.

COROLLARY 3.8.2. *Let $l \geqslant 4$. Then $\mathcal{G}(Y^\lambda) = P(\lambda)$.*

Thus we can determine $\mathcal{G}$ on Young, twisted Young and dual Specht modules, because these modules all have dual Specht filtrations. However, $\mathcal{G}(S^\lambda)$ is not known.

3.9

In [**20**], some conditions were given for when

$$\mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^i_{eAe}(eM, eN).$$

Theorem 3.8.1 gives such a stability result for A and $\mathcal{H}_q(d)$ extensions of modules with Weyl and dual Specht filtrations. Namely we have the following corollary.

COROLLARY 3.9.1. *Let $l \geqslant 4$. Suppose that $N_2 \in \mathrm{Mod}(A)$ has a Weyl filtration. Then*

$$\mathrm{Ext}^1_A(N_1, N_2) \cong \mathrm{Ext}^1_{\mathcal{H}_q(d)}(\mathcal{F}(N_1), \mathcal{F}(N_2)).$$

Proof. This is immediate from Theorem 2.4.2 and Theorem 3.8.1, since $N_2 = \mathcal{G}(\mathcal{F}(N_2))$ and $R^1\mathcal{G}(\mathcal{F}(N_2)) = 0$. □

In the classical case, an induction argument on the Sylow subgroups of Σ_d allows us to extend the stability of Ext^1 in Corollary 3.9.1 to higher Ext. The details can be found in [**20**], but we note the result here.

COROLLARY 3.9.2. *Let $p \geqslant 5$, and suppose that $N_2 \in \mathrm{mod}(S(n,d))$ has a Weyl filtration. Then, for $0 \leqslant i \leqslant p-2$,*

$$\mathrm{Ext}^i_{S(n,d)}(N_1, N_2) \cong \mathrm{Ext}^i_{k\Sigma_d}(\mathcal{F}(N_1), \mathcal{F}(N_2)).$$

4. *A new construction of Young modules*

4.1

Since $\mathcal{F}(T(\lambda)) = Y^{\lambda'} \otimes \mathrm{sgn}$, one might expect Young modules, or twisted Young modules, to play a role similar to that of tilting modules. In $\mathrm{Mod}(A)$, the tilting modules are exactly those modules with both good and Weyl filtrations. Young modules have both Specht and dual Specht filtrations, but they are not the only modules that do. For example, any simple Specht module has both a dual Specht and a Specht filtration, but simple Specht modules are not always Young modules. Still there are parallels, and in this section we show how the results from the last section lead to a new, cohomological, construction of Young modules, for which the motivation is a similar construction of tilting modules. First we need a result similar to [**20**, 6.4(b)].

PROPOSITION 4.1.1. *Let $l \geqslant 4$ and $\lambda, \mu \in \Lambda$. Then $\mathrm{Ext}^1_{\mathcal{H}_q(d)}(S^\mu, Y^\lambda) = 0$.*

Proof. Since $R^1\mathcal{G}(S_\mu) = 0$, the proposition is immediate from Theorems 2.4.2 and 3.4.2(ii). Namely,

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{H}_q(d)}(S^\mu, Y^\lambda) &\cong \mathrm{Ext}^1_{\mathcal{H}_q(d)}(Y^\lambda, S_\mu)\\ &\cong \mathrm{Ext}^1_A(P(\lambda), \Delta(\mu)) = 0.\end{aligned}$$ □

We remark that [**20**, 6.4(b)(iv)] claims incorrectly that $\mathrm{Ext}^1_{k\Sigma_d}(Y^\mu, S^\lambda) = 0$ (the argument given actually proves that $\mathrm{Ext}^1_{k\Sigma_d}(S^\lambda, Y^\mu) = 0$). For example, in [**3**], the nonprojective Young modules of defect 2 are determined. From those results, we calculate that, for $p = 5$ and $d = 13$,

$$\mathrm{Ext}^1_{k\Sigma_{13}}\left(Y^{(7,2,2,1,1)}, S^{(6,3,2,1,1)}\right) \neq 0.$$

4.2

We also need another proposition, for which the motivation is [**20**, 3.3c]. Let $\rhd$ denote the usual dominance order on partitions [**17**, 1.4.5].

PROPOSITION 4.2.1. *Let $l \geqslant 4$, and suppose that $\mu \not\trianglerighteq \lambda$. Then $\mathrm{Ext}^1_{\mathcal{H}_q(d)}(S^\mu, S^\lambda) = 0$.*

Proof.

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{H}_q(d)}(S^\mu, S^\lambda) &\cong \mathrm{Ext}^1_{\mathcal{H}_q(d)}(S^\mu \otimes \mathrm{sgn}, S^\lambda \otimes \mathrm{sgn})\\ &\cong \mathrm{Ext}^1_{\mathcal{H}_q(d)}(S_{\mu'}, S_{\lambda'})\end{aligned}$$

$$\begin{aligned} &\cong \operatorname{Ext}^1_A(\Delta(\mu'), \mathcal{G}(S_{\lambda'})) && \text{by Theorem 2.4.2} \\ &\cong \operatorname{Ext}^1_A(\Delta(\mu'), \Delta(\lambda')) && \text{by Theorem 3.4.2(ii).} \end{aligned}$$

However, $\operatorname{Ext}^1_A(\Delta(\mu'), \Delta(\lambda')) = 0$, since $\lambda' \not\trianglerighteq \mu'$, by [**19**, II, 2.14]. □

4.3

We can now begin the construction of the Young module Y^λ. We will follow the description of the Ringel–Donkin construction of $T(\lambda)$ as presented in [**1**]. Henceforth, assume that $l \geqslant 4$ and fix $\lambda \vdash d$. Define $W_0 = S^\lambda$. Choose λ_1 minimal with respect to $\triangleright$ such that

$$t_1 = \dim_k \operatorname{Ext}^1_{\mathcal{H}_q(d)}(S^{\lambda_1}, S^\lambda) \neq 0.$$

Let W_1 denote the corresponding extension:

$$0 \longrightarrow S^\lambda \longrightarrow W_1 \longrightarrow (S^{\lambda_1})^{\oplus t_1} \longrightarrow 0.$$

Notice that $\lambda_1 \triangleright \lambda$ by Proposition 4.2.1. Now choose λ_2 minimal such that

$$t_2 = \dim_k \operatorname{Ext}^1_{\mathcal{H}_q(d)}(S^{\lambda_2}, W_1) \neq 0.$$

Let W_2 denote the corresponding extension:

$$0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow (S^{\lambda_2})^{\oplus t_2} \longrightarrow 0.$$

We remark that $\lambda_1 \not\trianglerighteq \lambda_2$ by Proposition 4.2.1 and the minimality of λ_1. Continue in this way to construct a module W^λ with a filtration

$$0 \subset S^\lambda = W_0 \subset W_1 \subset \ldots \subset W_r = W^\lambda,$$

where

$$W_i / W_{i-1} \cong (S^{\lambda_i})^{\oplus t_i}.$$

It is clear from the construction that $\lambda_{i-1} \not\trianglerighteq \lambda_i$; that is, at each step, the new partition will not be $\trianglelefteq$ any of the previous partitions. However, there are only finitely many partitions of d, so the process must terminate. We conclude that W^λ is finite-dimensional and

$$\operatorname{Ext}^1_{\mathcal{H}_q(d)}(S^\tau, W^\lambda) = 0 \qquad \forall\, \tau \in \Lambda. \tag{4.3.1}$$

At this point in the corresponding construction of $T(\lambda)$, one could conclude immediately from (4.3.1) that $T(\lambda)$ has a Weyl filtration. However, we do not have an analogue of (2.5.2) for the Hecke algebra, so instead we must exploit the functor $\mathcal{G}$ to study W^λ.

4.4

Having constructed the module W^λ, we will now prove that it is isomorphic to the Young module Y^λ. First we recall a basic lemma from [**15**, 13.17] for the symmetric group and [**22**, Section 4.1, Exercise 4.11] for the Hecke algebra.

LEMMA 4.4.1. *Let $l \geqslant 3$.*

(i) *If $\lambda \not\trianglerighteq \mu$, then* $\operatorname{Hom}_{\mathcal{H}_q(d)}(S^\lambda, S^\mu) = 0$.

(ii) $\operatorname{Hom}_{\mathcal{H}_q(d)}(S^\lambda, S^\lambda) \cong k$.

Next we show that W^λ is indecomposable.

PROPOSITION 4.4.2. *W^λ is indecomposable.*

Proof. We prove that W_i is indecomposable by induction. $W_0 = S^\lambda$ is indecomposable whenever $l \geqslant 3$ by Lemma 4.4.1(ii), so the induction begins. Assume that W_t is indecomposable for all $t < i$. We will prove that W_i is indecomposable by showing that the only idempotent in $\mathrm{End}_{\mathcal{H}_q(d)}(W_i)$ is the identity.

Therefore choose an idempotent $f \in \mathrm{End}_{\mathcal{H}_q(d)}(W_i)$. By construction, we have a short exact sequence

$$0 \longrightarrow W_{i-1} \longrightarrow W_i \xrightarrow{\pi} (S^{\lambda_i})^{\oplus t_i} \longrightarrow 0, \tag{4.4.3}$$

where π is the natural projection.

If $f(W_{i-1}), \not\subset W_{i-1}$, then $\pi \circ f$ would be a nonzero map from W_{i-1} to $(S^{\lambda_i})^{\oplus t_i}$, which would contradict Lemma 4.4.1(i). Thus $f|_{W_{i-1}}$ is an idempotent in $\mathrm{End}_{H_q(d)}(W_{i-1})$. However, W_{i-1} is indecomposable by inductive hypothesis, so $f|_{W_{i-1}}$ is the identity map.

Define

$$\phi := f - 1 : W_i \longrightarrow W_i.$$

Since $W_{i-1} \in \ker \phi$, we get an induced map

$$\overline{\phi} : (S^{\lambda_i})^{\oplus t_i} \longrightarrow W_i,$$

where $\phi = 0$ if and only if $\overline{\phi} = 0$. Consider the long exact sequence resulting from applying $\mathrm{Hom}_{\mathcal{H}_q(d)}((S^{\lambda_i})^{\oplus t_i}, -)$ to (4.4.3). By construction of the W_i, the induced map

$$\mathrm{Hom}_{\mathcal{H}_q(d)}((S^{\lambda_i})^{\oplus t_i}, (S^{\lambda_i})^{\oplus t_i}) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}_q(d)}((S^{\lambda_i})^{\oplus t_i}, W_{i-1})$$

is an isomorphism.

Thus any map in $\mathrm{Hom}_{\mathcal{H}_q(d)}((S^{\lambda_i})^{\oplus t_i}, W_i)$ comes from a map in $\mathrm{Hom}_{\mathcal{H}_q(d)}((S^{\lambda_i})^{\oplus t_i}, W_{i-1})$, that is, has image in W_{i-1}. In particular, $\mathrm{Im}\,\overline{\phi} \subset W_{i-1}$, so $\mathrm{Im}(f-1) \subset W_{i-1}$. However, $f|_{W_{i-1}}$ is the identity, so

$$\begin{aligned} f - 1 &= f \circ (f-1) \quad \text{since } f \text{ is the identity on } \mathrm{Im}(f-1) \\ &= f \circ f - f \\ &= f - f \\ &= 0. \end{aligned}$$

Thus f is the identity map, as desired. $\square$

4.5

In this subsection we prove some nice homological properties of W^λ, which we will apply to show that it has a Specht filtration.

LEMMA 4.5.1. *Let $l \geqslant 4$.*

$$\mathrm{Ext}^1_{\mathcal{H}_q(d)}(V^{\otimes d}, W^\lambda) = 0 = \mathrm{Ext}^1_{\mathcal{H}_q(d)}(V^{\otimes d}, W^\lambda \otimes \mathrm{sgn}).$$

Proof. Use (4.3.1) and the well known fact that both $V^{\otimes d}$ and $V^{\otimes d} \otimes \mathrm{sgn}$ have Specht filtrations. $\square$

COROLLARY 4.5.2. *Let $M \in \mathrm{mod}(A)$.*

(i) $0 = R^1\mathcal{G}(W^\lambda) = R^1\mathcal{G}(W^\lambda \otimes \mathrm{sgn})$.
(ii) $\mathrm{Ext}^1_A(M, \mathcal{G}(W^\lambda)) \cong \mathrm{Ext}^1_{\mathcal{H}_q(d)}(\mathcal{F}(M), W^\lambda)$.
(iii) $\mathrm{Ext}^1_A(M, \mathcal{G}(W^\lambda \otimes \mathrm{sgn})) \cong \mathrm{Ext}^1_{\mathcal{H}_q(d)}(\mathcal{F}(M), W^\lambda \otimes \mathrm{sgn})$.

Proof. Part (i) is a restatement of Lemma 4.5.1. Parts (ii) and (iii) then follow from Theorem 2.4.2. □

4.6

In order to show that W^λ is a Young module, we first show that it has a dual Specht filtration.

LEMMA 4.6.1. *W^λ has a dual Specht filtration.*

Proof. From (4.3.1), we have

$$\begin{aligned} 0 &= \mathrm{Ext}^1_{\mathcal{H}_q(d)}(S^\tau, W^\lambda) \qquad \forall\, \tau \in \Lambda \\ &= \mathrm{Ext}^1_{\mathcal{H}_q(d)}(S_{\tau'}, W^\lambda \otimes \mathrm{sgn}) \\ &= \mathrm{Ext}^1_A(\Delta(\tau'), \mathcal{G}(W^\lambda \otimes \mathrm{sgn})) \quad \text{by Corollary 4.5.2(iii).} \end{aligned}$$

Thus $\mathcal{G}(W^\lambda \otimes \mathrm{sgn})$ has a good filtration by Proposition 2.5.1, so $W^\lambda \otimes \mathrm{sgn}$ has a Specht filtration and so W^λ has a dual Specht filtration. □

We just observed in proving Lemma 4.6.1 that $\mathcal{G}(W^\lambda \otimes \mathrm{sgn})$ has a good filtration. However, W^λ has a Specht filtration by construction, so $W^\lambda \otimes \mathrm{sgn}$ has a dual Specht filtration. Thus $\mathcal{G}(W^\lambda \otimes \mathrm{sgn})$ has a Weyl filtration by Theorem 3.5.1. Therefore $\mathcal{G}(W^\lambda \otimes \mathrm{sgn})$ has both a good and a Weyl filtration; thus it is a tilting module. Since $\mathcal{F} \circ G = \mathrm{Id}$, the module $W^\lambda \otimes \mathrm{sgn}$ must be isomorphic to $Y^\mu \otimes \mathrm{sgn}$ for some μ. However, the Specht filtration of W^λ implies that $\mu = \lambda$, that is, we have shown the following.

THEOREM 4.6.2. $W^\lambda \cong Y^\lambda$.

4.7

We remark that this construction works just as well if we start with S_λ and build the dual Specht filtration of Y^λ down from the top. Furthermore, this construction allows us to equate certain decomposition numbers with extensions between Specht modules. Specifically, we have the following theorem.

THEOREM 4.7.1. *Let $l \geqslant 4$. Fix λ and set*

$$L := \{\mu : \mathrm{Ext}^1_{\mathcal{H}_q(d)}(S^\mu, S^\lambda) \neq 0\}.$$

Let $L^{\min}$ be all the elements of L minimal with respect to $\rhd$. Then, for any $\mu \in L^{\min}$,

$$[Y^\lambda : S^\mu] = \dim_k \mathrm{Ext}^1_{H_q(d)}(S^\mu, S^\lambda).$$

Proof. This is clear from the construction of W^λ and the observation that, for $\mu_1, \mu_2 \in L^{\min}$, μ_1 and μ_2 are incomparable, so

$$0 = \mathrm{Ext}^1_{\mathcal{H}_q(d)}(S^{\mu_1}, S^{\mu_2}) = \mathrm{Ext}^1_{\mathcal{H}_q(d)}(S^{\mu_2}, S^{\mu_1}).$$

That is, putting the S^{μ_1} on top of S^λ first, or putting the S^{μ_2} on first, will result in the same module W^λ. □

By the well known reciprocity law,

$$(Y^\lambda : S^\mu) = [S^\mu : D_\lambda] = [\Delta(\mu) : L(\lambda)],$$

so Theorem 4.7.1 really does equate certain decomposition numbers and extensions between Specht modules.

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