ACADEMIC PRESS

Journal of Algebra 254 (2002) 422–440

JOURNAL OF Algebra

www.academicpress.com

Support varieties for modules over symmetric groups

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Received 21 January 2002

Communicated by George Glauberman

Dedicated to Professor Jonathan L. Alperin on the occasion of his 65th birthday

1. Introduction

1.1. In the late 1970s, Alperin [A] defined an invariant called the complexity of a module as a way to relate the modules with the complexes and resolutions that they admit. Several years later, Carlson [Ca1,Ca2] defined affine algebraic varieties corresponding to modules over group algebras. These varieties are subvarieties of the spectrum of the cohomology ring which was earlier described by Quillen [Q]. They are known in present day language as support varieties. It was discovered early on that the complexity of a module is equal to the dimension of the support variety of the module. Geometric methods involving support varieties have played a fundamental role in understanding the interplay between the modular representation theory and cohomology for finite groups. Despite substantial progress in this direction, there have been few explicit computations of support varieties for important classes of modules over certain groups.

The goal of this paper is to introduce methods and techniques for computing support varieties for modules over the symmetric group Σ_d. In the process, we will provide explicit computations of support varieties for certain classes

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1 Research of the author was supported in part by NSF grant DMS-0102019.

2 Research of the author was supported in part by NSF grant DMS-0102225.

0021-8693/02/$ – see front matter
PII: S0021-8693(02)00104-7

of modules. The paper is organized as follows. After setting up the notation in Section 1, we provide a definition of complexity and relative support varieties in Section 2. It will be advantageous to work with relative support varieties to relate the (ordinary) support varieties of different families of modules for the symmetric group. We also present some fundamental results on relative support varieties that will be used throughout the paper. In Section 3, the complexity and support varieties for the permutation and Young modules are determined. The varieties for these modules can be described by looking at the image of the restriction map on the variety of the trivial module over certain Young subgroups. The computation of the varieties for the Young modules are used in Section 4 to relate the varieties of the direct sums of irreducible modules and direct sums of Specht modules. For any module in a block for the symmetric group, we are able then to give a precise description of where the support of the module must be located. Later on in the section, we prove a formula which relates the computation of the support variety of a module to computing relative support varieties via branching over Young subgroups. The final section (Section 5) is devoted to computing the complexity and support varieties for a certain class of simple modules for the symmetric group called the completely splittable modules.

1.2. Notation. Let k be an algebraically closed field of characteristic $p > 0$. For any finite group G, let kG denote the group algebra of G. $\mathrm{Mod}(kG)$ will denote the category of all kG-modules, and $\mathrm{mod}(kG)$ will be the category of finite-dimensional kG-modules. If H is a subgroup of G, denoted $H \leqslant G$, and N is a kH-module then let $N{\uparrow}_H^G = kG \otimes_{kH} N$ be the induced module. On the other hand, if M is a kG-module then the restriction of M to kH will be denoted by $M{\downarrow}_H$.

Let d be a fixed positive integer and Σ_d be the symmetric group on d letters. We write $\lambda \models d$ for a composition of d and $\lambda \vdash d$ for a partition of d. For $\lambda \models d$, let Σ_λ be the corresponding Young subgroup of Σ_d, and $M^\lambda \cong k{\uparrow}_{\Sigma_\lambda}^{\Sigma_d}$ be the corresponding permutation module for Σ_d.

Other families of modules for the symmetric group can be constructed in the following way. Let $S(n,d)$ be the finite-dimensional associative k-algebra $\mathrm{End}_{k\Sigma_d}(V^{\otimes d})$, where V is the natural representation of the general linear group $\mathrm{GL}_n(k)$. This algebra is often referred to as the Schur algebra. It is well known that the category of modules for $S(n,d)$ is equivalent to the category of polynomial representations for $\mathrm{GL}_n(k)$ of homogeneous degree d. (See [Gr].)

Now suppose that $n \geqslant d$. Then there exists an idempotent $e \in S(n,d)$ such that $eS(n,d)e \cong k\Sigma_d$. The Schur functor $\mathcal{F}$ is the covariant exact functor from $\mathrm{mod}(S(n,d))$ to $\mathrm{mod}(k\Sigma_d)$ defined on objects by $\mathcal{F}(M) = eM$. The simple $S(n,d)$-modules are in bijective correspondence with partitions of d, and are denoted $L(\lambda)$ where $L(\lambda)$ has "highest weight" λ, in the sense that λ can be identified with a dominant polynomial weight of $\mathrm{GL}_n(k)$.

A partition $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ is called *p-restricted* if $\lambda_i - \lambda_{i+1} \leqslant p-1$ for all i. Let Λ_{res} be the set of all p-restricted partitions. The partition λ is *p-regular* if its transpose λ' is p-restricted, and we denote the set of p-regular partitions by Λ_{reg}. It is well known that $\mathcal{F}(L(\lambda))$ is non-zero if and only if $\lambda \in \Lambda_{\mathrm{res}}$ and

$$\bigl\{\mathcal{F}\bigl(L(\lambda)\bigr)\colon\ \lambda \in \Lambda_{\mathrm{res}}\bigr\}$$

is a complete set of simple $k\Sigma_d$-modules up to isomorphism. Set $D_\lambda = \mathcal{F}(L(\lambda))$ for $\lambda \in \Lambda_{\mathrm{res}}$.

The simple $k\Sigma_d$-modules are also indexed by Λ_{reg} by setting:

$$D^\lambda \cong D_{\lambda'} \otimes \mathrm{sgn} \quad \text{for any } \lambda \in \Lambda_{\mathrm{reg}}.$$

For each $\lambda \vdash d$, let $H^0(\lambda) = \mathrm{ind}_B^G \lambda$ be the induced module (see [Ja]) where $G = \mathrm{GL}_n(k)$ and B is the Borel subgroup, and let $I(\lambda)$ be the injective hull of $L(\lambda)$ in $\mathrm{Mod}(S(n,d))$. Set $S^\lambda = \mathcal{F}(H^0(\lambda))$ and $Y^\lambda = \mathcal{F}(I(\lambda))$. The modules $\{S^\lambda\colon\ \lambda \vdash d\}$ are called the Specht modules and the set $\{Y^\lambda\colon\ \lambda \vdash d\}$ are the Young modules. The indecomposable summands of the permutation modules M^λ consist of certain Young modules and every Young module appears as a direct summand of some permutation module.

The composition factors of these modules behave well with respect to $\trianglerighteq$, the usual dominance order on partitions. The Young modules all have filtrations by Specht modules and

$$S^\lambda = D^\lambda + \sum_{\mu \rhd \lambda} a_\mu D^\mu, \tag{1.1.1}$$

$$Y^\lambda = S^\lambda + \sum_{\mu \rhd \lambda} b_\mu S^\mu, \tag{1.1.2}$$

where the equalities in Eqs. (1.2.1) and (1.2.2) are of composition factors, and the term D^λ in Eq. (1.2.1) occurs only when $\lambda \in \Lambda_{\mathrm{reg}}$.

Each $\lambda \vdash d$ has a well-defined *p-core* $\widetilde{\lambda} \vdash d - pw$, where w is called the *weight* of λ. The Nakayama rule says the blocks of $k\Sigma_d$ are indexed by p-cores of partitions of d. The Specht modules S^λ and S^μ are in the same block if and only if $\widetilde{\lambda} = \widetilde{\mu}$, and similarly for the simple and Young modules. Thus the weight w is an invariant of the block. For $\lambda \vdash d$, let $\mathcal{B}_\lambda$ denote the block with $S^\lambda \in \mathcal{B}_\lambda$. Thus, $S^\mu \in \mathcal{B}_\lambda$ if and only if $\widetilde{\lambda} = \widetilde{\mu}$. This will be abbreviated by saying $\mu \in \mathcal{B}_\lambda$. In Section 5 we will also need the equivalent statement of the Nakayama rule in terms of residue contents of the Young diagrams. For details see [JK].

2. Complexity and support varieties

2.1. Let $\{d_n\}_{n \geqslant 0}$ be a sequence of non-negative integers. The rate of growth $r(d_\bullet)$ of this sequence is the smallest non-negative integer c for which there exists a positive real number C such that $d_n \leqslant C \cdot n^{c-1}$ for all $n \geqslant 1$. If no such d exists, set $r(d_\bullet) = \infty$.

Let $M \in \mathrm{mod}(kG)$ and let

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

be the minimal projective resolution of M. The *complexity* $c_G(M)$ of M is defined as $r(\dim_k P_\bullet)$ [A, Section 4].

2.2. Let G be a finite group. Set

$$H(G,k) = \begin{cases} H^{2\bullet}(G,k) & \text{if } \operatorname{char} k \neq 2, \\ H^\bullet(G,k) & \text{if } \operatorname{char} k = 2. \end{cases}$$

The algebra $H(G,k)$ is a commutative subalgebra of the cohomology ring $H^\bullet(G,k)$ and Evens [E1] proved that it is finitely generated. Set $V_G = \operatorname{Maxspec} H(G,k)$. The set V_G is an affine homogeneous variety and is often referred to as the *variety* of the group G.

Given $N, N' \in \mathrm{mod}(kG)$, define the *relative support variety* $V_G(N,N')$ as follows. The cup product gives $\mathrm{Ext}^\bullet_G(N,N')$ the structure of an $H(G,k)$-module [E2, p. 94]. Let $J(N,N')$ be the annihilator ideal in $H(G,k)$ for this action on $\mathrm{Ext}^\bullet_G(N,N')$. Set $V_G(N,N')$ equal to the closed subvariety of V_G defined by $J_G(N,N')$. The (*ordinary*) *support variety* $V_G(N)$ is obtained by setting $V_G(N) = V_G(N,N)$. The support varieties of modules are closed, conical subvarieties of V_G.

2.3. We now list some basic properties involving the notion of complexity and support varieties. Details can be found in [Ben2, Section 5.7].

2.3.1. *If* $N \in \mathrm{mod}(kG)$ *then* $c_G(N) = \dim V_G(N) = r(\dim_k \mathrm{Ext}^\bullet_G(N,N))$.

2.3.2. *If* $N \in \mathrm{mod}(kG)$ *and* $\{S_i \mid i = 1,2,\ldots,m\}$ *is a complete set of non-isomorphic simple modules for* kG *then*

$$c_G(N) = r\left(\dim_k \mathrm{Ext}^\bullet_G\left(N, \bigoplus_{i=1}^m S_i\right)\right) = r\left(\dim_k \mathrm{Ext}^\bullet_G\left(\bigoplus_{i=1}^m S_i, N\right)\right).$$

Moreover,

$$V_G(N) = V_G\left(N, \bigoplus_{i=1}^m S_i\right) = V_G\left(\bigoplus_{i=1}^m S_i, N\right).$$

2.3.3. *If* $N_1, N_2 \in \mathrm{mod}(kG)$ *then* $V_G(N_1 \oplus N_2) = V_G(N_1) \cup V_G(N_2)$.

2.3.4. *If* $N_1, N_2 \in \mathrm{mod}(kG)$ *then* $V_G(N_1 \otimes N_2) = V_G(N_1) \cap V_G(N_2)$.

2.3.5. *Let* $0 \to N_1 \to N_2 \to N_3 \to 0$ *be a short exact sequence in* $\operatorname{mod}(kG)$ *and let* $M \in \operatorname{mod}(kG)$. *If* Σ_3 *is the symmetric group on three letters and* $\sigma \in \Sigma_3$, *then*

(i) $V_G(N_{\sigma(1)}) \subseteq V_G(N_{\sigma(2)}) \cup V_G(N_{\sigma(3)})$.
(ii) $V_G(N_{\sigma(1)}, M) \subseteq V_G(N_{\sigma(2)}, M) \cup V_G(N_{\sigma(3)}, M)$.

2.3.6. *For any* $N \in \operatorname{mod}(kG)$, *we have* $V_G(N) \subseteq \bigcup V_G(S_i)$, *where the union runs over the set of composition factors* S_i *of* N.

2.3.7. *For* $N, N' \in \operatorname{mod}(kG)$, $V_G(N, N') \subseteq V_G(N) \cap V_G(N')$.

2.4. Let H be a subgroup of G. The inclusion map from H into G induces a restriction map $\widehat{\operatorname{res}}: H^\bullet(G, k) \to H^\bullet(H, k)$ on cohomology. This is turn induces a map of varieties $\operatorname{res}_{G,H}: V_H \to V_G$ with several nice properties:

2.4.1. *The map* $\operatorname{res}_{G,H}$ *is a finite map onto its image and maps closed sets to closed sets. Consequently, if* W *is a closed subset of* V_H *then* $\dim W = \dim \operatorname{res}_{G,H}(W)$.

2.4.2. *If* $K \leqslant L \leqslant G$ *then* $\operatorname{res}_{G,K} = \operatorname{res}_{G,L} \circ \operatorname{res}_{L,K}$.

The following proposition states how relative support varieties behave under induction for finite groups. This is a generalization of a result that can be found in [E2, Proposition 8.2.4].

2.4.3. Proposition. *Let* G *be a finite group and* $H \leqslant G$. *If* $M \in \operatorname{mod}(kG)$ *and* $N \in \operatorname{mod}(kH)$ *then*

(a) $V_G(N{\uparrow}_H^G, M) = \operatorname{res}_{G,H}(V_H(N, M{\downarrow}_H))$;
(b) $V_G(N{\uparrow}_H^G) = \operatorname{res}_{G,H}(V_H(N))$;
(c) $c_G(N{\uparrow}_H^G) = c_H(N)$.

Proof. (a) Let $\widehat{\operatorname{res}}: H^\bullet(G, k) \to H^\bullet(H, k)$ be as above. The isomorphism given by Frobenius reciprocity

$$\operatorname{Ext}^\bullet_G\big(N{\uparrow}_H^G, M\big) \cong \operatorname{Ext}^\bullet_H(N, M{\downarrow}_H)$$

implies that $\zeta \in J(N{\uparrow}_H^G, M)$ if and only if some power of $\widehat{\operatorname{res}}(\zeta)$ lies in $J(N, M{\downarrow}_H)$ (see [E2, Proposition 8.2.1]). Hence,

$$V_G\big(N{\uparrow}_H^G, M\big) = \operatorname{res}_{G,H}\big(V_H(N, M{\downarrow}_H)\big).$$

(b) Set $M = N{\uparrow}_H^G$. Then by part (a),

$$V_G\big(N{\uparrow}_H^G\big) = V_G\big(N{\uparrow}_H^G, N{\uparrow}_H^G\big) = \operatorname{res}_{G,H}\big(V_H\big(N, \big(N{\uparrow}_H^G\big){\downarrow}_H\big)\big).$$

Since N is a direct summand of $(N\uparrow_H^G)\downarrow_H$, it follows that

$$V_H(N) = V_H(N,N) \subseteq V_H\big(N, \big(N\uparrow_H^G\big)\downarrow_H\big).$$

But, by 2.3.7, $V_H(N,(N\uparrow_H^G)\downarrow_H) \subseteq V_H(N)$. Therefore,

$$\mathrm{res}_{G,H}\big(V_H\big(N,\big(N\uparrow_H^G\big)\downarrow_H\big)\big) = \mathrm{res}_{G,H}\big(V_H(N)\big).$$

(c) This follows immediately from part (b). □

The next proposition will be used throughout this paper. The proof relies on facts from Section 2.2 and Proposition 2.4.3.

2.4.4. Proposition. *Let G be a finite group and $H \leqslant G$ with $M \in \mathrm{mod}(kG)$ and $N \in \mathrm{mod}(kH)$. Suppose that*

(i) $M \mid N\uparrow_H^G$;
(ii) $N \mid M\downarrow_H$.

Then $V_G(M) = \mathrm{res}_{G,H}(V_H(N))$. Moreover, $c_G(M) = c_H(N)$.

Proof. From (i) and 2.3.3, we have $V_G(M) \subseteq V_G(N\uparrow_H^G)$. By Proposition 2.4.3(b), we have $V_G(N\uparrow_H^G) = \mathrm{res}_{G,H}(V_H(N))$, thus $V_G(M) \subseteq \mathrm{res}_{G,H}(V_H(N))$. On the other hand, from (ii) and 2.3.3, we have $V_H(N) \subseteq V_H(M\downarrow_H)$. It follows that

$$\mathrm{res}_{G,H}\big(V_H(N)\big) \subseteq \mathrm{res}_{G,H}\big(V_H(M\downarrow_H)\big) \subseteq V_G(M).$$

Hence, $V_G(M) = \mathrm{res}_{G,H}(V_H(N))$. The statement about the complexity follows immediately from 2.4.1 by taking dimensions. □

3. Permutation and Young modules

3.1. In this section we will use properties of complexity and support varieties plus the theory of Young vertices to give a simple formula for the complexities of the modules $\{Y^\lambda\}$ and $\{M^\lambda\}$. This is accomplished by first determining their support varieties as images of the map $\mathrm{res}_{\Sigma_d,\Sigma_\rho}$ applied to $V_{\Sigma\rho}(k)$ for a particular Young subgroup Σ_ρ. The support variety $V_G(k)$ of the trivial module is explicitly given by the Quillen Stratification Theorem, which we describe briefly now.

For E an elementary abelian p-group, $H^\bullet(E,k)$ is a polynomial ring [Ben1, Section 3.5]. The variety V_E is a vector space of dimension $r = \mathrm{rank}(E)$. According to [Ben2, Proposition 5.6.1],

$$V_G = V_G(k) = \bigcup_{E \leqslant G} \mathrm{res}_{G,E}\big(V_E(k)\big) \tag{3.1.1}$$

where the union is taken over all elementary abelian subgroups E of G. Thus, if $r_p(G)$ is the maximal rank of an elementary abelian p-subgroup of G then

$$c_G(k) = \dim\bigl(V_G(k)\bigr) = r_p(G). \tag{3.1.2}$$

The decomposition in (3.1.1) can be refined further in the following way. Define

$$V_E^+ = V_E \Big\backslash \bigcup_{E'<E} \operatorname{res}_{E,E'} V_{E'}$$

so V_E^+ is V_E with hyperplanes defined over $\mathbb{F}_p$ removed [Ben1, p. 173]. Let $V_{G,E}^+ = \operatorname{res}_{G,E}(V_E^+)$. Then the variety $V_G(k)$ is the disjoint union of locally closed subvarieties $V_{G,E}^+$, one for each conjugacy class of elementary abelian subgroups $E \leqslant G$.

3.2. We first recall the following well-known fact.

3.2.1. *The p-rank of Σ_d is $[d/p]$ where $[\,]$ is the greatest integer function.*

We can now determine the complexity and support varieties for the permutation modules M^λ.

3.2.2. Proposition. *Let $\lambda = (\lambda_1, \ldots, \lambda_s) \models d$ and M^λ be a permutation module for Σ_d. Then*

(a) $V_{\Sigma_d}(M^\lambda) = \operatorname{res}_{\Sigma_d,\Sigma_\lambda}(V_{\Sigma_\lambda}(k))$;
(b) $c_{\Sigma_d}(M^\lambda) = \sum_{i=1}^{s}[\lambda_i/p]$.

Proof. Part (a) follows immediately from Proposition 2.4.3(b) since $M^\lambda \cong k\uparrow_{\Sigma_\lambda}^{\Sigma_d}$. Part (b) follows from part (a) and 3.2.1 since $\operatorname{res}_{\Sigma_d,\Sigma_\lambda}$ preserves dimension and $\dim(V_{\Sigma_\lambda}(k))$ is determined by Eq. (3.1.2). $\square$

3.3. To describe the complexity and support varieties of Young modules we will need the theory of Young vertices due to Grabmeier [G]. We remark that a lower bound for the complexity of the Young modules was given in [EN] and used to determine the representation type of the blocks for the Hecke algebra of type A.

Notice that any $\lambda \vdash d$ has a unique p-adic expansion of the form

$$\lambda = \sum_{i=0}^{s} \lambda_{(i)} p^i \tag{3.3.1}$$

where $\lambda_{(i)} \in \Lambda_{\mathrm{res}}$. Define the partition

$$\rho(\lambda) = \bigl(\bigl(p^s\bigr)^{b_s}, \bigl(p^{s-1}\bigr)^{b_{s-1}}, \ldots, (1)^{b_0}\bigr) \tag{3.3.2}$$

where $\lambda_{(i)} \vdash b_i$. Then $\Sigma_{\rho(\lambda)}$ is called the *Young vertex* of Y^λ and Y^λ is a *trivial source module*, in analogy with the usual theory of vertices and sources. In particular $k \mid (Y^\lambda\downarrow_{\Sigma_{\rho(\lambda)}})$ and $\Sigma_{\rho(\lambda)}$ is the minimal Young subgroup such that Y^λ is a summand of $M^{\rho(\lambda)}$. Notice that:

3.3.1. *The p-rank of the Young vertex is $r_p(\Sigma_{\rho(\lambda)}) = \sum_{i=0}^{s} ib_i$.*

If λ is not p-restricted then one can successively strip horizontal rim p-hooks from λ to obtain a p-restricted partition. The following theorem demonstrates that the complexity of the Young module Y^λ can be obtained combinatorially as the number of such hooks removed.

3.3.2. Theorem. *Let $\lambda \vdash d$ with Y^λ the corresponding Young module for Σ_d and $\rho(\lambda)$ as in Eq.* (3.3.2). *Then*

(a) $V_{\Sigma_d}(Y^\lambda) = \operatorname{res}_{\Sigma_d,\Sigma_{\rho(\lambda)}}(V_{\Sigma_{\rho(\lambda)}}(k))$;
(b) $c_{\Sigma_d}(Y^\lambda) = \sum_{i=0}^{s} ib_i$.

Proof. Part (a) follows from Proposition 2.4.4 by setting $N = k$, $H = \Sigma_{\rho(\lambda)}$ and $G = \Sigma_d$. In order to prove (b) take the dimension on both sides of (a) and recall from 2.4.1 that $\operatorname{res}_{\Sigma_d,\Sigma_{\rho(\lambda)}}$ preserves dimension. But the $\dim_k(V_{\Sigma_{\rho(\lambda)}}(k)) = r_p(\Sigma_{\lambda(\rho)})$ is given by 3.3.1. $\square$

We remark that this theorem agrees with the well-known fact that Y^λ is projective exactly when λ is p-restricted. Furthermore, from Theorem 3.3.2(b) it is easy to see that for a block $\mathcal{B}$ of weight w, there are Young modules in $\mathcal{B}$ of every possible complexity $\{0, 1, \ldots, w\}$.

Recall that a module is called *periodic* if it admits a periodic projective resolution. Non-projective periodic modules are exactly those with complexity one [E2, 8.4.4]. Thus Theorem 3.3.2 immediately yields:

3.3.3. Corollary. *A non-projective Young module Y^λ is periodic if and only if λ is of the form $(\mu_1 + p, \mu_2, \ldots, \mu_s)$ where $(\mu_1, \mu_2, \ldots, \mu_s)$ is p-restricted.*

Proof. The complexity is one exactly when the p-adic expansion of λ has the form $\mu + (1)p$. $\square$

3.4. In the next section we will see the support varieties for modules in a block all sit inside $V_{\Sigma_d}(Y^\lambda)$ for Y^λ having a distinguished Young vertex. To do this we now observe that the set of Young vertices for Young modules in a block have nice ordering properties. First a lemma:

3.4.1. Lemma. *Let $d = \sum_{i=0}^{z} c_i p^i$ be the unique p-adic expansion of d, so $0 \leqslant c_i < p$. Suppose $d = \sum_{i=0}^{z} a_i p^i$ is another expansion, with $0 \leqslant a_i$. Then*

$$\Sigma_{(1^{a_0}, p^{a_1}, (p^2)^{a_2}, \ldots, (p^z)^{a_z})} \leqslant \Sigma_{(1^{c_0}, p^{c_1}, (p^2)^{c_2}, \ldots, (p^z)^{c_z})}.$$

Proof. It is clear that $\sum_{i=0}^{t} c_i p^i < p^{i+1}$ for any $0 \leqslant t \leqslant z$. This immediately implies that

$$\sum_{i=t}^{z} a_i p^i \leqslant \sum_{i=t}^{z} c_i p^i \quad \forall t\colon\ 0 \leqslant t \leqslant z. \tag{3.4.1}$$

From Eq. (3.4.1) it is clear that the Young subgroups embed as desired. □

Now suppose $\mathcal{B}_\mu$ is a block of $k\Sigma_d$ with weight w and p-core $\widetilde{\mu} \vdash d - pw$. Let $\sum_{i=1}^{z} c_i p^i$ be the p-adic expansion of pw. Define

$$\rho := \rho(w) = \left(\left(p^z\right)^{c_z}, \left(p^{z-1}\right)^{c_{z-1}}, \ldots, p^{c_1}, 1^{d-pw}\right) \vdash d. \tag{3.4.2}$$

Let $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots)$. Notice that Σ_ρ is the Young vertex for Y^μ where $\mu = (\widetilde{\mu}_1 + pw, \widetilde{\mu}_2, \ldots)$. For every other $\lambda \in \mathcal{B}_\mu$, we have $\mu \rhd \lambda$ and the Young vertex of Y^λ is of the form

$$\tau = \left(\left(p^z\right)^{a_z}, \left(p^{z-1}\right)^{a_{z-1}}, \ldots, p^{a_1}, 1^{a_0}\right)$$

where $a_0 \geqslant d - pw$ and $d = \sum_{i=0}^{z} a_i p^i$. Thus

$$pw = \bigl(a_0 - (d - pw)\bigr) + \sum_{i=1}^{z} a_i p^i$$

is an expansion of pw and Lemma 3.4.1 immediately implies:

3.4.2. Proposition. *Let $\lambda \in \mathcal{B}_\mu$ and let Y^λ have a Young vertex $\Sigma_{\rho(\lambda)}$. Let ρ be as in Eq.* (3.4.2). *Then*

$$\Sigma_{\rho(\lambda)} \geqslant \Sigma_\rho.$$

Thus the Young vertices for the Young modules in a block are all contained in a unique maximal vertex Σ_ρ, which is the vertex for the Young module $Y^{\widetilde{\mu}+(pw)}$. In the next section we use this to give a precise description of where the support varieties for modules in the block are located.

4. Support varieties and branching

4.1. We begin by showing that the relative support varieties for the direct sum of simple, Specht and Young modules are indeed equal.

4.1.1. Theorem. *Let $M \in \operatorname{mod}(k\Sigma_d)$. The following varieties are equal*:

(a) $V_{\Sigma_d}(M)$;
(b) $V_{\Sigma_d}(\bigoplus_{\lambda\in\Lambda_{\mathrm{reg}}} D^\lambda, M)$;
(c) $V_{\Sigma_d}(\bigoplus_{\lambda\vdash d} S^\lambda, M)$;
(d) $V_{\Sigma_d}(\bigoplus_{\lambda\vdash d} Y^\lambda, M)$.

Proof. (a) = (b). From (2.3.2), we have $V_{\Sigma_d}(M) = V_{\Sigma_d}(\bigoplus_{\lambda\in\Lambda_{\mathrm{reg}}} D^\lambda, M)$.

(c) ⊆ (a), (d) ⊆ (a). These inclusions follow from (2.3.7).

(b) ⊆ (c). This will be proved by using induction on the dominance order of partitions. Set $W = V_{\Sigma_d}(\bigoplus_{\lambda\vdash d} S^\lambda, M)$. Let λ be maximal with respect to $\trianglelefteq$. Then $S^\lambda = D^\lambda$ and $V_{\Sigma_d}(D^\lambda, M) \subseteq W$. Now suppose that for every $\mu \rhd \tau$, we know $V_{\Sigma_d}(D^\mu, M) \subseteq W$. We need to show that $V_{\Sigma_d}(D^\tau, M) \subseteq W$. By Eq. (1.2.1) there exists a short exact sequence of the form

$$0 \to N \to S^\tau \to D^\tau \to 0 \tag{4.1.1}$$

with N having composition factors of the form D^μ with $\mu \rhd \tau$. Therefore, by 2.3.5(ii)

$$V_{\Sigma_d}\big(D^\tau, M\big) \subseteq V_{\Sigma_d}\big(S^\tau, M\big) \cup V_{\Sigma_d}(N, M) \subseteq W.$$

Thus, $V_{\Sigma_d}(\bigoplus_{\tau\in\Lambda_{\mathrm{reg}}} D^\tau, M) \subseteq W$.

(c) ⊆ (d). This statement will be proved in a similar fashion as above. Set $X = V_{\Sigma_d}(\bigoplus_{\lambda\vdash d} Y^\lambda, M)$. Again let λ be maximal with respect to $\trianglelefteq$ so $Y^\lambda = S^\lambda$ and $V_{\Sigma_d}(Y^\lambda, M) \subseteq X$. Suppose that for any $\mu \rhd \tau$, $V_{\Sigma_d}(S^\mu, M) \subseteq X$. It will suffice to show that $V_{\Sigma_d}(S^\tau, N) \subseteq X$. By Eq. (1.2.2) there is a short exact sequence of the form

$$0 \to S^\tau \to Y^\tau \to Z \to 0 \tag{4.1.2}$$

with Z having a Specht filtration with factors of the form S^μ with $\mu \rhd \tau$. Consequently, by 2.3.5(ii)

$$V_{\Sigma_d}(S^\tau, M) \subseteq V_{\Sigma_d}(Y^\lambda, M) \cup V_{\Sigma_d}(Z, M) \subseteq X. \qquad \square$$

Let $\operatorname{Stmod}(kG)$ be the stable module category [Ben1, Section 2.1]. The argument above shows that $\bigoplus_{\lambda\vdash d} S^\lambda$ and $\bigoplus_{\lambda\vdash d} Y^\lambda$ generate $\operatorname{Stmod}(k\Sigma_d)$.

4.2. The preceding result along with our computation for the support variety of Young modules can be used to provide an explicit description for the location of the support varieties for modules in a block of $k\Sigma_d$.

4.2.1. Corollary. *Let $\mathcal{B}_\mu$ be a block of $k\Sigma_d$ of weight w and let M be a finite-dimensional module in $\mathcal{B}_\mu$. Let ρ be as in Eq.* (3.4.2). *Then*

(a) $V_{\Sigma_d}(\bigoplus_{\lambda\in\mathcal{B}_\mu} D^\lambda) = V_{\Sigma_d}(\bigoplus_{\lambda\in\mathcal{B}_\mu} S^\lambda) = V_{\Sigma_d}(\bigoplus_{\lambda\in\mathcal{B}_\mu} Y^\lambda)$;
(b) $V_{\Sigma_d}(\bigoplus_{\lambda\in\mathcal{B}_\mu} D^\lambda) = \operatorname{res}_{\Sigma_d,\Sigma_\rho}(V_{\Sigma_\rho}(k))$;
(c) $V_{\Sigma_d}(M) \subseteq \operatorname{res}_{\Sigma_d,\Sigma_\rho}(V_{\Sigma_\rho}(k))$;
(d) $c_{\Sigma_d}(M) \leqslant w$.

Proof. (a) Let N be equal to either $\bigoplus_{\lambda\in\mathcal{B}_\mu} S^\lambda$ or $\bigoplus_{\lambda\in\mathcal{B}_\mu} Y^\lambda$. By 2.3.6, $V_{\Sigma_d}(N) \subseteq V_{\Sigma_d}(\bigoplus_{\lambda\in\mathcal{B}_\mu} D^\lambda)$. On the other hand, by Theorem 4.1.1 and 2.3.7,

$$\begin{aligned} V_{\Sigma_d}\Bigl(\bigoplus_{\lambda\in\mathcal{B}_\mu} D^\lambda\Bigr) &\subseteq V_{\Sigma_d}\Bigl(\bigoplus_{\lambda\vdash d} S^\lambda, \bigoplus_{\lambda\in\mathcal{B}_\mu} D^\lambda\Bigr) = V_{\Sigma_d}\Bigl(N, \bigoplus_{\lambda\in\mathcal{B}_\mu} D^\lambda\Bigr) \\ &\subseteq V_{\Sigma_d}(N). \end{aligned}$$

(b) From part (a) we have $V_{\Sigma_d}(\bigoplus_{\lambda\in\mathcal{B}_\mu} D^\lambda) = V_{\Sigma_d}(\bigoplus_{\lambda\in\mathcal{B}_\mu} Y^\lambda)$. Furthermore, by Proposition 3.4.2,

$$V_{\Sigma_d}\Bigl(\bigoplus_{\lambda\in\mathcal{B}_\mu} Y^\lambda\Bigr) = V_{\Sigma_d}\bigl(Y^{\widetilde{\mu}+(pw)}\bigr) = \operatorname{res}_{\Sigma_d,\Sigma_\rho}\bigl(V_{\Sigma_\rho}(k)\bigr).$$

(c) This follows from (b) because for any M in $\mathcal{B}_\mu$,

$$V_{\Sigma_d}(M) \subseteq V_{\Sigma_d}\Bigl(\bigoplus_{\lambda\in\mathcal{B}_\mu} D^\lambda\Bigr).$$

(d) The dimension of $V_{\Sigma_\rho}(k) = w$, so by part (c) $c_{\Sigma_d}(M) \leqslant w$ for any M in $\mathcal{B}_\mu$. □

4.3. In the representation theory of the symmetric group, one of the fundamental questions is how does a $k\Sigma_d$-module M decompose on restriction to Σ_λ. Answers to questions of this type are often referred to as "branching rules." Kleshchev has proved important results on branching of the simple $k\Sigma_d$ modules on restriction to $k\Sigma_{d-1}$ [K1]. The next theorem shows that the computation of support varieties for $k\Sigma_d$-modules can be reduced to looking at how the modules branch over Young subgroups Σ_λ.

4.3.1. Theorem. *Let $M \in \operatorname{mod}(k\Sigma_d)$. Then*

(a) $V_{\Sigma_d}(M) = \bigcup_{\lambda\models d} \operatorname{res}_{\Sigma_d,\Sigma_\lambda}(V_{\Sigma_\lambda}(k, M))$;
(b) $c_{\Sigma_d}(M) = \max_{\lambda\models d}\{r(\operatorname{Ext}^\bullet_{\Sigma_\lambda}(k, M))\}$.

Proof. Part (b) follows immediately from part (a). From Proposition 2.4.3(a) and 2.3.7, we have for $\lambda \models d$:

$$\operatorname{res}_{\Sigma_d,\Sigma_\lambda}\bigl(V_{\Sigma_\lambda}(k, M)\bigr) = V_{\Sigma_d}\bigl(M^\lambda, M\bigr) \subseteq V_{\Sigma_d}(M).$$

Therefore, $\bigcup_{\lambda\models d}\operatorname{res}_{\Sigma_d,\Sigma_\lambda}(V_{\Sigma_\lambda}(k,M))\subseteq V_{\Sigma_d}(M)$. On the other hand, by 2.3.2 and Theorem 4.1.1,

$$V_{\Sigma_d}(M)=V_{\Sigma_d}\Bigl(\bigoplus_{\lambda\in\Lambda_{\mathrm{reg}}}D^\lambda,M\Bigr)=V_{\Sigma_d}\Bigl(\bigoplus_{\lambda\vdash d}Y^\lambda,M\Bigr).$$

Now, by 2.3.3 and Proposition 2.4.3(i),

$$\begin{aligned}V_{\Sigma_d}\Bigl(\bigoplus_{\lambda\vdash d}Y^\lambda,M\Bigr)&\subseteq V_{\Sigma_d}\Bigl(\bigoplus_{\lambda\models d}M^\lambda,M\Bigr)=\bigcup_{\lambda\models d}V_{\Sigma_d}\bigl(M^\lambda,M\bigr)\\&\subseteq\bigcup_{\lambda\models d}\operatorname{res}_{\Sigma_d,\Sigma_\lambda}\bigl(V_{\Sigma_\lambda}(k,M)\bigr).\qquad\square\end{aligned}$$

4.4. We should remark that one can give an alternate proof of Theorem 4.3.1(b), by using the Schur functor $\mathcal{F}$. This proof will does not rely on the ordering properties of the Specht and Young modules given in Eqs. (1.2.1) and (1.2.2). The functor $\mathcal{F}$ admits a right adjoint functor, $\mathcal{G}$, defined by

$$\mathcal{G}(N)=\operatorname{Hom}_{k\Sigma_d}\bigl(e\bigl(S(n,d)\bigr),N\bigr)=\operatorname{Hom}_{k\Sigma_d}\bigl(V^{\otimes d},N\bigr).$$

The functor $\mathcal{G}$ is a left inverse to $\mathcal{F}$. By using these two functors one can construct a first-quadrant Grothendieck spectral sequence [DEN, 2.2]:

$$E_2^{i,j}=\operatorname{Ext}^i_{S(n,d)}\bigl(M,\operatorname{Ext}^j_{k\Sigma_d}\bigl(V^{\otimes d},N\bigr)\bigr)\Rightarrow\operatorname{Ext}^{i+j}_{k\Sigma_d}\bigl(\mathcal{F}(M),N\bigr),$$

where $M\in\operatorname{Mod}(S(n,d))$ and $N\in\operatorname{Mod}(k\Sigma_d)$. When $M=\mathcal{G}(N)$ the spectral sequence becomes

$$E_2^{i,j}=\operatorname{Ext}^i_{S(n,d)}\bigl(\mathcal{G}(N),\operatorname{Ext}^j_{k\Sigma_d}\bigl(V^{\otimes d},N\bigr)\bigr)\Rightarrow\operatorname{Ext}^{i+j}_{k\Sigma_d}(N,N).$$

Let

$$D_n=\sum_{i+j=n}\operatorname{Ext}^i_{S(n,d)}\bigl(\mathcal{G}(N),\operatorname{Ext}^j_{k\Sigma_d}\bigl(V^{\otimes d},N\bigr)\bigr).$$

Since E_∞ is a subquotient of E_2,

$$c_{\Sigma_d}(N)=r\bigl(\dim_k\operatorname{Ext}^\bullet_{k\Sigma_d}(N,N)\bigr)\leqslant r(\dim_k D_\bullet).\tag{4.4.1}$$

There exists a finite projective resolution $P_\bullet\to\mathcal{G}(M)$ because the Schur algebra $S(n,d)$ has finite global dimension. Then $r(\dim_k P_\bullet)=0$. Consequently,

$$\begin{aligned}\dim_k D_n&\leqslant\sum_{i+j=n}\dim_k\operatorname{Hom}_{S(n,d)}\bigl(P_i,\operatorname{Ext}^j_{\Sigma_d}\bigl(V^{\otimes d},N\bigr)\bigr)\\&\leqslant\sum_{i+j=n}(\dim_k P_i)\otimes_k\bigl(\dim_k\operatorname{Ext}^j_{\Sigma_d}\bigl(V^{\otimes d},N\bigr)\bigr).\end{aligned}$$

It follows that

$$\begin{aligned} r(\dim_k D_\bullet) &\leqslant r(\dim_k P_\bullet) + r\big(\dim_k \operatorname{Ext}^\bullet_{\Sigma_d}\big(V^{\otimes d}, N\big)\big) \\ &= r\big(\dim_k \operatorname{Ext}^\bullet_{\Sigma_d}\big(V^{\otimes d}, N\big)\big) \\ &\leqslant c_{\Sigma_d}(N). \end{aligned} \tag{4.4.2}$$

The statement of Theorem 4.3.1(b) follows from Eqs. (4.4.1) and (4.4.2) because as a $k\Sigma_d$-module, $V^{\otimes d} = \bigoplus_{\lambda \vDash d} M^\lambda$.

5. Completely splittable modules

5.1. We now determine the complexity and support varieties of the completely splittable modules, defined below:

5.1.1. Definition. A simple $k\Sigma_d$-module D^λ is called *completely splittable* if and only if the restriction $D^\lambda\downarrow_{\Sigma_\mu}$ to any Young subgroup $\Sigma_\mu \leqslant \Sigma_d$ is semisimple. When D^λ is completely splittable we will also say that λ is completely splittable.

It was shown in [H] that almost every completely splittable $k\Sigma_d$-module D^λ occurs as a direct summand of $D^\mu\uparrow_{\Sigma_{d-1}}^{\Sigma_d}$ for some completely splittable $k\Sigma_{d-1}$-module D^μ. In this case we will see that

$$\operatorname{res}_{\Sigma_d,\Sigma_{d-1}}\big(V_{\Sigma_{d-1}}\big(D^\mu\big)\big) = V_{\Sigma_d}\big(D^\lambda\big). \tag{5.1.1}$$

This will reduce the problem to determining the support varieties of the minimal modules, defined in [H]. We will show the minimal modules all have the maximum possible complexity by proving they have dimension not divisible by p.

We begin by recalling the main results of [K2] on completely splittable modules. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ define:

$$h(\lambda) := s \quad \text{and} \quad \chi(\lambda) := \lambda_1 - \lambda_s + h(\lambda).$$

Then

5.1.2. [K2] *D^λ is completely splittable if and only if $\chi(\lambda) \leqslant p$.*

It is clear from the definition that restricting a completely splittable module to Σ_{d-1} will give a direct sum of completely splittable Σ_{d-1}-modules. Kleshchev determined this decomposition. For a removable node A of $\lambda \vdash d$, let $\lambda_A \vdash d-1$ denote λ with the node A removed. Then

5.1.3. [K2] *Let D^λ be completely splittable. Then*

$$D^\lambda\downarrow_{\Sigma_{d-1}} = \bigoplus D^{\lambda_A}$$

where the sum is over all removable nodes A with $\chi(\lambda_A) \leqslant p$.

5.2. We next recall the definition of minimal modules.

5.2.1. Definition. A completely splittable $k\Sigma_d$-module D^λ is *minimal* if there does not exist $\mu \vdash d-1$, with D^μ a completely splittable $k\Sigma_{d-1}$-module, such that $D^\mu\uparrow_{\Sigma_{d-1}}^{\Sigma_d} \cong D^\lambda \oplus M$ where M has no summand in the same block as D^λ. If D^λ is minimal, we also say λ is minimal.

It turns out that if D^λ is completely splittable and D^λ is not minimal, then we can obtain μ by removing a node from λ. That is, there is some removable node A of λ such that D^{λ_A} is completely splittable and $D^\lambda \oplus M \cong D^{\lambda_A}\uparrow_{\Sigma_{d-1}}^{\Sigma_d}$ with M in a different block then D^λ. Also $D^{\lambda_A} \mid (D^\lambda\downarrow_{\Sigma_{d-1}})$. In this case Proposition 2.4.4 proves:

5.2.2. Lemma. *Let λ be not minimal and choose λ_A as above. Then*

(a) $V_{\Sigma_d}(D^\lambda) = \operatorname{res}_{\Sigma_d,\Sigma_{d-1}}(V_{\Sigma_{d-1}}(D^{\lambda_A}))$;
(b) $c_{\Sigma_d}(D^\lambda) = c_{\Sigma_{d-1}}(D^{\lambda_A})$.

5.3. Lemma 5.2.2 indicates that the problem of computing support varieties for completely splittable modules reduces to calculating the support variety for minimal completely splittable modules. To make this precise we need a few more details from [H].

Recall that for a node $A = (i, j)$ in the diagram of a partition λ, the *p-residue* of the node, denoted $\operatorname{res} A$, is defined to be $j - i \bmod p$. Then the alternate version of the Nakayama rule states that $\lambda \in \mathcal{B}_\mu$ if and only if λ and μ have the same number of nodes of each p-residue $0, 1, \ldots, p-1$.

Often a completely splittable module D^λ can be obtained by induction from more than one completely splittable $k\Sigma_{d-1}$-module. The next lemma determines when this happens:

5.3.1. Lemma [H, Lemma 4.2]. *Let D^λ be a completely splittable $k\Sigma_d$-module, and A a removable node of λ. Then*

$$D^{\lambda_A}\uparrow_{\Sigma_{d-1}}^{\Sigma_d} \cong D^\lambda \oplus M$$

where M is in a different block than D^λ, unless

(i) *$\chi(\lambda) = p$ and A is the lowest removable node of λ, or*
(ii) $\operatorname{res} A$ *is equal to the residue of the lowest addable node of λ.*

Since we plan to reduce the calculation of $V_{\Sigma_d}(D^\lambda)$ to the case where λ is minimal, the next definition is natural:

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \\ 5 & & & & & & & & \end{array} \qquad \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ 6 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 1 & 2 & \\ 4 & 5 & 6 & 0 & & \\ 3 & & & & & \end{array}$$

$$\lambda = (9,8) \qquad \lambda = (6,6,5,4)$$
$$\hat{\lambda} = (7,7) \qquad \hat{\lambda} = (5,5,2,2)$$

Fig. 1. Two examples of minimal 7-cores.

5.3.2. Definition. Given a completely splittable $k\Sigma_d$-module D^λ, the *minimal core* of λ, denoted $\hat{\lambda}$, is obtained by successively removing nodes from λ which do not satisfy Lemma 5.3.1, (i) or (ii), until no such nodes can be removed.

It is shown in [H] that $\hat{\lambda}$ is well-defined, minimal, and can be easily obtained from λ. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ be completely splittable. To obtain $\hat{\lambda}$ from λ, simply remove removable nodes, never removing any of residue equal to the residue of the bottom addable node, and never allowing χ to be $> p$. This process will terminate at $\hat{\lambda}$.

Two examples are given in Fig. 1, with the minimal core drawn inside the partition. The residues of all nodes in λ and the bottom addable node of λ are labeled.

5.4. We now use the minimal cores and Lemma 5.2.2 to formalize the reduction of determining the complexity and support variety of a completely splittable module to the minimal case.

5.4.1. Theorem. *Let D^λ be a completely splittable $k\Sigma_d$-module and let $\hat{\lambda}$ be the minimal core of λ with $\hat{\lambda} \vdash t$. Then*

(a) $V_{\Sigma_d}(D^\lambda) = \mathrm{res}_{\Sigma_d,\Sigma_t}(V_{\Sigma_t}(D^{\hat{\lambda}}))$;
(b) $c_{\Sigma_d}(D^\lambda) = c_{\Sigma_t}(D^{\hat{\lambda}})$.

Proof. This follows immediately from Lemmas 5.3.1 and 5.2.2 since we obtain $\hat{\lambda}$ by successively removing nodes from λ that do not satisfy Lemma 5.3.1. We also use 2.4.2. $\square$

5.5. We must still determine the complexity and support variety of the minimal modules. We will need a few basic facts about defect groups for blocks and p-divisibility of the modules in the blocks. First:

5.5.1. [JK, 6.2.45]. *Let $\mathcal{B}$ be a p-block of the $k\Sigma_d$ with weight w. Then a defect group of $\mathcal{B}$ is isomorphic to the Sylow p-subgroup of Σ_{wp}.*

If the defect group of a block has order p^d, then d is called the defect of the block. The defect can be determined from the dimensions of the simple modules in the block, namely:

5.5.2. [I, 15.42]. *For any group G where $|G| = p^a t$ with $(p,t) = 1$ and a block $\mathcal{B}$ with defect d, then p^{a-d} is the largest power of p which divides the dimensions of all the simple modules in the block, and hence the dimensions of all the modules in the block.*

The aforementioned facts give the following proposition.

5.5.3. Proposition. *Let M be an indecomposable $k\Sigma_{mp}$-module that is not in the principal block. Then $p \mid \dim M$.*

Proof. Only the principal block of $k\Sigma_{mp}$ has weight m. Thus, by Lemma 5.5.1, the other blocks have defect group strictly smaller than the p-Sylow subgroup of Σ_{mp}. As a result, $a - d > 0$ from Lemma 5.5.2 and p divides the dimension of M. □

We need a little more information about minimal modules.

5.5.4. Theorem [H, Theorem 3.3]. *Suppose D^λ is a completely splittable $k\Sigma_d$ module. Then $d = mp$ for some m and D^λ is in the principal block of $k\Sigma_{mp}$.*

Furthermore:

5.5.5. Lemma [H, Lemma 3.5]. *Let $d = mp$. There are $p-1$ minimal partitions of d. They have $1, 2, \dots, p-1$ parts. The top removable node of the one with i parts has residue $p-i$.*

The minimal partitions for $p = 5$ and $n = 5, 10, 15$ are illustrated in Fig. 2, with the 5-residues labeled. Notice there is an obvious bijection between minimal partitions of $(m-1)p$ and of mp given by adding a rim p-hook with head in the first row. This bijection preserves the number of parts and the residue of the top removable node.

5.6. We now investigate the branching behavior of the minimal modules. Applying 5.1.3 repeatedly we obtain:

5.6.1. Theorem. *For $1 \leqslant i \leqslant p-1$ and $m \geqslant 2$, let λ_i be the minimal partition of mp with i parts and let $\widetilde{\lambda}_i$ be the minimal partition of $(m-1)p$ with i parts. Then*:

$$D^{\lambda_i}\downarrow_{\Sigma_{(m-1)p}} \cong \binom{p-2}{i-1} D^{\widetilde{\lambda}_i} \oplus U$$

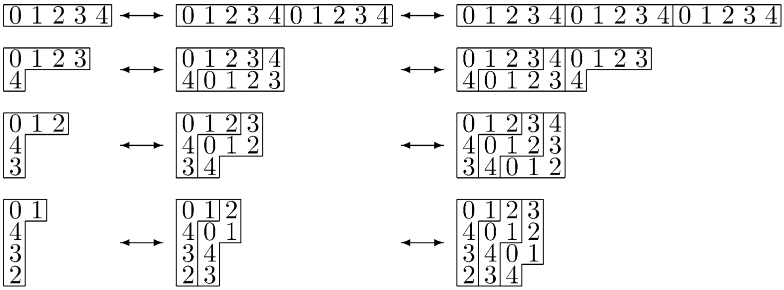

Fig. 2. Minimal partitions for $p = 5$ and $n = 5, 10, 15$.

where U is not in the principal block of $k\Sigma_{(m-1)p}$.

Proof. We know $D^{\lambda_i}\downarrow_{\Sigma_{(m-1)p}}$ is a direct sum of completely splittable modules. To prove Theorem 5.6.1 we must show that the only minimal $k\Sigma_{(m-1)p}$-module which occurs is $D^{\widetilde{\lambda}_i}$, and that it occurs with the correct multiplicity.

We need to prove the component of $D^{\lambda_i}\downarrow_{\Sigma_{(m-1)p}}$ which lies in the principal block is correct. By 5.1.3 and the Nakayama rule, we must count the number ways to remove p nodes successively from λ_i so that at each step we have a partition with $\chi \leqslant p$, and so that we remove exactly one node of each residue.

It is clear from Fig. 2 that the p nodes which are successively removed must make up the rim hook with head in the first row. To remove any other set of p nodes with distinct residues would force χ to be $> p$ at some point. Thus the only minimal module that occurs in the decomposition of $D^{\lambda_i}\downarrow_{\Sigma_{(m-1)p}}$ is $D^{\widetilde{\lambda}_i}$, and we must prove its multiplicity is $\binom{p-2}{i-1}$. By 5.1.3, this multiplicity is the number of ways to remove the nodes in the rim hook with head in the first row while maintaining $\chi \leqslant p$. Equivalently this is the number of paths from $\widetilde{\lambda}_i$ to λ_i in the graph Y defined in [K2, Definition 2.2].

The first node removed must be the top removable node since the other removable node, if there is one, leaves $\chi = p + 1$ when removed. It is also easy to determine that the node of residue 0 must be the last one removed if χ is to stay $\leqslant p$. There will always be $\binom{p-2}{i-1}$ ways to legally remove the remaining $p - 2$ nodes. An example should make it clear why this is the case:

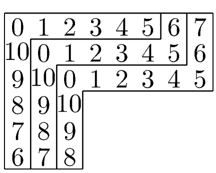

Fig. 3. λ_5 for $mp = 33$.

Figure 3 illustrates the case $p = 11$, $m = 3$ and $i = 6$. We must remove the rim hook with head at $(1, 8)$ while maintaining $\chi \leqslant 11$. The node $(3, 8)$ of residue 5 must be removed first. The node $(3, 3)$ of residue 0 must be removed last. The nodes of residues 4, 3, 2, 1 must be removed in that order. Also the node 7 must be removed before 8 in order to maintain $\chi \leqslant 11$. So the nodes 6, 7, 8, 9, 10 must be removed in that order.

We have seen the sequence of residues of removed nodes must begin with 5 and end with 0. Also it must have subsequences 4, 3, 2, 1 and 6, 7, 8, 9, 10, but there are no other restrictions. That is, any sequence of $\{0, 1, \ldots, 10\}$ that starts with 5, ends with 0 and has subsequences 4, 3, 2, 1 and 6, 7, 8, 9, 10 will give a legal partition with $\chi \leqslant 11$ at each step. For example we could remove the nodes in the order 5, 6, 4, 7, 8, 3, 2, 9, 1, 10, 0.

The total number of such sequences is clearly $\binom{9}{5}$. We have a sequence that is nine terms in length (not counting the first and last which are determined). Once we place the five numbers 6, 7, 8, 9, 10 the positions of 4, 3, 2, 1 are forced.

In the general situation instead of five numbers 6, 7, 8, 9, 10, we have $i - 1$ numbers corresponding to the last node in each row except that row containing the top removable node of λ. And instead of 9 positions we will have $p - 2$ positions, so there are $\binom{p-2}{i-1}$ possible paths. This completes the proof. $\square$

5.7. We can now show that p does not divide the dimension of minimal modules:

5.7.1. Theorem. *Let D^λ be a minimal $k\Sigma_{mp}$-module. Then $p \nmid \dim_k D^\lambda$.*

Proof. We prove this by induction on m. For $m = 1$ the minimal modules correspond to partitions $(p-i, 1^i)$. The principal block of $k\Sigma_p$ is well understood and $\dim_k D^{(p-i,1^i)} = \binom{p-2}{i}$. This implies that $p \nmid \dim_k D^{(p-i,1^i)}$.

Now p does not divide $\binom{p-2}{i-1}$. But Corollary 5.5.3 implies that p divides the dimension of U in Theorem 5.6.1. Hence, the theorem follows from Theorem 5.6.1 by induction. $\square$

5.7.2. Corollary. *The complexity of a completely splittable $k\Sigma_d$-module D^λ is the weight of the minimal core $\hat{\lambda}$.*

Proof. We know $c_{\Sigma_d}(D^\lambda) = c_{\Sigma_{mp}}(D^{\hat{\lambda}})$ where $\hat{\lambda} \vdash mp$. Since $D^{\hat{\lambda}}$ is in the principal block of $k\Sigma_{mp}$, $\hat{\lambda}$ has weight m. But $p \nmid \dim_k D^{\hat{\lambda}}$ so its complexity is equal to the p-rank of Σ_{mp}, which is m. $\square$

5.7.3. Corollary. *Let D^λ be completely splittable and $\hat{\lambda} \vdash mp$. Then*

$$V_{\Sigma_d}\bigl(D^\lambda\bigr) = \operatorname{res}_{\Sigma_d,\Sigma_{mp}}\bigl(V_{\Sigma_{mp}}\bigl(D^{\hat{\lambda}}\bigr)\bigr) = \operatorname{res}_{\Sigma_d,\Sigma_{mp}}\bigl(V_{\Sigma_{mp}}(k)\bigr).$$

Proof. The first equality is just Theorem 5.4.1(i) and the second follows because $p \nmid \dim D^{\hat{\lambda}}$. □

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