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The Ext^1-quiver for completely splittable representations of the symmetric group

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Abstract. Kleshchev has recently [7] classified those modules for the symmetric group which have semisimple restriction to any Young subgroup. We determine $\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu)$ where D^λ and D^μ are $K\Sigma_d$-modules of this type, called completely splittable. As a corollary of this and recent work of Kleshchev and Nakano, we can determine $\mathrm{Ext}^1_{\mathrm{GL}_n(K)}((L(\lambda), L(\mu))$ for certain simple $\mathrm{GL}_n(K)$-modules $L(\lambda)$ and $L(\mu)$.

1 Introduction

Let Σ_d denote the symmetric group on d letters. The complex irreducible Σ_d-modules correspond bijectively with partitions λ of d, and we denote by S^λ the irreducible module corresponding to λ. We work over an algebraically closed field K of positive characteristic $p > 2$. The simple $K\Sigma_d$-modules are indexed by p-regular partitions, and we denote the corresponding simple module by D^λ. These modules can also be indexed by column p-regular partitions, and for λ column p-regular we denote the corresponding simple module by D_λ. For a comprehensive treatment of the theory, see [4].

For any composition $\mu = (\mu_1, \mu_2, \dots \mu_k)$ of d there is a standard Young subgroup defined by

$$\Sigma_\mu = \Sigma_{\mu_1} \times \Sigma_{\mu_2} \times \cdots \times \Sigma_{\mu_k} < \Sigma_d.$$

We will consider the following class of modules:

Definition 1.1. An irreducible $K\Sigma_d$-module D^λ is called *completely splittable* if and only if the restriction $D^\lambda{\downarrow_{\Sigma_\mu}}$ to any Young subgroup $\Sigma_\mu < \Sigma_d$ is semisimple. We will also say that λ is completely splittable.

In [8], Kleshchev and Nakano obtained several results about the cohomology of completely splittable modules, suggesting that obtaining the Ext^1-quiver for these

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modules should be possible. In this paper we apply Kleshchev's branching theorems to prove the following result:

Theorem 1.2. *Let D^λ and D^μ be completely splittable and in the same p-block of $K\Sigma_d$. Then*

$$\operatorname{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu) = \begin{cases} K & \text{if } h_{11}(\lambda) \geqslant p, h_{11}(\mu) \geqslant p \\ & h_{21}(\lambda) < p,\ h_{21}(\mu) < p \\ & \text{and } |h(\lambda) - h(\mu)| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $h(\lambda)$ is the height of λ and

$$h_{ij}(\lambda_1, \lambda_2, \ldots, \lambda_k) = \lambda_i + \lambda'_j + 1 - i - j$$

is the (i, j) hook length.

This theorem together with the work of [8] gives a corresponding result for $\mathrm{GL}_n(K)$ when $n \geqslant d$. Let $L(\lambda)$ denote the simple, polynomial $\mathrm{GL}_n(K)$-module with highest weight λ. Let m denote the Mullineaux map on p-regular partitions defined by

$$D^\lambda \otimes \operatorname{sgn} \cong D^{m(\lambda)}.$$

We have the following result:

Corollary 1.3. *Let D^λ and D^μ be completely splittable. Then*

$$\operatorname{Ext}^1(L(m(\lambda)'), L(m(\mu)')) = \begin{cases} K & \text{if } h_{11}(\lambda) \geqslant p, h_{11}(\mu) \geqslant p \\ & h_{21}(\lambda) < p,\ h_{21}(\mu) < p \\ & \text{and } |h(\lambda) - h(\mu)| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The Ext^1 in Corollary 1.3 is taken in the category $M(n, d)$ of polynomial $\mathrm{GL}_n(K)$-modules of homogeneous degree d.

2 Notation and preliminaries

We follow the notation of Kleshchev's excellent survey paper [6]. We write $\lambda \vdash d$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ a partition of d. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. We do not distinguish between λ and its Young diagram

$$\lambda = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid j \leqslant \lambda_i\}.$$

The *conjugate* partition of λ, denoted by λ', is obtained by interchanging the rows and columns in the Young diagram for λ.

For $\lambda \vdash d$ let

$$\sigma_s(\lambda) = \sum_{i=1}^{s} \lambda_s.$$

Then we say that λ *dominates* μ, and we write $\lambda \trianglerighteq \mu$, if $\sigma_s(\lambda) \geqslant \sigma_s(\mu)$ for every $s \in \mathbb{N}$. If $\lambda \neq \mu$ we write $\lambda \triangleright \mu$. This is the usual dominance order on partitions.

The *removable* (resp. *addable*) nodes are those which can be removed (resp. added) to the diagram λ to produce a diagram for a partition of $d-1$ (resp. $d+1$). For a removable node A, we write λ_A for λ with the node A removed, so that $\lambda_A \vdash d-1$. Similarly for $\lambda^B \vdash d+1$, where B is an addable node. The *p-residue* of a node $A = (i, j)$ is defined as $\operatorname{res} A = j - i \pmod p$. The *residue content* of λ is

$$\operatorname{cont}(\lambda) = (c_0, c_1, \dots, c_{p-1})$$

where c_α is the number of nodes of λ with p-residue α.

The Nakayama conjecture (now a theorem) says that D^λ and D^μ belong to the same p-block if and only if $\operatorname{cont}(\lambda) = \operatorname{cont}(\mu)$, and similarly for S^λ and S^μ. We denote this by $\lambda \sim \mu$. We assume familiarity with this and the equivalent characterization that $\lambda \sim \mu$ if and only if λ and μ have the same p-core. See [4] for details.

The Nakayama conjecture allows us to define *Robinson's α-induction* and *α-restriction functors*. For a $K\Sigma_d$-module M in a fixed block corresponding to the residue content $(c_0, c_1, \dots, c_{p-1})$, and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$, we define $\operatorname{Ind}^\alpha M$ as the direct summand of $\operatorname{Ind} M$ in the block of $K\Sigma_{d+1}$ corresponding to residue content $(c_0, \dots, c_\alpha + 1, c_{\alpha+1}, \dots, c_{p-1})$. We define $\operatorname{Res}_\alpha M$ similarly. For full details, see [4, 6.3.16].

We need Kleshchev's notion of normal and good nodes. A removable node A of λ is *normal* if for every addable node B above A with $\operatorname{res} B = \operatorname{res} A$, there exists a removable node $C(B)$ strictly between A and B with $\operatorname{res} C(B) = \operatorname{res} A$, and $C(B) \neq C(B')$ for $B \neq B'$. A removable node is called *good* if it is the lowest among the normal nodes of a fixed residue. An addable node B is called *good addable* if it is good as a removable node of λ^B.

We now recall the main results of [7] on completely splittable modules. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ define

$$h(\lambda) := k \quad \text{and} \quad \chi(\lambda) := \lambda_1 - \lambda_k + h(\lambda).$$

Theorem 2.1 ([7]). *D^λ is completely splittable if and only if $\chi(\lambda) \leqslant p$.*

We remark that $\chi(\lambda)$ is the length of the hook with foot at (k, λ_k). We remark further that Theorem 2.1 holds as well for $p = 2$, but since there are no non-trivial completely splittable modules in this case, we can assume that $p > 2$ for this paper.

It is clear from the definition that restricting a completely splittable module to Σ_{d-1} will give a direct sum of completely splittable Σ_{d-1}-modules. Kleshchev determined this decomposition:

Theorem 2.2 ([7]). *Let D^λ be completely splittable. Then*

$$D^\lambda\downarrow_{\Sigma_{d-1}} = \bigoplus D^{\lambda_A},$$

where the sum is over all removable nodes A with $\chi(\lambda_A) \leqslant p$.

We will make extensive use of induced modules, and the next several results describe how some of the simple and Specht modules induce. The first two are easy applications of the classical branching theorem.

Lemma 2.3. *Let $\{B_1, B_2, \ldots, B_s\}$ be the addable nodes for λ of residue α. Then* $\mathrm{Ind}^\alpha S^\lambda$ *has a filtration with composition factors $\{S^{\lambda^{B_i}}\}$.*

Proof. This follows from the classical branching theorem [3, 9.3] and the fact that $\mathrm{Ind}^\alpha S^\lambda$ has a filtration by Specht modules [3, 17.14].

Lemma 2.3 has an immediate corollary:

Lemma 2.4. $S^\lambda \cong \mathrm{Ind}^{\mathrm{res}\,A} S^{\lambda_A}$ *if and only if A is the unique addable node of λ_A of residue* $\mathrm{res}\,A$.

Inducing the simple modules is not as easy, but Kleshchev [5], [6] has determined the head and socle of $\mathrm{Ind}\,D^\lambda$ and a criterion for when the induced module is semisimple. We state these results below:

Lemma 2.5 ([5]). *Let B be a good addable node for λ. Then*

$$\mathrm{soc}(\mathrm{Ind}^{\mathrm{res}\,B} D^\lambda) \cong \mathrm{head}(\mathrm{Ind}^{\mathrm{res}\,B} D^\lambda) \cong D^{\lambda^B}.$$

If there is no good addable node of λ of residue α then $\mathrm{Ind}^\alpha D^\lambda = 0$.

Because there is at most one good addable node of each residue, Lemma 2.5 makes sense. Kleshchev has determined exactly when $\mathrm{Ind}\,D^\lambda$ is completely reducible:

Lemma 2.6 ([6]). $\mathrm{Ind}^{\Sigma_{d+1}} D^\lambda$ *is completely reducible if and only if the number of normal nodes of λ is one less than the number of good addable nodes for λ, in which case*

$$\mathrm{Ind}^{\Sigma_{d+1}} D^\lambda \cong \bigoplus_{B \text{ good addable}} D^{\lambda^B}.$$

If the Specht module S^λ is α-induced from Σ_{d-1} as in Lemma 2.4, and if D^λ is completely splittable, then D^λ is also α-induced from Σ_{d-1}:

Lemma 2.7. *Let $\chi(\lambda) \leqslant p$. If $S^\lambda \cong \mathrm{Ind}^{\mathrm{res}\,A} S^{\lambda_A}$, then $D^\lambda \cong \mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A}$.*

Proof. Since $\chi(\lambda) \leqslant p$, λ_A must be p-regular. Also, there are no removable or addable nodes of λ of residue $\mathrm{res}\,A$ except A. Thus A is good addable for λ_A. We apply $\mathrm{Ind}^{\mathrm{res}\,A}$ to

$$0 \to R \to S^{\lambda_A} \to D^{\lambda_A} \to 0$$

to obtain

$$0 \to \mathrm{Ind}^{\mathrm{res}\,A} R \to S^\lambda \to \mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A} \to 0.$$

But $[S^\lambda : D^\lambda] = 1$ and so Lemma 2.5 applied to λ_A implies the result.

3 Minimal modules

We wish to determine $\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu)$ for completely splittable modules D^λ and D^μ by using induction and the Eckmann–Shapiro lemma. It is not surprising then that it is useful to determine which completely splittable modules arise by α-inducing a completely splittable module from Σ_{d-1} to Σ_d. This motivates the following definition:

Definition 3.1. A completely splittable module D^λ is *minimal* if there does not exist $\mu \vdash d-1$, with D^μ completely splittable, such that $D^\lambda \cong \mathrm{Ind}^\alpha D^\mu$ for some α. If D^λ is minimal, we also say that λ is minimal.

Notice that if D^λ and D^μ are completely splittable and D^λ is not minimal, then there is some removable node A of λ such that D^{λ_A} is completely splittable and $D^\lambda \cong \mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A}$. In this case

$$\begin{aligned}\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu) &\cong \mathrm{Ext}^1_{K\Sigma_d}(\mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A}, D^\mu)\\ &\cong \mathrm{Ext}^1_{K\Sigma_{d-1}}(D^{\lambda_A}, \mathrm{Res}_{\mathrm{res}\,A} D^\mu). \qquad (1)\end{aligned}$$

Now $\mathrm{Res}_{\mathrm{res}\,A} D^\mu$ is either 0 or completely splittable, and so Equation 1 implies that unless both partitions are minimal we can determine $\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu)$ from knowledge of the Ext^1-quiver for completely splittable $K\Sigma_{d-1}$-modules. Recall here that the modules D^λ are self-dual so that

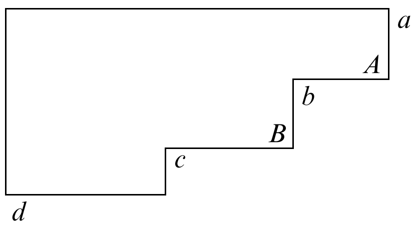
$$\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu) \cong \mathrm{Ext}^1_{K\Sigma_d}(D^\mu, D^\lambda).$$

Thus our next goal will be to classify the minimal modules and study their extensions.

We begin classifying the minimal modules with the following lemma:

Lemma 3.2. *If D^λ is minimal then λ is either of the form $(\lambda_1^{a_1})$ or $(\lambda_1^{a_1}, \lambda_2^{a_2})$.*

Proof. Suppose that $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_t^{a_t})$ is minimal and $t > 2$. Then the diagram of λ has the form

$\lambda =$


where A and B are the top two of the at least three removable nodes and a, b and c are the top three of the at least four addable nodes. Note that the residues of a, b, c, A, and B are all distinct because $\chi(\lambda) \leqslant p$. Thus at least one of A or B (whichever has residue not equal to $\operatorname{res} d$) will be the unique addable node of its residue when it is removed from λ. Suppose, for example, that $\operatorname{res} A \neq \operatorname{res} d$. Since $\chi(\lambda_A) \leqslant \chi(\lambda_B) = \chi(\lambda) \leqslant p$, we know that D^{λ_A} is completely splittable. Then Lemmas 2.4 and 2.7 imply that

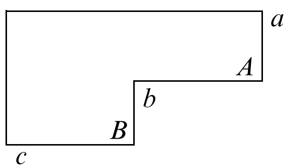
$$D^\lambda \cong \operatorname{Ind}^{\operatorname{res} A} D^{\lambda_A},$$

and so λ is not minimal, contradicting our assumption. The same argument works with D^{λ_B} if $\operatorname{res} B \neq \operatorname{res} d$. Thus the assumption that $t > 2$ contradicts the assumption that D^λ is minimal.

We can now completely classify the minimal modules:

Theorem 3.3. *A completely splittable $K\Sigma_d$-module D^λ is minimal if and only if $p|d$ and D^λ is in the principal block $B_0(K\Sigma_d)$.*

Proof. Let D^λ be completely splittable and minimal. By Lemma 3.2, we can assume that λ has at most two distinct parts. Suppose first that $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2})$ for $\lambda_1 \neq \lambda_2$. We label the addable and removable nodes of λ as follows:

$\lambda =$


We calculate (congruences are modulo p):

$$\begin{aligned} \operatorname{res} A &\equiv \lambda_1 - a_1 \\ \operatorname{res} B &\equiv \lambda_2 - a_1 - a_2 \\ \operatorname{res} a &\equiv \lambda_1 \\ \operatorname{res} b &\equiv \lambda_2 - a_1 \\ \operatorname{res} c &\equiv -a_1 - a_2 \\ \chi(\lambda) &= \lambda_1 - \lambda_2 + a_1 + a_2. \end{aligned} \tag{2}$$

Notice that Lemmas 2.4 and 2.7 imply that $D^\lambda \cong \mathrm{Ind}^{\operatorname{res} A} D^{\lambda_A}$ unless $\operatorname{res} A = \operatorname{res} c$. But λ is minimal and λ_A is completely splittable, and so we can assume that $\operatorname{res} A = \operatorname{res} c$. Because $\chi(\lambda) \leqslant p$ we know that $\operatorname{res} A \neq \operatorname{res} B$, and hence $\operatorname{res} B \neq \operatorname{res} c$. Using Lemmas 2.4 and 2.7 again, we conclude that $\operatorname{res} B = \operatorname{res} a$ (for otherwise $D^\lambda \cong \mathrm{Ind}^{\operatorname{res} B} D^{\lambda_B}$).

Setting $\operatorname{res} B = \operatorname{res} a$ and $\operatorname{res} A = \operatorname{res} c$, a simple calculation using Equation 2 shows that $\chi(\lambda) = p$ and $\operatorname{res} b = 0$. But $\operatorname{res} b = 0$ if and only if $\lambda_2 \equiv a_1$, and $\operatorname{res} A = \operatorname{res} c$ if and only if $\lambda_1 \equiv -a_2$. Since D^λ is minimal it follows that $\lambda_2 \equiv a_1$ and $\lambda_1 \equiv -a_1$, and hence

$$d = a_1\lambda_1 + a_2\lambda_2 \equiv -a_1 a_2 + a_1 a_2 = 0$$

so that $p|d$.

Before proceeding further, we verify that $p|d$, $\chi(\lambda) = p$ and $\operatorname{res} b = 0$ are sufficient conditions for the partition $(\lambda_1^{a_1}, \lambda_2^{a_2})$ to be minimal:

Lemma 3.4. *Let* $(\lambda_1^{a_1}, \lambda_2^{a_2}) \vdash mp$ *satisfy* $\chi(\lambda) = p$ *and* $\operatorname{res} b = 0$ (*i.e.* $\lambda_2 \equiv a_1$). *Then* λ *is minimal.*

Proof. First suppose that $m = 1$, so that λ must be a hook. Then $D^{\lambda_A} \cong S^{\lambda_A}$ and $D^{\lambda_B} \cong S^{\lambda_B}$ (because λ_A and λ_B partition $p-1$, and $K\Sigma_{p-1}$ is semisimple). Then λ_A has two addable nodes of residue $\operatorname{res} A$ and so $\mathrm{Ind}^{\operatorname{res} A} D^{\lambda_A}$ is not simple. Similarly $\mathrm{Ind}^{\operatorname{res} B} D^{\lambda_B}$ is not simple, so that λ must be minimal.

Next suppose that $m > 1$. Then λ_B is not completely splittable, and so to prove that λ is minimal we must verify that $D^\lambda \not\cong \mathrm{Ind}^{\operatorname{res} A} D^{\lambda_A}$.

If $\lambda_1 - \lambda_2 = 1$ then λ_A has one normal node, the top removable, and one good addable node, A. Thus $\mathrm{Ind}^{\operatorname{res} A} D^{\lambda_A} \cong \mathrm{Ind}\, D^{\lambda_A}$ is not simple by Lemmas 2.5 and 2.6.

If $\lambda_1 - \lambda_2 > 1$ then λ_A has two normal nodes, the top two removable, and two good addable nodes, A and b. The node b has residue $0 \neq \operatorname{res} A$ and $\mathrm{Ind}^0 D^{\lambda_A}$ is simple by Lemmas 2.4 and 2.7. But $\mathrm{Ind}\, D^{\lambda_A}$ is not semisimple by Lemma 2.6, and so $\mathrm{Ind}^{\operatorname{res} A} D^{\lambda_A}$ is not simple. This completes the proof that λ is minimal.

The rim hook with foot at B has p nodes and removing it preserves $\chi = p$ and preserves $\operatorname{res} b = 0$, and so preserves minimality by Lemma 3.4. Of course, it also preserves membership in B_0. Thus $\lambda \in B_0$ by induction.

Now suppose that λ is rectangular, i.e. $\lambda = (\lambda_1^{a_1})$ with $a_1 < p$, and let A be the one removable node of λ. Unless $p|\lambda_1$, the partition λ_A will have a unique addable node of residue $\operatorname{res} A$ and, as in the previous case, $D^\lambda \cong \mathrm{Ind}^{\operatorname{res} A} D^{\lambda_A}$. Therefore the minimality of λ implies that $p|\lambda_1$, so that $p|d$ and $\lambda \in B_0$. An argument as in Lemma 3.4 confirms that $\lambda = (cp^{a_1})$ is indeed minimal.

To complete the proof of Theorem 3.3 we must show that a completely splittable module $D^\lambda \in B_0(K\Sigma_{mp})$ must be minimal. This too follows by induction. The base case $m = 1$ is just $\lambda = (p - r, 1^r)$ for $0 \leqslant r \leqslant p - 2$. In this case D^λ is minimal by Lemma 3.4. Suppose then that $m > 1$ and D^λ is completely splittable and in $B_0(K\Sigma_{mp})$. Since $\chi(\lambda) \leqslant p$ and the rim of λ must have at least p nodes (for otherwise λ is a p-core, so not in the principal block), there must be a rim p-hook with head at $(1, \lambda_1)$. Strip off this rim hook to get $\bar{\lambda}$. Clearly $\chi(\bar{\lambda}) \leqslant p$, so that $D^{\bar{\lambda}}$ is completely splittable and in $B_0(K\Sigma_{(m-1)p})$, and hence is minimal by induction. We leave it to the reader to verify that there is only one way to add a hook to a minimal partition such that the result is completely splittable and such that the head of the hook ends up in the first row. (See for example Figure 1.) Doing so preserves $\chi(\lambda) = p$ and $\operatorname{res} b = 0$, and hence produces another minimal partition. So we conclude that λ must be minimal, completing the proof of Theorem 3.3.

We now determine how many minimal modules there are, and some information we will need to determine the extensions between them.

Lemma 3.5. *Let $d = pm$. There are $p - 1$ minimal partitions of d. They have $1, 2, \ldots, p - 1$ parts. The top removable node of the one with i parts has residue $p - i$.*

Proof. The result is clear for $m = 1$: the partitions are just the p-regular hooks. Now observe that removing the rim p-hook with head $(1, \lambda_1)$ or adding a rim p-hook with foot at $(k, \lambda_k + 1)$ gives a bijection between minimal partitions of mp and $(m + 1)p$ that preserves the number of parts and the residue of the top removable node. The bijection for $p = 5$ and $n = 5, 10, 15$ is illustrated in Figure 1, with the p-residues labelled.

We will need a few more technical results about minimal partitions before we can begin determining the Ext^1-quiver.

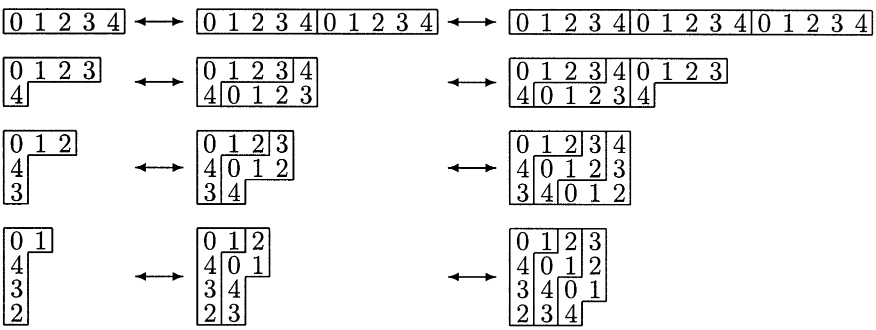


Figure 1. Minimal partitions for $p = 5$ and $n = 5, 10, 15$

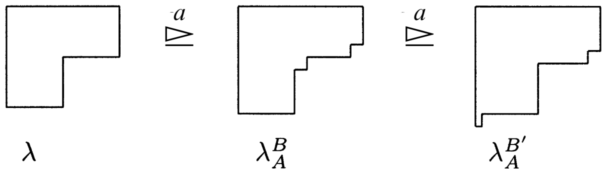


Figure 2. Neighbors of λ in the dominance order

Lemma 3.6. *Let $\lambda \vartriangleright \mu$ be minimal partitions of $d = mp$ with $m > 1$. Then $\mu \ntrianglerighteq \lambda_A^B$ for any removable node A and addable node B of λ.*

Proof. First we recall the well known fact [4, 1.4.10] that partitions $\rho \vartriangleright \tau$ are neighbors in the dominance order if and only if τ is obtained from ρ by moving single node down to the next available location. There are few cases to check, since minimal partitions have at most two removable nodes and at most three addable nodes. First note that $m > 1$ implies that λ_A^B is never minimal. Let $\overset{a}{\trianglerighteq}$ denote partitions which are neighbors in the dominance order. The first case is when A is the top removable node of λ. There are two choices for B, and we have illustrated the possible λ_A^B in Figure 2.

Since the three partitions drawn are all adjacent in the dominance order, and only λ is minimal, there cannot be a minimal partition μ with $\lambda \vartriangleright \mu \trianglerighteq \lambda_A^B$. The other case, when A is the lower removable node of λ is similar.

The final observation that we need about minimal partitions is clear from Figure 1:

Lemma 3.7. *The minimal partitions are totally ordered by $\trianglerighteq$ and $\lambda \vartriangleright \mu$ if and only if λ has fewer parts.*

4 Ext^1 for minimal modules

Recall from Equation 1 that if we can determine $\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu)$ for minimal modules D^λ and D^μ, we should be able to determine it for all completely splittable modules. We prove the following result:

Theorem 4.1. *Let $\lambda \neq \mu$ be minimal partitions of mp and $m > 1$. Then*

$$\mathrm{Ext}^1_{K\Sigma_{mp}}(D^\lambda, D^\mu) = 0.$$

Proof. By Lemma 3.7 we can assume that $\lambda \vartriangleright \mu$. Let A be the top removable node of λ. By Lemma 3.5, the top removable node of μ has residue different from $\operatorname{res} A$. But, since $m > 1$, removing the bottom removable node of μ (if it has two), leaves a partition with $\chi = p + 1$. Thus by Lemma 2.2,

$$\mathrm{Res}_{\operatorname{res} A} D^\mu = 0 \tag{3}$$

We assumed that λ is minimal and so $D^{\lambda} \not\cong \mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A}$. However, A is good addable for λ_A, and so by Lemma 2.5 we have

$$\mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A} = \begin{matrix} D^{\lambda} \\ H \\ D^{\lambda} \end{matrix}. \tag{4}$$

Equation 3 together with the Eckmann–Shapiro lemma proves that

$$\mathrm{Ext}^1_{K\Sigma_{mp}}(\mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A}, D^{\mu}) = 0. \tag{5}$$

Consider the exact sequence

$$0 \to D^{\lambda} \to \mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A} \to \begin{matrix} D^{\lambda} \\ H \end{matrix} \to 0.$$

Apply $\mathrm{Hom}_{K\Sigma_{mp}}(-, D^{\mu})$ to this to obtain a long exact sequence:

$$0 \to \mathrm{Ext}^1_{K\Sigma_{mp}}\left(\begin{matrix} D^{\lambda} \\ H \end{matrix}, D^{\mu}\right) \to \mathrm{Ext}^1_{K\Sigma_{mp}}(\mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A}, D^{\mu}) \to \cdots,$$

then use (5) to conclude that

$$\mathrm{Ext}^1_{K\Sigma_{mp}}\left(\begin{matrix} D^{\lambda} \\ H \end{matrix}, D^{\mu}\right) = 0. \tag{6}$$

Similarly we apply $\mathrm{Hom}_{K\Sigma_{mp}}(-, D^{\mu})$ to

$$0 \to H \to \begin{matrix} D^{\lambda} \\ H \end{matrix} \to D^{\lambda} \to 0$$

to obtain a long exact sequence and to conclude, by (6), that

$$\mathrm{Ext}^1_{K\Sigma_{mp}}(D^{\lambda}, D^{\mu}) \cong \mathrm{Hom}_{K\Sigma_{mp}}(H, D^{\mu}). \tag{7}$$

But H is a subquotient of $\mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A}$, and so it is a subquotient of $\mathrm{Ind}^{\mathrm{res}\,A} S^{\lambda_A}$. However $\mathrm{Ind}^{\mathrm{res}\,A} S^{\lambda_A}$ has a Specht series with factors S^{λ} and perhaps $S^{\lambda_A^B}$ if $\mathrm{res}\,A = \mathrm{res}\,B$, by Lemma 2.3 (where B is the lowest addable node of λ). Since a Specht module S^{τ} has a composition factor D^{ρ} only if $\rho \trianglerighteq \tau$, Lemma 3.6 implies there are no composition factors D^{μ} in H. Thus $\mathrm{Hom}_{K\Sigma_{mp}}(H, D^{\mu}) = 0$, and the proof is complete by Equation 7.

Often a completely splittable module D^{λ} can be obtained by α-induction from more than one completely splittable $K\Sigma_{d-1}$-module. The next lemma determines when this happens.

Lemma 4.2. *Let D^λ be completely splittable, and A a removable node of λ. Then $D^\lambda \cong \mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A}$ unless*

(i) *$\chi(\lambda) = p$ and A is the lowest removable node of λ, or*

(ii) *$\mathrm{res}\,A$ is equal to the residue of the lowest addable node of λ.*

Proof. Suppose that we are not in case (i) or case (ii). Since we are not in case (i), A is not the lowest removable node of λ, and so $\chi(\lambda_A) \leqslant p$, and D^{λ_A} is completely splittable. Since we are not in case (ii), A is the unique addable node for λ_A with residue $\mathrm{res}\,A$. So Lemmas 2.4 and 2.7 prove that $D^\lambda \cong \mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A}$.

We remark that if A satisfies (i) or (ii) from Lemma 4.2, then a calculation similar to the proof of Lemma 3.4 shows that $D^\lambda \not\cong \mathrm{Ind}^{\mathrm{res}\,A} D^{\lambda_A}$.

Since we plan to reduce the calculation of $\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu)$ to the case where λ and μ are minimal, the next definiton is natural:

Definition 4.3. Given D^λ completely splittable, the *minimal core* of λ, denoted by $\hat{\lambda}$, is obtained by successively removing nodes from λ which do not satisfy Lemma 4.2 (i) or (ii) until no further such nodes can be removed. The remark following Lemma 4.2 implies that $\hat{\lambda}$ is indeed minimal.

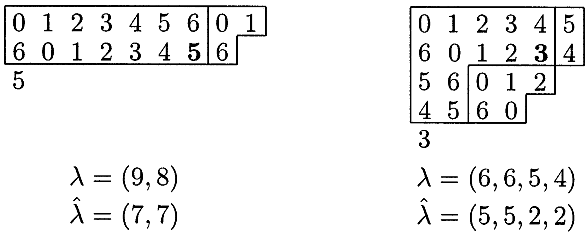
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be completely splittable. It is easy to see from a few examples that $\hat{\lambda}$ is well defined and easy to determine. Starting with λ, simply remove nodes, never removing any of residue equal to the residue of the bottom addable node, and never allowing χ to be greater than p.

More explicitly, suppose that the bottom addable node of λ has residue α. Let (u, v) be the node of residue α farthest to the right. This node is uniquely determined because $h(\lambda) \leqslant p - 1$. If there is no such node, $\hat{\lambda}$ is empty. If $u = k$ then $\hat{\lambda} = (v^k)$. If $u < k$ then $\hat{\lambda}$ is the partition with rim consisting of the following nodes:

$$(1, v), (2, v), \ldots, (u, v), (u, v-1), (u, v-2), \ldots, (u, v-p+k),$$
$$(u+1, v-p+k), \ldots, (k, v-p+k), (k, v-p+k-1), \ldots, (k, 1).$$

In particular, note that $h(\lambda) = h(\hat{\lambda})$ when $\hat{\lambda} \neq \varnothing$.

Two examples are given in Figure 3, with the node (u, v) in bold, and the minimal



$\lambda = (9, 8)$ $\quad$ $\lambda = (6, 6, 5, 4)$

$\hat{\lambda} = (7, 7)$ $\quad$ $\hat{\lambda} = (5, 5, 2, 2)$

Figure 3. Two examples of minimal 7-cores

core drawn inside the partition. The residues of all nodes in λ and the bottom addable node of λ are labelled.

We now use the minimal cores and Equation 1 to formalize the reduction of determining Ext^1 between two minimal modules to the minimal case:

Theorem 4.4. *Let D^λ and D^μ be completely splittable and in the same block. Then $\hat{\lambda}$ and $\hat{\mu}$ partition the same number r and*

$$\mathrm{Ext}^1_{K\Sigma_r}(D^{\hat{\lambda}}, D^{\hat{\mu}}) \cong \mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu).$$

Proof. We first remark that since $\lambda \sim \mu$, by Theorem 3.3 either λ and μ are both minimal or neither is. So we will assume λ and μ are not minimal, since otherwise $\hat{\lambda} = \lambda$ and $\hat{\mu} = \mu$.

To prove this claim we will use the abacus notation for partitions and the characterization of blocks by p-cores. See [4] for details. We make a series of observations about the abacus representations of λ and μ.

(i). Since λ has less than p parts, we will represent it on an abacus with p beads. We label the runners $0, 1, \ldots, p-1$ from left to right.

(ii). Notice that $\chi(\lambda)$ can be read off the abacus display. It is equal to $s - t + 1$, where t is the position of the first bead after the first gap and s is the position of the last bead.

(iii). Since $\chi(\lambda) \leqslant p$, by (ii), the core of λ (also represented with p beads) has at most two beads on any runner and must have a bead on runner 0.

(iv). λ is not minimal, so not in the principal block of $K\Sigma_{mp}$. Thus λ does not have an empty p-core, and so some runner has two beads. Since $\chi(\lambda) \leqslant p$, if runner i has no beads, then runners to the right of i have at most one bead by (ii).

Of course, (i)–(iv) hold for μ as well. Moreover $\lambda \sim \mu$, and so the abacus displays for λ and μ have the same number of beads on each runner.

Now suppose that the lowest addable node for λ has residue i. Then in the abacus display for λ, the first gap occurs on runner i and so (ii) implies that runner i has at most one bead. Similarly, if the lowest addable node of μ has residue j then runner j has at most one bead.

Consider the abacus diagram for the p-core $\tilde{\lambda} = \tilde{\mu}$. Note that (i) and (iv) imply that runner $p-1$ cannot have two beads. We consider separately the case where it has no beads and where it has one bead.

Case 1. Suppose that runner $p-1$ has no beads. Let runner $k < p-1$ be the leftmost runner with two beads. Then (even if $k = 0$) both of λ and μ must have a removable node of residue k. Furthermore k is not equal to i or j since runners i and j have at most one bead.

Case 2. Suppose that runner $p-1$ has one bead. There must be some runner with no beads, and by (iv) all runners to the right of it have at most one bead. Thus we can choose a runner k such that runner k has one bead and runner $k-1$ has no beads. Again both λ and μ must have a removable node of residue k. Since runner $k-1$ has no beads, the partitions have no addable nodes of residue k so that k is not equal to i or j. The two cases are illustrated in Figure 4:

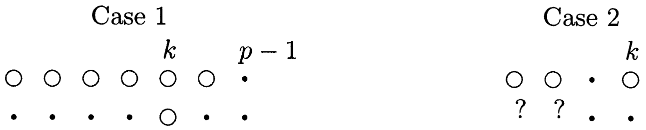


Figure 4. Possible abacus configurations for $\tilde{\lambda}$

In both cases λ and μ must have a removable node of residue k. Suppose that $\chi(\lambda) = p$ and the node of residue k is the bottom removable node for λ. Then removing this node will increase $\chi(\lambda)$ from p to $p+1$, and λ will have the abacus configuration shown in Figure 5.

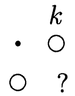


Figure 5. Runners k and $k-1$ for λ

But the bead configuration for runners $k-1$ and k in the abacus display of λ illustrated in Figure 5 is inconsistent with Figure 4 when λ is completely splittable. In Case 1, runner k has two beads, and so Figure 5 together with (ii) above proves $\chi(\lambda) > p$, contradicting the assumption that λ is completely splittable. In Case 2, runner $k-1$ has no beads and so the configuration shown in Figure 5 cannot occur. The same applies for μ.

Thus we have found removable nodes A_λ of λ and A_μ of μ, both with residue k not equal to the residue of the bottom addable node of λ or μ respectively. Moreover $\chi(\lambda_{A_\lambda}) \leqslant p$ and $\chi(\mu_{A_\mu}) \leqslant p$, so that these nodes are not the lowest removable nodes for λ and μ. Thus A_λ is the unique addable node of λ_{A_λ} of residue $\operatorname{res} A_\lambda$, and similarly for A_μ.

Lemma 4.2 implies that

$$D^\lambda \cong \mathrm{Ind}^k D^{\lambda_{A_\lambda}} \quad \text{and} \quad D^\mu \cong \mathrm{Ind}^k D^{\mu_{A_\mu}}. \tag{8}$$

We also have

$$\mathrm{Res}_k D^\lambda \cong D^{\lambda_{A_\lambda}} \quad \text{and} \quad \mathrm{Res}_k D^\mu \cong D^{\mu_{A_\mu}}. \tag{9}$$

Equations 8 and 9 give us

$$\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu) \cong \mathrm{Ext}^1_{K\Sigma_{d-1}}(D^{\lambda_{A_\lambda}}, D^{\mu_{A_\mu}}). \tag{10}$$

But clearly $\hat{\lambda} = \widehat{\lambda_{A_\lambda}}$ and $\hat{\mu} = \widehat{\mu_{A_\mu}}$, and so the theorem follows from Equation 10 by induction.

Before we prove Theorem 1.2, we need to deal with self-extensions and with the base case of the induction. Self-extensions have been handled previously:

Lemma 4.5. $\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\lambda) = 0$ *for* D^λ *completely splittable.*

Proof. This was proven in [9] for partitions with less than p parts, and this clearly holds for λ since $\chi(\lambda) \leqslant p$ and λ is p-regular.

We have determined $\mathrm{Ext}^1_{K\Sigma_{mp}}(D^\lambda, D^\mu) = 0$ for λ and μ minimal partitions of mp with $m > 1$. The minimal partitions for $d = p$ are exactly the p-regular hooks. So we need to handle $\mathrm{Ext}^1_{K\Sigma_p}(D^\lambda, D^\mu)$ for λ and μ hook partitions of p. But this is just the principal block of $K\Sigma_p$, and the following fact is well known (see e.g. [10]):

Lemma 4.6. *Let* $\lambda = (p-r, 1^r)$ *and* $\mu = (p-s, 1^s)$ *for* $0 \leqslant r \leqslant p-2$. *Then*

$$\mathrm{Ext}^1_{K\Sigma_p}(D^\lambda, D^\mu) = \begin{cases} K & \textit{if } |r-s| = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

We call the two hooks *adjacent* when $|r-s| = 1$. Thus for completely splittable modules, $\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu)$ will be zero unless $\hat{\lambda}$ and $\hat{\mu}$ are adjacent hook partitions of p.

Everything is now in place to prove Theorem 1.2. We know that

$$\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu) = 0$$

unless $\hat{\lambda}$ and $\hat{\mu}$ are adjacent hook partitions of p. Let the bottom addable node of λ have residue α. It is not difficult to see from our description of the minimal core that $\hat{\lambda} = \varnothing$ exactly when λ has no nodes of residue α. This is equivalent to $h_{11}(\lambda) < p$. For $\hat{\lambda}$ to be a hook partition of p we need λ to have exactly one node of residue α. This is equivalent to the condition that $h_{21}(\lambda) < p$. We know that $h(\hat{\lambda}) = h(\lambda)$ if it is non-empty, and so the criterion for $\hat{\lambda}$ and $\hat{\mu}$ to be adjacent is that $|h(\lambda) - h(\mu)| = 1$. Thus we have proven Theorem 1.2.

We remark that if d is large relative to p, the conditions on $\lambda \vdash d$ stated in Theorem 1.2 are incompatible with λ being completely splittable. We make this precise in the following corollary:

Corollary 4.7. *Let* D^λ *and* D^μ *be completely splittable* $K\Sigma_d$*-modules, and suppose that* $d > p + p^2/4$. *Then*

$$\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu) = 0.$$

Proof. We suppose that $\mathrm{Ext}^1_{K\Sigma_d}(D^\lambda, D^\mu) \neq 0$ and prove that $d \leqslant p + p^2/4$. Theorem 1.2 implies that $\hat{\lambda}$ and $\hat{\mu}$ are adjacent hook partitions of p, so assume that $\hat{\lambda} = (p-r, 1^r)$ where $r > 0$. Since $h(\lambda) = h(\hat{\lambda})$, the lowest addable node of λ is the same

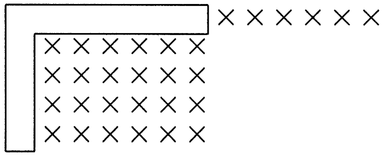


Figure 6. Possible nodes for λ with 11-core $(7, 1^4)$

as the lowest addable node of $\hat{\lambda}$, and has residue $p - r - 1$. From our description (after Definition 4.3) of how to obtain the minimal core, we see that λ has no nodes of residue $p - r - 1$ except for the node $(1, p - r)$. In particular the node $(2, p - r + 1)$ is not in λ. This restriction together with the requirement that $\chi(\lambda) \leqslant p$ forces all nodes of λ not in $\hat{\lambda}$ to be contained in the region marked with $\times$'s in Figure 6, where we have illustrated the case $p = 11$ and $r = 4$.

It is easy to see from Figure 6 that the maximum number of nodes that λ can have is $p + (r+1)(p - (r+1))$. This expression achieves a maximum value of $p + p^2/4$ when $r = p/2$. Thus $d \leqslant p + p^2/4$ as desired.

5 Application to $\mathbf{GL}_n(K)$

We describe here how to prove Corollary 1.3. For a comprehensive discussion of the background for this section, see [2].

Let $M(n, d)$ denote the polynomial representations of $\mathrm{GL}_n(K)$ of homogeneous degree d, for $d \leqslant n$. The simple modules in $M(n, d)$ are indexed by partitions λ of d and denoted by $L(\lambda)$. The Schur functor $\mathscr{F}$ is a covariant, exact functor from $M(n, d)$ to $K\Sigma_d$-mod such that

$$\mathscr{F}(L(\lambda)) = \begin{cases} D_\lambda & \text{if } \lambda \text{ is column } p\text{-regular} \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Because Equation 11 is in terms of D_λ and Theorem 1.2 is in terms of D^λ, we recall here how the two parametrizations of simple $K\Sigma_d$-modules are related:

$$\begin{aligned} D_\lambda &\cong D^{\lambda'} \otimes \mathrm{sgn} \cong D^{m(\lambda')} \\ D^\lambda &\cong D_{\lambda'} \otimes \mathrm{sgn} \cong D_{(m(\lambda))'}. \end{aligned} \tag{12}$$

Now $\mathscr{F}$ has a right adjoint functor $\mathscr{G}$, and this adjointness sometimes extends to Ext^1, as in the following result:

Theorem 5.1 ([1]).

$$\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\mu) \cong \mathrm{Ext}^1_{M(n,d)}(L(\lambda), \mathscr{G}(D_\mu)).$$

This theorem is especially useful for completely splittable modules, as the functor $\mathscr{G}$ behaves well on them:

Theorem 5.2 ([8]). *Let D_λ be non-trivial and completely splittable. Then*

$$\mathscr{G}(D_\lambda) = L(\lambda).$$

Theorems 1.2, 5.1, and 5.2 and Equation 12 immediately imply Corollary 1.3. Although Theorem 5.2 applies only to non-trivial modules, Corollary 1.3 still holds when $D^\lambda \cong D^\mu \cong K$, by the well-known fact that $\mathrm{Ext}^1_{M(n,d)}(L(\lambda), L(\lambda)) = 0$.

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