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Fixed-point functors for symmetric groups and Schur algebras

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Abstract

Let Σ_d be the symmetric group. For $1 < m < d$ let $\mathcal{F}_m$ be the functor which takes a Σ_d-module U to the space of fixed points U^{Σ_m}, which is naturally a module for Σ_{d-m}. This functor was previously used by the author to study cohomology of the symmetric group, but little is known about it. This paper initiates a study of $\mathcal{F}_m$. First, we relate it to James' work on row and column removal and decomposition numbers for the Schur algebra. Next, we determine the image of dual Specht modules, permutation and twisted permutation modules, and some Young and twisted Young modules under $\mathcal{F}_m$. In particular, $\mathcal{F}_m$ acts as first row removal on dual Specht modules S_λ with $\lambda_1 = m$ and as first column removal on twisted Young and twisted permutation modules corresponding to partitions with m parts. Finally, we prove that determining $\mathcal{F}_m$ on the Young modules is equivalent to determining the decomposition numbers for the Schur algebra.

1. Notation and preliminaries

We will assume familiarity with the representation theory of the symmetric group Σ_d and of the Schur algebra $S(n,d)$ as found in [3,7,11]. Let k be an algebraically closed field of characteristic $p > 2$. We write $\lambda \vdash d$ for $\lambda = (\lambda_1, \dots, \lambda_r)$ a partition of d and $\lambda \models d$ for a composition of d. Let $\Lambda^+(n,d)$ denote the set of partitions of d with at most n parts and

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let $\Lambda(n,d)$ denote the set of compositions of d with at most n parts. We do not distinguish between λ and its Young diagram:

$$\lambda = \{(i,j) \in \mathbf{N} \times \mathbf{N} \mid j \leqslant \lambda_i\}.$$

A partition λ is *p-regular* if there is no i such that $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1} \neq 0$. It is *p-restricted* if its conjugate partition, denoted λ', is p-regular. We write $\trianglerighteq$ for the usual dominance order on partitions. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ we write $\bar{\lambda}$ for λ with its first row removed, i.e.,

$$\bar{\lambda} = (\lambda_2, \dots, \lambda_r) \vdash d - \lambda_1.$$

We write $\hat{\lambda}$ for λ with its first column removed, i.e.,

$$\hat{\lambda} = (\lambda_1 - 1, \lambda_2, -1, \dots, \lambda_r - 1) \vdash d - r.$$

The complex simple Σ_d-modules are the Specht modules $\{S^\lambda \mid \lambda \vdash d\}$. Simple $k\Sigma_d$-modules can be indexed by p-restricted partitions or by p-regular partitions. Both

$$\{D^\lambda := S^\lambda/\mathrm{rad}(S^\lambda) \mid \lambda \text{ is } p\text{-regular}\} \quad \text{and} \quad \{D_\lambda = \mathrm{soc}(S^\lambda) \mid \lambda \text{ is } p\text{-restricted}\}$$

are complete sets of nonisomorphic simple $k\Sigma_d$-modules. The two indexings are related by $D^\lambda \cong D_{\lambda'} \otimes \mathrm{sgn}$, where sgn is the one-dimensional signature representation. We recall that

$$S^\lambda \otimes \mathrm{sgn} \cong S_{\lambda'}, \tag{1.1}$$

where S_μ denotes the dual of the Specht module S^μ.

We will also consider the Young modules $\{Y^\lambda \mid \lambda \vdash d\}$, the permutation modules $\{M^\lambda \mid \lambda \in \Lambda(n,d)\}$, and their twisted versions obtained by tensoring with sgn. If λ is p-restricted, then Y^λ is the projective cover of D_λ. All these modules, and the $S(n,d)$-modules below, are described in [11].

Let $V = k^n$ be the natural module for the general linear group $GL_n(k)$. Then $V^{\otimes d}$ is a $GL_n(k)$ -module with the diagonal action and a $k\Sigma_d$-module by place permutation. The Schur algebra $S(n,d)$ is defined by

$$S(n,d) := \mathrm{End}_{k\Sigma_d}(V^{\otimes d}).$$

The actions of $GL_n(k)$ and Σ_d commute, so we get a map $GL_n(k) \to S(n,d)$. This map identifies the category mod-$S(n,d)$ with the category of homogeneous polynomial representations of $GL_n(k)$ of degree d.

For $\lambda \in \Lambda^+(n,d)$ we denote the irreducible $S(n,d)$-module with highest weight λ by $L(\lambda)$. We also write $\Delta(\lambda)$ and $\nabla(\lambda)$ (sometimes written $V(\lambda)$ and $H^0(\lambda)$) for the standard and costandard modules with highest weight λ, respectively. $P(\lambda)$ will be the projective cover of $L(\lambda)$, $I(\lambda)$ the injective hull, and $T(\lambda)$ the corresponding tilting module. Finally, $S^\lambda(V)$ and $\Lambda^\lambda(V)$ will be the symmetric and exterior powers.

Let $I(n,d)$ denote the set of all d-tuples $\mathbf{i} = (i_1, i_2, \ldots, i_d)$ with $i_s \in \{1, 2, \ldots, n\}$. The symmetric group Σ_d acts on the right on $I(n,d)$ by

$$\mathbf{i}\pi = \big(i_{\pi(1)}, i_{\pi(2)} \ldots, i_{\pi(d)}\big).$$

This action extends to an action on $I(n,d) \times I(n,d)$, and we write $(\mathbf{i}, \mathbf{j}) \sim (\mathbf{k}, \mathbf{l})$ if $\mathbf{k} = \mathbf{i}\pi$ and $\mathbf{l} = \mathbf{j}\pi$ for some $\pi \in \Sigma_d$. Let Ω be a set of equivalence class representatives under $\sim$. Recall [3] that $S(n,d)$ has a basis $\{\xi_{\mathbf{i},\mathbf{j}}\}$ indexed by Ω. For $\mathbf{i} \in I(n,d)$ define the weight of $\mathbf{i}$ by $\mathrm{wt}(\mathbf{i}) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda(n,d)$, where λ_s is the number of times s appears in $\mathbf{i}$. Then $\xi_{\mathbf{i},\mathbf{i}}$ is an idempotent, usually denoted ξ_λ.

1.1. The Schur and adjoint Schur functor

Henceforth assume $n \geqslant d$. Let $\omega = (1^d) \in \Lambda(n,d)$ and let e denote the idempotent $\xi_\omega \in S(n,d)$. Then $eS(n,d)e \cong k\Sigma_d$, and the Schur functor $\mathcal{F} : \text{mod-}S(n,d) \to \text{mod-}k\Sigma_d$ is defined by $\mathcal{F}(U) := eU$. This is an exact, covariant functor with

$$\begin{aligned} &\mathcal{F}\big(\nabla(\lambda)\big) = S^\lambda, &&\mathcal{F}\big(\Delta(\lambda)\big) = S_\lambda, &&\mathcal{F}\big(L(\lambda)\big) = D_\lambda \text{ or } 0,\\ &\mathcal{F}\big(P(\lambda)\big) = Y^\lambda, &&\mathcal{F}\big(I(\lambda)\big) = Y^\lambda, &&\mathcal{F}\big(T(\lambda)\big) = Y^{\lambda'} \otimes \text{sgn}. \end{aligned} \tag{1.2}$$

The Schur functor admits a right adjoint functor $\mathcal{G} : \text{mod-}k\Sigma_d \to \text{mod-}S(n,d)$ defined by

$$\mathcal{G}(N) := \mathrm{Hom}_{k\Sigma_d}\big(V^{\otimes d}, N\big) \cong \mathrm{Hom}_{eS(n,d)e}\big(eS(n,d), N\big).$$

The two definitions are equivalent since $eS(n,d)e \cong k\Sigma_d$ and $eS(n,d) \cong V^{\otimes d}$. The module $V^{\otimes d}$ is not injective as a $k\Sigma_d$-module. Thus the functor $\mathcal{G}$ is only left exact, and so has higher right derived functors

$$R^i\mathcal{G}(N) = \mathrm{Ext}^i_{k\Sigma_d}\big(V^{\otimes d}, N\big).$$

We now collect a few results about $\mathcal{G}$ and $R^1\mathcal{G}$. Recall that we are assuming $p > 2$ throughout:

Proposition 1.1.

(i) [10, 3.2] $\mathcal{G}(S_\lambda) \cong \Delta(\lambda)$. *In particular,* $\mathcal{G}(k) \cong \Delta(d)$.
(ii) [5, 3.8.2] $\mathcal{G}(Y^\lambda) \cong P(\lambda)$.
(iii) [10, 6.4] *For* $p > 3$, $R^1\mathcal{G}(S_\lambda) \cong 0$.

1.2. Decomposition numbers, Specht filtrations, and dual Specht filtrations

We collect here material that will be used repeatedly in later sections. For a thorough treatment see Martin's book [11].

We say a $k\Sigma_d$-module U has a *Specht filtration* if it has a filtration with successive quotients isomorphic to Specht modules. Similarly, we will say U has a *dual Specht filtration*. For $S(n,d)$-modules we refer to a *Weyl filtration* (by $\Delta(\mu)$'s) or a *good filtration* (by $\nabla(\mu)$'s). The multiplicities in a good or Weyl filtration are independent of the choice of filtration. The same holds for Specht and dual Specht filtrations in mod-$k\Sigma_d$ when $p > 3$ (but not for $p = 2$ or 3) by work in [5].

The Young modules Y^λ are self-dual and are known to have both Specht and dual Specht filtrations. If $[Y^\mu : S_\lambda]$ denotes the multiplicity of S_λ in a dual Specht filtration of Y^μ and $[\Delta(\lambda) : L(\mu)]$ denotes a decomposition number for $S(n,d)$, then a well-known reciprocity theorem [11, p. 118] gives

$$\bigl[\Delta(\lambda) : L(\mu)\bigr] = \bigl[I(\mu) : \nabla(\lambda)\bigr] = \bigl[Y^\mu : S^\lambda\bigr] = \bigl[Y^\mu : S_\lambda\bigr]. \tag{1.3}$$

So knowledge of the decomposition numbers for $S(n,d)$ is equivalent to knowledge of multiplicities in dual Specht filtrations of Young modules, a fact we will use repeatedly later.

The category mod-$S(n,d)$ is a highest weight category. In particular, $[\Delta(\lambda) : L(\lambda)] = 1$ and $[\Delta(\lambda) : L(\mu)] = 0$ unless $\lambda \trianglerighteq \mu$. This triangular structure in the decomposition matrix, together with reciprocity (1.3), will be very useful to us, since it gives a triangular structure to the matrix of filtration multiplicities $[Y^\mu : S_\lambda]$. In particular, suppose we know a $k\Sigma_d$-module is a direct sum of Young modules. If we know the multiplicities in a dual Specht filtration of the module, then we can determine the multiplicities of the Young module summands.

The permutation module M^λ is a direct sum of Young modules

$$M^\lambda \cong Y^\lambda \bigoplus_{\mu \rhd \lambda} K_{\lambda\mu} Y^\mu.$$

The *p-Kostka numbers* $K_{\lambda\mu}$ are not known. However, Young's rule (see [7, Chapter 14]) gives a nice formula for the multiplicities in a Specht or dual Specht filtration of M^λ. Thus if we know the decomposition numbers for $S(n,d)$, then Young's rule together with reciprocity let us determine the p-Kostka numbers.

2. Relating the fixed-point functor to James' idempotent

In [4] we proved some theorems on extensions between simple modules for the symmetric group when the first row or column of the corresponding partitions is removed. The proof used the fixed-point functor, which we now define. Throughout the paper we will consider Σ_m as the subgroup of Σ_d fixing $\{m+1, m+2, \ldots, d\}$ and Σ_{d-m} as the subgroup of Σ_d fixing $\{1, 2, \ldots, m\}$. Then Σ_{d-m} commutes with $\Sigma_m \leqslant \Sigma_d$. So for a $k\Sigma_d$-module U, the space of Σ_m-fixed points is a Σ_{d-m}-submodule of U. Thus we can define

$$\mathcal{F}_m : \text{mod-}k\Sigma_d \to \text{mod-}k\Sigma_{d-m}$$

by

$$\mathcal{F}_m(U) := U^{\Sigma_m} \cong \mathrm{Hom}_{k\Sigma_m}(k, U) \cong \mathrm{Hom}_{k\Sigma_d}\bigl(M^{(m,1^{d-m})}, U\bigr).$$

The proofs in [4] were motivated by, but did not really use, James' work in [8]. In this paper we will first relate James' paper to the functor $\mathcal{F}_m$. It will often be convenient to think of $\mathcal{F}_m$ as first restricting to $\Sigma_m \times \Sigma_{d-m} \leqslant \Sigma_d$, and then taking the largest subspace on which Σ_m acts trivially.

In order to describe James' work, we need some more notation. Fix $1 < m < d$ and let $I^*(n,d)$ be the subset of $I(n,d)$ consisting of those d-tuples with $i_{d-m+1} = i_{d-m+2} = \cdots = i_d = 1$ and $i_k \neq 1$ for $1 \leqslant k \leqslant d-m$. James defined an idempotent $\eta = \Sigma\, \xi_{\mathbf{i},\mathbf{i}}$, the sum being over distinct elements $\xi_{\mathbf{i},\mathbf{i}}$ with $\mathbf{i} \in I^*(n,d)$. James also defined

$$S_1 := \mathrm{span}\bigl\{\xi_{\mathbf{i},\mathbf{j}} \bigm| \mathbf{i},\mathbf{j} \in I^*(n,d)\bigr\} \subset S(n,d).$$

Then S_1 is a subalgebra of $\eta S(n,d)\eta$, and James proved [8] that $S_1 \cong S(n-1, d-m)$. Thus we have the following (only partly commutative) diagram of functors, where $\mathcal{F}$ and $\mathcal{G}$ are as above while $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are the corresponding Schur and adjoint Schur functors for the smaller symmetric group Σ_{d-m}. Also we use $\mathcal{J}$ to denote multiplication by η followed by restriction to S_1.

$$\begin{tikzcd}
\text{mod-}S(n,d) \arrow[r, shift left, "\mathcal{F}"] \arrow[d, "\eta\cdot"'] \arrow[dd, bend right=60, "\mathcal{J}"'] & \text{mod-}k\Sigma_d \arrow[l, shift left, "\mathcal{G}"] \arrow[dd, "\mathcal{F}_m"'] \\
\text{mod-}\eta S(n,d)\eta \arrow[d, "\text{res}"'] & \\
\text{mod-}S_1 \arrow[r, shift left, "\widetilde{\mathcal{F}}"] & \text{mod-}k\Sigma_{d-m} \arrow[l, shift left, "\widetilde{\mathcal{G}}"]
\end{tikzcd}$$

Our first theorem is that James' functor $\mathcal{J}$ is closely related to the fixed-point functor $\mathcal{F}_m$.

Theorem 2.1. *Let $U \in \text{mod-}k\Sigma_d$ and let the functors be as in the diagram above. Then*

$$\mathcal{F}_m(U) \cong \widetilde{\mathcal{F}}\bigl(\mathcal{J}\bigl(\mathcal{G}(U)\bigr)\bigr).$$

Proof. We begin the proof with a well-known lemma:

Lemma 2.2. *Recall that $S(n,d) \cong \mathrm{End}_{k\Sigma_d}(V^{\otimes d})$. Then*

(i) $V^{\otimes d} \cong \bigoplus_{\lambda\in\Lambda(n,d)} M^\lambda$ *as $k\Sigma_d$-modules.*
(ii) $\xi_{\mathbf{i},\mathbf{j}} \in S(n,d)$ *corresponds to an element in* $\mathrm{Hom}_{k\Sigma_d}(M^\lambda, M^\mu)$ *where* $\lambda = \mathrm{wt}(\mathbf{i})$ *and* $\mu = \mathrm{wt}(\mathbf{j})$.

(iii) ξ_λ *corresponds to projection onto* M^λ.
(iv) η *corresponds to projection onto*

$$\bigoplus_{\substack{\lambda\in\Lambda(n,d)\\ \lambda_1=m}} M^\lambda.$$

Proof. To see (i), recall that $V^{\otimes d}$ has standard basis $\{e_{\mathbf{i}} := e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_d} \mid \mathbf{i}\in I(n,d)\}$. Under this identification, M^λ is spanned by the basis vectors $\{e_{\mathbf{i}} \mid \mathrm{wt}(\mathbf{i})=\lambda\}$. Parts (ii) and (iii) are immediate from the description in [3, 2.6a] of the action of $\xi_{\mathbf{i},\mathbf{j}}$ on $V^{\otimes d}$. Part (iv) follows from (iii). $\square$

To prove the theorem, let $U\in\text{mod-}k\Sigma_d$. Let $\mathbf{i}=(m+1,m+2,\dots,d,1,1,\dots,1)\in I^*(n,d)$ and let $\tilde{e}=\xi_{\mathbf{i},\mathbf{i}}$. Then $\tilde{e}$ is an idempotent, $\tilde{e}S_1\tilde{e}\cong k\Sigma_{d-m}$, and the Schur functor $\widetilde{\mathcal{F}}$ is multiplication by $\tilde{e}$.

Now $\mathcal{G}(U)\cong\mathrm{Hom}_{k\Sigma_d}(V^{\otimes d},U)$ is a left $S(n,d)$-module with action obtained from the right action of $S(n,d)$ on $V^{\otimes d}$. Thus the action of $S(n,d)$ (and hence of $\eta S(n,d)\eta$ and of S_1) is given by precomposing functions. That is if $f: V^{\otimes d}\to V^{\otimes d}$ is in $S(n,d)$ and $g: V^{\otimes d}\to U$ is in $\mathcal{G}(U)$, then $fg=g\circ f: V^{\otimes d}\to U$. So Lemma 2.2(iv) gives

$$\eta\mathcal{G}(U)\cong\mathrm{Hom}_{k\Sigma_d}\Bigg(\bigoplus_{\substack{\lambda\in\Lambda(n,d)\\ \lambda_1=m}} M^\lambda, U\Bigg).$$

But $\tilde{e}$ is projection onto $M^{(m,1^{d-m})}$, so we get

$$\tilde{e}\eta\mathcal{G}(U)\cong\mathrm{Hom}_{k\Sigma_d}\big(M^{(m,1^{d-m})},U\big)\cong\mathcal{F}_m(U). \qquad \square$$

3. Some general properties of $\mathcal{F}_m$

In this brief section we collect a few general properties of $\mathcal{F}_m$ which will be useful in determining how $\mathcal{F}_m$ acts on specific modules. We first remark that $\mathcal{F}_m$ has a left adjoint functor $\mathcal{G}_m : \text{mod-}k\Sigma_{d-m}\to\text{mod-}k\Sigma_d$ given by

$$\mathcal{G}_m(U) := \mathrm{Ind}_{\Sigma_m\times\Sigma_{d-m}}^{\Sigma_d}(k\otimes U).$$

The functor $\mathcal{G}_m$ is exact, but $\mathcal{F}_m$ is exact only when $m<p$, and is left exact in general. Thus $\mathcal{F}_m$ has higher right derived functors

$$R^i\mathcal{F}_m(U)=\mathrm{Ext}^i_{k\Sigma_d}\big(M^{(m,1^{d-m})},U\big).$$

A key fact in understanding $\mathcal{F}_m$ is that $R^1\mathcal{F}_m$ vanishes on dual Specht modules. A closely related fact (that the first higher right derived functor of the adjoint Schur functor $\mathcal{G}$ vanishes on dual Specht modules) played a key role in [5]. Some of the results below require the assumption $p > 3$. This seems to be similar to what was observed in [5]. Specifically when $p = 3$ it is possible for $H^1(\Sigma_d, S_\lambda)$ to be nonzero, and this has a dramatic effect on the results.

Proposition 3.1. *Let $p > 3$ and $\lambda \vdash d$. Then $R^1\mathcal{F}_m(S_\lambda) = 0$.*

Proof. We have

$$R^1\mathcal{F}_m(S_\lambda) \cong \mathrm{Ext}^1_{k\Sigma_d}\bigl(M^{(m,1^{d-m})}, S_\lambda\bigr) \cong \mathrm{Ext}^1_{k\Sigma_d}\bigl(S^\lambda, M^{(m,1^{d-m})}\bigr).$$

But the permutation modules M^μ are direct sums of Young modules, and $\mathrm{Ext}^1_{k\Sigma_d}(S^\lambda, Y^\mu)$ is always zero when $p > 3$ [10, 6.4b], so the result follows. □

Proposition 3.1 is false when $p = 3$. For example if $d = 5$ and $m = 3$, then

$$R^1\mathcal{F}_3\bigl(S_{(1^5)}\bigr) \cong \mathrm{Ext}^1_{k\Sigma_3}\bigl(k, \mathrm{Res}_{\Sigma_3}\bigl(S^{(1^5)}\bigr)\bigr) \cong \mathrm{Ext}^1_{k\Sigma_3}(k, \mathrm{sgn}) \neq 0.$$

Finally we recall that $\mathcal{F}_m$ was used in [4], where its image was determined on the modules S^λ, S_λ, and D_λ if $\lambda_1 \leqslant m < p$, i.e., when $\mathcal{F}_m$ is exact. In the case when $m = \lambda_1 < p$, the functor $\mathcal{F}_m$ "removes the first row" from S^λ, S_λ, and D_λ (i.e., maps S_λ to $S_{\bar{\lambda}}$, etc.). In the general case considered here (with m arbitrary), we will see that $\mathcal{F}_m$ acts as first row or first column removal only on the twisted modules, namely $S^\lambda \otimes \mathrm{sgn}$, $Y^\lambda \otimes \mathrm{sgn}$, and $M^\lambda \otimes \mathrm{sgn}$.

4. $\mathcal{F}_m$ on dual Specht modules

In this section we show that $\mathcal{F}_m$ behaves very nicely on dual Specht modules S_λ. We will need the notions of semistandard λ-tableaux of type μ and the basis of M^λ given by λ-tabloids, which are described in the book [7]. First we need a lemma.

Lemma 4.1. *The dimension of $\mathcal{F}_m(S_\lambda)$ is the number of semistandard λ-tableaux of type $(m, 1^{d-m})$. In particular, if $\lambda_1 < m$, then $\mathcal{F}_m(S_\lambda) = 0$.*

Proof. Since

$$\mathcal{F}_m(S_\mu) \cong \mathrm{Hom}_{k\Sigma_d}\bigl(M^{(m,1^{d-m})}, S_\lambda\bigr) \cong \mathrm{Hom}_{k\Sigma_d}\bigl(S^\lambda, M^{(m,1^{d-m})}\bigr),$$

the result follows from [7, Theorem 13.13], where a basis for $\mathrm{Hom}_{k\Sigma_d}(S^\lambda, M^\mu)$ is described for arbitrary λ and μ. In particular, if $\lambda_1 < m$ then there is no such tableau, so $\mathcal{F}_m(S_\lambda) = 0$. □

Thus we must consider $\mathcal{F}_m(S_\lambda)$ for $\lambda_1 \geqslant m$. The case $\lambda_1 = m$ follows from our work in Section 2 together with work of James:

Theorem 4.2. *Let $\lambda_1 = m$. Then $\mathcal{F}_m(S_\lambda) \cong S_{\bar{\lambda}}$.*

Proof. James showed in [8] that $\mathcal{J}(\Delta(\lambda)) \cong \Delta(\bar{\lambda})$. So

$$\begin{aligned} \mathcal{F}_m(S_\lambda) &\cong \widetilde{\mathcal{F}}\big(\mathcal{J}\big(\mathcal{G}(S_\lambda)\big)\big) \quad \text{by Theorem 2.1} \\ &\cong \widetilde{\mathcal{F}}\big(\mathcal{J}\big(\Delta(\lambda)\big)\big) \quad \text{by Proposition 1.1(i)} \\ &\cong \widetilde{\mathcal{F}}\big(\Delta(\bar{\lambda})\big) \\ &\cong S_{\bar{\lambda}} \quad \text{by (1.2).} \qquad \square \end{aligned}$$

In order to describe $\mathcal{F}_m(S_\lambda)$ in general, we must discuss skew diagrams and the corresponding skew Specht modules. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash d$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_s) \vdash t$ for $t < d$. Suppose $\mu_i \leqslant \lambda_i$ for all i (where μ_i is interpreted as 0 for $i > s$). Then the *skew diagram* $\lambda \backslash \mu$ is defined as

$$\lambda \backslash \mu := \big\{(i, j) \in \mathbf{N} \times \mathbf{N} \,\big|\, 1 \leqslant i \leqslant r,\ \mu_i \leqslant j \leqslant \lambda_i\big\}.$$

To each skew diagram there is associated a skew Specht module $S^{\lambda\backslash\mu}$ and its dual, which we denote $S_{\lambda\backslash\mu}$. Their construction can be found in [9], where it is shown that $S^{\lambda\backslash\mu}$ has a Specht filtration, and the filtration multiplicities are determined combinatorially. We will not describe the details of the construction here, but we will need the following from the paper of James and Peel.

Proposition 4.3 [9, Theorem 3.1]. *Let $\lambda \vdash d$. When restricted to $\Sigma_m \times \Sigma_{d-m}$, the module S^λ has a chain of submodules with factors isomorphic to $S^\beta \otimes S^{\lambda\backslash\beta}$, where each partition β of m such that $\lambda\backslash\beta$ exists occurs exactly once. If $m \leqslant \lambda_1$ (so $\lambda\backslash(m)$ exists), the filtration can be chosen so that $S^\beta \otimes S^{\lambda\backslash(m)}$ occurs on the top.*

James and Peel constructed a filtration of S^λ, but of course taking duals gives a filtration of S_λ by modules $S_\beta \otimes S_{\lambda\backslash\beta}$ with $S_{(m)} \otimes S_{\lambda\backslash(m)}$ as a submodule. We also need the following well-known lemma.

Lemma 4.4 [7, 13.17].

$$\operatorname{Hom}_{\Sigma_d}(k, S_\lambda) \cong \begin{cases} k, & \text{if } \lambda = (d), \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

We can now determine $\mathcal{F}_m(S_\lambda)$ in general.

Theorem 4.5. *Let $\lambda \vdash d$. Then*

$$\mathcal{F}_m(S_\lambda) \cong \begin{cases} S_{\lambda\backslash(m)}, & \text{if } \lambda_1 > m, \\ S_{\bar{\lambda}}, & \text{if } \lambda_1 = m, \\ 0, & \text{if } \lambda_1 < m. \end{cases}$$

Furthermore, if $p > 3$, then $\mathcal{F}_m$ takes modules with dual Specht filtrations to modules with dual Specht filtrations.

Proof. Only the case $\lambda_1 > m$ is not handled by Lemma 4.1 or Theorem 4.2. It is clear from the description of $\mathcal{F}_m$ that one way to get $\mathcal{F}_m(U)$ is to first restrict U to $\Sigma_m \times \Sigma_{d-m}$, and then take the largest subspace on which Σ_m acts trivially, which is a Σ_{d-m}-submodule. But we know the filtration in Proposition 4.3 can be chosen so $S_{(m)} \otimes S_{\lambda\setminus(m)}$ occurs as a submodule. Lemma 4.4 guarantees that no other S_β's have any Σ_m fixed-points, so the first part of result is immediate.

The second part follows by induction on the number of dual Specht modules in the filtration. We know from the work of James and Peel that $\mathcal{F}_m(S_\lambda) \cong S_{\lambda\setminus(m)}$ has a dual Specht module filtration. Now suppose U has a dual Specht filtration. Then we have

$$0 \to S_\mu \to U \to N,$$

where N has a dual Specht filtration. Applying $\mathcal{F}_m$ gives

$$0 \to \mathcal{F}_m(S_\mu) \to \mathcal{F}_m(U) \to \mathcal{F}_m(N) \to R^1\mathcal{F}_m(S_\mu).$$

However if $p > 3$, then $R^1\mathcal{F}_m(S_\mu) = 0$ by Proposition 3.1(iii), and $\mathcal{F}_m(N)$ has a dual Specht filtration by inductive hypothesis, so $\mathcal{F}_m(U)$ has a dual Specht filtration. $\square$

We close this section with a few observations. First, the statement corresponding to Lemma 4.4 is definitely false for S^λ. This will explain the much greater difficulty in determining $\mathcal{F}_m(S^\lambda)$. Also $\mathcal{F}_m(S_\lambda) \cong \mathrm{Hom}_{\Sigma_d}(M^{(m,1^{d-m})}, S_\lambda)$ has a basis of semistandard homomorphisms indexed by semistandard λ-tableau of type $(m, 1^{d-m})$. (See [7, Chapter 13] for details.) These tableaux are in obvious bijection with the set of standard $\lambda\setminus(m)$ tableaux, which index a basis for $S_{\lambda\setminus(m)}$. In this case the obvious bijection between the two bases does not extend to a $k\Sigma_{d-m}$-homomorphism.

5. $\mathcal{F}_m$ on permutation and twisted permutation modules

In [7, 13.19] James gives a basis for $\mathrm{Hom}_{\Sigma_d}(M^\lambda, M^\mu)$ indexed by row-standard λ-tableaux of type μ. So the dimension of $\mathcal{F}_m(M^\mu)$ is known, and it is not hard to determine the module structure.

Theorem 5.1.

$$\mathcal{F}_m\bigl(M^\mu\bigr) \cong \bigoplus_{\substack{\tau \models d-m \\ \tau_i \leqslant \mu_i \forall i}} M^\tau.$$

Proof. Since M^μ is a permutation module on μ-tabloids, a basis of $\mathcal{F}_m(M^\mu)$ is given by orbit sums of μ-tabloids under the action of Σ_m. It is easy to determine how Σ_{d-m} acts on these orbit sums. In particular, for $\tau \models d-m$, the summand M^τ in the theorem has a basis given by orbit sums of tabloids with exactly τ_i elements of $\{m+1, m+2, \ldots, d\}$ in row i. Is is elementary to check the permutation action on these orbit sums is exactly the action on τ-tabloids. $\square$

As an example of Theorem 5.1:

$$\mathcal{F}_3\big(M^{321}\big) = M^3 \oplus M^{210} \oplus M^{201} \oplus M^{120} \oplus M^{021} \oplus M^{111}$$
$$\cong M^3 \oplus \big(M^{21}\big)^{\oplus 4} \oplus M^{111}.$$

The analysis of $\mathcal{F}_m$ on twisted permutation modules is different. We begin with a lemma:

Lemma 5.2.

(i) $\mathrm{sgn} \cong S^{(1^d)}$.

(ii) $\mathrm{Hom}_{k\Sigma_d}\big(S^{(1^d)}, M^\lambda\big) \cong \begin{cases} k, & \textit{if } \lambda \in \Lambda(n,d) \textit{ has } d \textit{ ones and } n-d \textit{ zeroes,} \\ 0, & \textit{otherwise.} \end{cases}$

(iii) $\mathrm{Res}^{\Sigma_d}_{\Sigma_m}\big(M^\lambda\big) \cong \bigoplus_{\substack{\tau \in \Lambda(d,m) \\ \tau_i \leqslant \lambda_i}} \big(M^\tau\big)^{\oplus \dim M^{\lambda\setminus\tau}}$.

where $\dim M^{\lambda\setminus\tau}$ *is* $d!/(\Pi(\lambda_i - \tau_i)!)$.

Proof. Part (i) is well-known and (ii) is immediate from [7, 13.13]. Part (iii) can be seen either from Mackey's theorem or directly by considering the action of Σ_m on λ-tabloids. $\square$

We can now determine how $\mathcal{F}_m$ behaves on twisted permutation modules. In general, the image is much smaller than on permutation modules. In particular, if λ has m parts then $\mathcal{F}_m$ just sends $M^\lambda \otimes \mathrm{sgn}$ to $M^{\hat{\lambda}} \otimes \mathrm{sgn}$. In general:

Theorem 5.3. *For* $\lambda \vdash d$ *we have*

$$\mathcal{F}_m\big(M^\lambda \otimes \mathrm{sgn}\big) \cong \bigoplus_{\substack{\rho \models d-m \\ \lambda_i - 1 \leqslant \rho_i \leqslant \lambda_i}} M^\rho \otimes \mathrm{sgn}.$$

In particular, if λ *has fewer than* m *parts than* $\mathcal{F}_m(M^\lambda \otimes \mathrm{sgn}) = 0$ *and if* λ *has exactly* m *parts, then* $\mathcal{F}_m(M^\lambda \otimes \mathrm{sgn}) \cong M^{\hat{\lambda}} \otimes \mathrm{sgn}$.

Proof.

$$\begin{aligned}\mathcal{F}_m\big(M^\lambda \otimes \mathrm{sgn}\big) &\cong \mathrm{Hom}_{k\Sigma_m}\big(k, M^\lambda \otimes \mathrm{sgn}\big) \\ &\cong \mathrm{Hom}_{k\Sigma_m}\big(S^{(1^m)}, M^\lambda\big) \quad \text{since } S^{(1^m)} \cong \mathrm{sgn} \\ &\cong \mathrm{Hom}_{k\Sigma_m}\Bigg(S^{(1^m)}, \bigoplus_{\substack{\tau \models m \\ \tau_i \leqslant \lambda_i}} \big(M^\tau\big)^{\oplus \dim M^{\lambda\setminus\tau}}\Bigg) \quad \text{by Lemma 5.1(iii)}. \qquad (5.1)\end{aligned}$$

By Lemma 5.2(ii), the only τ which contribute to the direct sum in (5.1) are those with m ones. For each such τ, $\mathrm{Hom}_{k\Sigma_d}(S^{(1^d)}, M^\tau)$ is one-dimensional. So we get a space of dimension equal to that of M^ρ, where $\rho = \lambda\backslash\tau$ has $\lambda_i - 1 \leqslant \rho_i \leqslant \lambda_i$. Each map in (5.1) corresponds to a one-dimensional subspace of M^λ on which Σ_m acts by the sign representation, and the action of Σ_{d-m} is just its action on M^λ. Then it is just a matter of checking that the action of Σ_{d-m} on these maps is the same as the action on M^ρ. We leave this to the reader.

When λ has fewer than m parts, there is no such ρ, so $\mathcal{F}_m(M^\lambda \otimes \mathrm{sgn}) = 0$. When λ has exactly m parts, then $\hat{\lambda}$ is the only such ρ and $\mathcal{F}_m(M^\lambda \otimes \mathrm{sgn}) \cong M^{\hat{\lambda}} \otimes \mathrm{sgn}$. □

It is known that $\mathcal{G}(M^\lambda) \cong S^\lambda(V)$ and $\mathcal{G}(M^\lambda \otimes \mathrm{sgn}) \cong \Lambda^\lambda(V)$. Thus an alternative way to prove Theorems 5.1 and 5.3 would be to analyze how James' idempotent acts on the standard bases of $S^\lambda(V)$ and $\Lambda^\lambda(V)$, then apply Theorem 2.1. Combinatorially the analysis is of similar difficulty to our proof.

Since we are discussing semistandard homomorphisms, we will take the opportunity to observe that the tools being used in this paper give a very short proof of a theorem originally proved by James via a long combinatorial argument. In [7, Chapter 13], James constructed a basis for $\mathrm{Hom}_{\Sigma_d}(M^\lambda, M^\mu)$ indexed by row-standard λ-tableaux of type μ. He then showed that those maps corresponding to semistandard tableaux, when restricted to $S^\lambda \subseteq M^\lambda$, give a basis for $\mathrm{Hom}_{\Sigma_d}(S^\lambda, M^\mu)$. This indirectly shows that every element of $\mathrm{Hom}_{k\Sigma_d}(S^\lambda, M^\mu)$ extends to a map on M^λ. James remarked that he knew no direct proof of this fact, so we give one below when $p > 3$.

Proposition 5.4 [7, 13.15]. *Suppose $p > 3$. Then every element of $\mathrm{Hom}_{k\Sigma_d}(S^\lambda, M^\mu)$ can be extended to an element of $\mathrm{Hom}_{k\Sigma_d}(M^\lambda, M^\mu)$.*

Proof. We have

$$0 \to S^\lambda \to M^\lambda \to Q \to 0,$$

where Q has a Specht filtration. Apply $\mathrm{Hom}_{k\Sigma_d}(-, M^\mu)$ to the sequence to get

$$\cdots \to \mathrm{Hom}_{k\Sigma_d}\big(M^\lambda, M^\mu\big) \xrightarrow{\pi} \mathrm{Hom}_{k\Sigma_d}\big(S^\lambda, M^\mu\big) \to \mathrm{Ext}^1_{k\Sigma_d}\big(Q, M^\mu\big) \to \cdots. \tag{5.2}$$

But

$$\begin{aligned} \dim_k \mathrm{Ext}^1_{k\Sigma_d}\big(Q, M^\mu\big) &= \dim_k \mathrm{Ext}^1_{k\Sigma_d}\big(M^\mu, Q^*\big) \\ &\leqslant \dim_k \mathrm{Ext}^1_{k\Sigma_d}\big(V^{\otimes d}, Q^*\big) \quad \text{by Lemma 2.2(i)} \\ &= \dim_k R^1\mathcal{G}\big(Q^*\big) \\ &= 0 \quad \text{by Proposition 1.1(iii)}, \end{aligned}$$

since Q^* has a dual Specht filtration. Thus the map π in (5.2) must be a surjection, and the proposition follows. □

6. $\mathcal{F}_m$ on Young and twisted Young modules

Having determined how the fixed point functor acts on permutation modules and twisted permutation modules, we now turn to their direct summands, namely the Young modules Y^λ and twisted Young modules $Y^\lambda \otimes \mathrm{sgn}$. We will see, as in the previous section, that the twisted case is better, in the sense that when λ has m parts, $\mathcal{F}_m$ acts as first column removal on $Y^\lambda \otimes \mathrm{sgn}$. No such result holds for $\mathcal{F}_m(Y^\lambda)$. Indeed, we will show that determining all the $\mathcal{F}_m(Y^\lambda)$ is essentially equivalent to determining the decomposition matrix for the Schur algebra, a very difficult problem indeed!

We begin with a lemma made easy by the calculations in the previous section.

Lemma 6.1. *$\mathcal{F}_m(Y^\lambda)$ is a direct sum of Young modules for $k\Sigma_{d-m}$ while $\mathcal{F}_m(Y^\lambda \otimes \mathrm{sgn})$ is a direct sum of twisted Young modules for $k\Sigma_{d-m}$.*

Proof. Since Y^λ is a direct summand of M^λ, this is immediate from Theorems 5.1 and 5.3.

We can determine $\mathcal{F}_m(Y^\lambda \otimes \mathrm{sgn})$ precisely in some cases:

Theorem 6.2. *Let $p > 3$. If λ has fewer than m parts, then $\mathcal{F}_m(Y^\lambda \otimes \mathrm{sgn}) = 0$. If λ has exactly m parts, then*

$$\mathcal{F}_m\big(Y^\lambda \otimes \mathrm{sgn}\big) \cong Y^{\hat{\lambda}} \otimes \mathrm{sgn}.$$

Proof. From Lemma 6.1 we know $\mathcal{F}_m(Y^\lambda \otimes \mathrm{sgn})$ is a direct sum of twisted Young modules. Twisted Young modules have filtrations by both Specht and dual Specht modules, and the matrix giving the multiplicities $[Y^\lambda : S_\mu]$ is triangular. Thus, if we can prove $\mathcal{F}_m(Y^\lambda \otimes \mathrm{sgn})$ has a filtration by dual Specht modules with the same multiplicities as in a dual Specht filtration of $Y^{\hat{\lambda}} \otimes \mathrm{sgn}$, we can conclude that it is indeed $Y^{\hat{\lambda}} \otimes \mathrm{sgn}$. So let

$$Y^\lambda \otimes \mathrm{sgn} = S^\lambda \otimes \mathrm{sgn} + \sum_{\mu \rhd \lambda} a_{\lambda\mu}\big(S^\mu \otimes \mathrm{sgn}\big) = S_{\lambda'} + \sum_{\mu \rhd \lambda} a_{\lambda\mu} S_{\mu'}, \tag{6.1}$$

where by the “sum” in (6.1), we mean the modules have filtrations with the factors listed in the sum. Now, appealing to the second part of Theorem 4.5, it makes sense to apply $\mathcal{F}_m$ to both sides of (6.1). Notice first that if λ has fewer than m parts, then λ' and all the μ'’s in the sum have first row less than m, so the corresponding $S_{\mu'}$’s are annihilated by $\mathcal{F}_m$. So $Y^\lambda \otimes \mathrm{sgn}$ has a filtration by dual Specht modules, all of which are annihilated by $\mathcal{F}_m$. Any such module must be annihilated by $\mathcal{F}_m$. So now assume λ has m parts. Then applying $\mathcal{F}_m$ to (6.1) gives

$$\mathcal{F}_m\big(Y^\lambda \otimes \mathrm{sgn}\big) = S_{\overline{\lambda'}} + \sum_{\mu_1 = m} a_{\lambda\mu} S_{\overline{\mu'}}. \tag{6.2}$$

Now suppose

$$Y^{\hat{\lambda}} \otimes \mathrm{sgn} = S^{\hat{\lambda}} \otimes \mathrm{sgn} + \sum_{\tau \rhd \hat{\lambda}} b_{\hat{\lambda}\tau}\bigl(S^{\tau} \otimes \mathrm{sgn}\bigr) = S_{\hat{\lambda}'} + \sum_{\tau \rhd \hat{\lambda}} b_{\hat{\lambda}\tau} S_{\tau'}. \tag{6.3}$$

Since $\tau \rhd \hat{\lambda}$ and $\hat{\lambda}$ has $\leqslant m$ parts, then τ has $\leqslant m$ parts. Thus all the τ appearing in (6.3) are of the form $\hat{\mu}$ for some $\mu \vdash d$ with $\mu_1 = m$. This fact plus the observation that $\hat{\lambda}' = \overline{\lambda'}$ for any λ lets us rewrite (6.3) as

$$Y^{\hat{\lambda}} \otimes \mathrm{sgn} = S_{\overline{\lambda'}} + \sum_{\mu_1 = m} b_{\hat{\lambda}\hat{\mu}} S_{\overline{\mu'}}. \tag{6.4}$$

However by (1.3) the $a_{\lambda\mu}$ are decomposition numbers for $S(n,d)$. Specifically,

$$a_{\lambda\mu} = \bigl[\Delta(\mu) : L(\lambda)\bigr].$$

But James proved in [8] that first column removal preserves decomposition numbers (essentially just by tensoring with the determinant). Thus $a_{\lambda\mu} = b_{\hat{\lambda}\hat{\mu}}$, so the sums in (6.2) and (6.4) coincide, as desired. $\square$

We remark that if λ is p-restricted, then a twisted Young module is also a Young module. In particular, $Y^{\lambda} \otimes \mathrm{sgn} \cong Y^{m(\lambda)}$ where $m(\lambda)$ is the Mullineux map given by $D_{\lambda} \otimes \mathrm{sgn} \cong D_{m(\lambda)}$. Thus we have determined how $\mathcal{F}_m$ behaves on projective Young modules. We are not sure whether the theorem holds for $p = 3$. The problem is that applying $\mathcal{F}_m$ to "sums" like (6.1) is not justified, since $R^1\mathcal{F}_m$ may be nonzero on dual Specht modules.

We also remark that Theorems 5.3 and 6.2 give a column removal theorem for p-Kostka numbers, which is already known.

Theorem 6.3. *Let λ, $\mu \vdash d$ both have m parts. Then*

$$\bigl[M^{\mu} : Y^{\lambda}\bigr] = \bigl[M^{\hat{\mu}} : Y^{\hat{\lambda}}\bigr].$$

Proof. As we mentioned, the result would follow from Theorems 5.3 and 6.2 since for Y^{λ} a summand of M^{μ}, λ cannot have more parts than μ. However, it is easier to recall the well-known fact that the p-Kostka number $[M^{\mu} : Y^{\lambda}]$ is the same as the dimension of the μ-weight space in $L(\lambda)$, so tensoring with the determinant gives the result. $\square$

The row-removal version of Theorem 6.3 has been conjectured by Henke [6, Conjecture 6.3].

We wish now to convince the reader that determining the multiplicities of Young modules for Σ_{d-m} in $\mathcal{F}_m(Y^{\lambda})$ is a very difficult problem by showing it is equivalent to knowing the decomposition numbers for the Schur algebra.

For the base case in the inductive proof of the main theorem of this section, we will need to know $[Y^\lambda : S_{(d)}]$. Recall from (1.3) that

$$\left[Y^\mu : S_{(d)}\right] = \left[\nabla(d) : L(\mu)\right].$$

These numbers are known from work of Doty. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_s) \vdash d$. Define a sequence of nonnegative integers $\alpha_i(\mu)$ as follows. First write each μ_i out in base p. Then add them all together. For $i \geqslant 1$, let α_i be the number that is "carried" to the top of the p^i column during the addition. For example, let $p = 3$ and $\mu = (5, 5, 2)$. Then adding $5 + 5 + 2$ base three gives

$$\begin{array}{r} {\scriptstyle 1\ \ 2\ \ \ } \\ 1\ 2 \\ 1\ 2 \\ +\ 0\ 2 \\ \hline 1\ 1\ 0 \end{array}$$

and $\alpha(\mu) = (2, 1)$. Doty calls this the *carry pattern* of μ. Then we have the following lemma.

Lemma 6.4 [1, Section 2.4]. *The multiplicity $[\nabla(d) : L(\mu)]$ is either one or zero. It is one precisely when μ is maximal among all partitions of d with the same carry pattern as μ.*

As an aside, we point out here that Doty's work plus the work in [5] allows a determination of which Young modules have a fixed point.

Proposition 6.5.

$$\left[Y^\mu : S_{(d)}\right] = \dim_k \operatorname{Hom}_{k\Sigma_d}\left(k, Y^\mu\right).$$

Proof. We know $[Y^\mu : S_{(d)}] = [\Delta(d) : L(\mu)]$ (which is known by the previous lemma). But

$$\begin{aligned} \left[\Delta(d) : L(\mu)\right] &= \dim \operatorname{Hom}_{S(n,d)}\left(P(\mu), \Delta(d)\right) \\ &= \dim \operatorname{Hom}_{S(n,d)}\left(P(\mu), \mathcal{G}(k)\right) \quad \text{by Proposition 1.1(i)} \\ &= \dim \operatorname{Hom}_{k\Sigma_d}\left(Y^\mu, k\right) \end{aligned}$$

by the adjointness of $\mathcal{G}$ and $\mathcal{F}$. But Y^μ is self-dual so the result follows. □

Now we will show that determining $\mathcal{F}_m(Y^\lambda)$ is essentially equivalent to determining the decomposition numbers for $S(n, d)$. Specifically:

Theorem 6.6. *Let $p > 3$. Suppose we know the decomposition numbers for $S(n, r)$ for $r \leqslant d$. Then we can determine $\mathcal{F}_m(Y^\lambda)$ for all m and all $\lambda \vdash d$. Conversely suppose we* *know $\mathcal{F}_m(Y^\lambda)$ for all m and all partitions $\lambda \vdash r$, for $r \leqslant d$. Then we can determine all the decomposition numbers for $S(n,r)$ for $r \leqslant d$.*

Proof. First suppose we know all $[\Delta(\lambda) : L(\mu)]$, or equivalently by (1.3), suppose we know all the multiplicities $[Y^\mu : S_\lambda]$ in dual Specht filtrations of the Young modules. We will calculate $\mathcal{F}_m(Y^\lambda)$ inductively. $Y^{(d)} \cong k$ so $\mathcal{F}_m(Y^{(d)}) = Y^{(d-m)}$. Now assume we know $\mathcal{F}_m(Y^\tau)$ for all $\tau \rhd \mu$, we must determine $\mathcal{F}_m(Y^\mu)$.

We know how to write $\mathcal{F}_m(M^\mu)$ as a direct sum of permutation modules. However, Young's rule [7, Chapter 14] tells us the filtration multiplicities in a Specht filtration of any permutation module, so we can write

$$\mathcal{F}_m\big(M^\mu\big) = \sum_{\lambda \vdash d-m} c_{\mu\lambda} S_\lambda, \tag{6.5}$$

where by the summation we again mean the module has a dual Specht filtration with the factors in the sum. We also know

$$M^\mu \cong Y^\mu \bigoplus_{\tau \rhd \mu} K_{\mu\tau}\big(Y^\tau\big), \tag{6.6}$$

where the p-Kostka numbers $K_{\mu\tau}$ can be determined from the decomposition numbers. So we have a Specht series of $\mathcal{F}_m(M^\mu)$. We also know $\mathcal{F}_m(Y^\tau)$ for all $\tau \rhd \mu$. So Eqs. (6.5) and (6.6) together with knowledge of the $\mathcal{F}_m(Y^\tau)$'s let us calculate the Specht filtration multiplicities in $\mathcal{F}_m(Y^\mu)$. But we already know that $\mathcal{F}_m(Y^\mu)$ is a direct sum of Young modules for $k\Sigma_{d-m}$, so the decomposition numbers of $S(n, d-m)$ let us determine precisely which Young modules.

Conversely suppose we know $\mathcal{F}_m(Y^\mu)$ for all m, and for all $\mu \vdash r$ with $r \leqslant d$. Inductively we can assume we know the decomposition numbers for $S(n,r)$ with $r < d$, and we will obtain the decomposition numbers for $S(n,d)$. That is, we need to get all the $[Y^\mu : S_\lambda]$. We will proceed inductively on λ. Lemma 6.4 provides the base case of the induction, i.e., give us $[Y^\mu : S_{(d)}]$. Now assume we have calculated $[Y^\mu : S_\lambda]$ for all μ and for all $\lambda \rhd \tau$. We must determine $[Y^\mu : S_\tau]$. We will actually simultaneously get $[Y^\mu : S_\sigma]$ for all σ with $\sigma_1 = \tau_1$. Write

$$Y^\mu = \sum_{\rho_1 < \tau_1} a_{\mu\rho} S_\rho + \sum_{\rho_1 = \tau_1} b_{\mu\rho} S_\rho + \sum_{\rho_1 > \tau_1} c_{\mu\rho} S_\rho, \tag{6.7}$$

where we know the $\{c_{\mu\rho}\}$ by inductive hypothesis. Now apply $\mathcal{F}_{\tau_1}$ to (6.7) and use Theorem 4.5 to get

$$\mathcal{F}_m\big(Y^\mu\big) = \sum_{\rho_1 = \tau_1} b_{\mu\rho} S_{\bar{\rho}} + \sum_{\rho_1 > \tau_1} c_{\mu\rho}\big(S_{\rho\setminus(\tau_1)}\big). \tag{6.8}$$

The dual Specht filtration multiplicities in the right-hand sum in (6.8) are known since James and Peel [9] give a dual Specht filtration of the $S_{\rho\setminus(\tau_1)}$. Also by assumption we can write $\mathcal{F}_m(Y^\mu)$ as a direct sum of Young modules, and we can then write out a dual

Specht filtration for it, since we are assuming we know the decomposition numbers for $S(n, d-\tau_1)$. Equating the multiplicities on both sides of (6.8) lets us solve for all the $\{b_{\mu\rho}\}$ with $\rho_1 = \tau_1$ as desired. $\square$

It is elementary to carry out either of the computations described in the proof of the theorem. In Appendix A we present the images of Y^λ for $d = 10$, $p = 5$ (where all the decomposition numbers are known) under $\mathcal{F}_5$. It would be interesting to get a more explicit relation between the data in Appendix A and decomposition numbers.

7. $\mathcal{F}_m$ on simple modules and Specht modules

Determining $\mathcal{F}_m(D_\lambda)$ and $\mathcal{F}_m(S^\lambda)$ seems to be very difficult. Indeed, we suspect, although cannot prove, that either or both problem may be as difficult as determining decomposition numbers, similar to Theorem 6.6. The only thing we can say comes from the following lemma:

Lemma 7.1 [2, Lemma 2.3]. *Let λ be p-restricted. Then $\mathcal{G}(D_\lambda)$ has simple socle isomorphic to $L(\lambda)$.*

Thus we have:

Proposition 7.2. *Let λ be p-restricted and $\lambda_1 = m$. Then*

(i) $D_{\bar{\lambda}} \subseteq \mathrm{soc}(\mathcal{F}_m(D_\lambda))$.
(ii) *If $\mathcal{G}(D_\lambda)$ is simple, then $\mathcal{F}_m(D_\lambda) \cong D_{\bar{\lambda}}$.*
(iii) [4, 5.5] *If $m < p$, then $\mathcal{F}_m(D_\lambda) \cong D_{\bar{\lambda}}$.*

Proof. Parts (i) and (ii) are both immediate from Theorem 2.1 since James proved in [8] that $\mathcal{J}(L(\lambda)) \cong L(\bar{\lambda})$. $\square$

We have some evidence that $\mathcal{G}(D_\lambda)$ is simple about half the time, usually either $\mathcal{G}(D_\lambda)$ or $\mathcal{G}(D_\lambda \otimes \mathrm{sgn})$ is simple. A case where $\mathcal{G}(D_\lambda)$ is known to be simple is if D_λ is a completely splittable module [10].

We cannot say much about $\mathcal{F}_m(S^\lambda)$ either. In [4] it was shown that if $\lambda_1 = m < p$, then $\mathcal{F}_m(S^\lambda) \cong S^{\bar{\lambda}}$. The corresponding statement to Theorem 4.5 is definitely false. That is, even if $\lambda_1 < m$, it is possible for $\mathcal{F}_m(S^\lambda)$ to be nonzero. The filtration of S^λ as a $\Sigma_m \times \Sigma_{d-m}$ module given in Theorem 4.3 would not have any terms of the form $S^{(m)} \otimes S^{\lambda \setminus (m)}$. But this does not rule out Σ_m fixed points since nontrivial Specht modules S^β can still have fixed points. We renew our conjecture from [4]:

Conjecture 7.3. *$\mathcal{F}_m(S^\lambda)$ has a Specht filtration as a $k\Sigma_m$-module.*

Perhaps a better understanding of the filtration given in Proposition 4.3 would be useful to attack the conjecture. If $S^\beta \otimes S^{\lambda\setminus\beta}$ occurs in the filtration, it is known whether S^β has a fixed point under Σ_m. The question is whether this fixed point "drops" to the bottom of the filtration, and whether it takes all of $S^{\lambda\setminus\beta}$ with it, so to speak. This seems to be difficult to determine.

Another possibility is to consider S^λ as a subspace of M^λ. We know precisely a basis for $\mathcal{F}_m(M^\lambda)$ as a subspace of M^λ so to determine $\mathcal{F}_m(S^\lambda)$ we just need to know $S^\lambda \cap \mathcal{F}_m(M^\lambda)$. But S^λ as a subspace of M^λ is given by the kernel intersection theorem of James [7, 17.18]. However, it seems to be very hard to combinatorially determine the subspace of $\mathcal{F}_m(M^\lambda)$ which lies in the intersection of the kernels.

Appendix A. $\mathcal{F}_5$ on Young modules for Σ_{10}

Below are the images of the Young modules for Σ_{10} under the functor $\mathcal{F}_5$ in characteristic $p = 5$. Young modules not listed are annihilated by the functor.

$$\mathcal{F}_5(Y^{10}) \cong Y^5, \qquad \mathcal{F}_5(Y^{541}) \cong (Y^{41})^{\oplus 2} \oplus Y^{31^2},$$
$$\mathcal{F}_5(Y^{91}) \cong Y^5 \oplus Y^{41}, \qquad \mathcal{F}_5(Y^{532}) \cong Y^{32},$$
$$\mathcal{F}_5(Y^{82}) \cong Y^{41} \oplus Y^{32}, \qquad \mathcal{F}_5(Y^{531^2}) \cong (Y^{31^2})^{\oplus 2} \oplus Y^{21^3},$$
$$\mathcal{F}_5(Y^{81^2}) \cong Y^{41} \oplus Y^{31^2}, \qquad \mathcal{F}_5(Y^{52^21}) \cong Y^{2^21},$$
$$\mathcal{F}_5(Y^{73}) \cong Y^{41} \oplus Y^{32}, \qquad \mathcal{F}_5(Y^{521^3}) \cong (Y^{21^3})^{\oplus 2} \oplus Y^{1^5},$$
$$\mathcal{F}_5(Y^{721}) \cong Y^{32} \oplus Y^{31^2} \oplus Y^{2^21}, \qquad \mathcal{F}_5(Y^{51^5}) \cong (Y^{1^5})^{\oplus 2},$$
$$\mathcal{F}_5(Y^{71^3}) \cong Y^{31^2} \oplus Y^{21^3}, \qquad \mathcal{F}_5(Y^{4^22}) \cong (Y^{41})^{\oplus 2},$$
$$\mathcal{F}_5(Y^{64}) \cong (Y^{41})^{\oplus 2} \oplus Y^{32}, \qquad \mathcal{F}_5(Y^{43^2}) \cong Y^{32} \oplus Y^{31^2},$$
$$\mathcal{F}_5(Y^{631}) \cong (Y^{32})^{\oplus 2} \oplus (Y^{31^2})^{\oplus 2} \oplus Y^{2^21}, \qquad \mathcal{F}_5(Y^{4321}) \cong Y^{31^2},$$
$$\mathcal{F}_5(Y^{62^2}) \cong Y^{32} \oplus Y^{2^21}, \qquad \mathcal{F}_5(Y^{42^21^2}) \cong Y^{21^3},$$
$$\mathcal{F}_5(Y^{621^2}) \cong Y^{2^21} \oplus Y^{21^3}, \qquad \mathcal{F}_5(Y^{41^6}) \cong Y^{1^5},$$
$$\mathcal{F}_5(Y^{61^4}) \cong Y^{21^3} \oplus Y^{1^5}, \qquad \mathcal{F}_5(Y^{3^31}) \cong Y^{41},$$
$$\mathcal{F}_5(Y^{5^2}) \cong Y^5 \oplus Y^{41}, \qquad \mathcal{F}_5(Y^{3^22^2}) \cong Y^{41}.$$

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