Representation type of Schur superalgebras

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Abstract. Let $S(m|n, d)$ be the Schur superalgebra whose supermodules correspond to the polynomial representations of the supergroup $\mathrm{GL}(m|n)$ of degree d. In this paper we determine the representation type of these algebras (that is, we classify the ones which are semisimple, have finite, tame and wild representation type). Moreover, we prove that these algebras are in general not quasi-hereditary and have infinite global dimension.

1   Introduction

1.1   A central problem in the study of finite-dimensional algebras is to understand the structure of the indecomposable modules. As a first step one would like to know how many indecomposable modules a given algebra admits. Any finite-dimensional algebra can be classified into one of three categories: finite, tame, or wild representation type. A finite-dimensional algebra A has *finite representation type* if A has finitely many indecomposable modules up to isomorphism. If A is not of finite type then A is of *infinite representation type*. Algebras of infinite representation type are either of *tame representation type* or of *wild representation type*. For algebras of tame representation type one has a chance of classifying all the indecomposable modules up to isomorphism.

There has been much progress in the past ten years in determining the representation type of important classes of finite-dimensional algebras. The Schur algebras and the q-Schur algebra are finite-dimensional algebras which arise in the representation theory of the general linear groups and symmetric groups. A complete classification of the representation type of these algebras was given by Doty, Erdmann, Martin and Nakano [10], [11], [13], [14]. These algebras are fundamental examples of quasi-hereditary algebras (or equivalently highest weight categories) which were introduced by Cline, Parshall and Scott [8]. The blocks for the ordinary and parabolic BGG category $\mathcal{O}$ for finite-dimensional complex semisimple Lie algebras are other important examples of highest weight categories. Results pertaining to the represent-

ation type for these blocks were obtained independently by Futorny, Nakano and Pollack [16], and Brüstle, König and Mazorchuk [6] for the ordinary category $\mathcal{O}$ and more recently for the parabolic category $\mathcal{O}$ by Boe and Nakano [3]. For quasi-hereditary algebras the projective modules admit filtrations by certain standard modules. This information allows one to determine the structure of the projective modules which in turn can lead to an expression of the algebra via quiver and relations. In many cases this information can be used to determine the representation type of the algebra.

Other important classes of finite-dimensional algebras are quasi-Frobenius algebras, where all projective modules are injective. These algebras all have infinite global dimension (except when the algebra is semisimple). For group algebras, restricted enveloping algebras and Frobenius kernels, homological information involving the theory of complexity and support varieties can be used to deduce information about the representation type of these algebras. For Hecke algebras of type A, the representation type of the blocks was given by Erdmann and Nakano in [15]. Ariki and Mathas [2] classified the representation type for Hecke algebras of type B using Fock space methods. Ariki [1] recently has extended this classification to Hecke algebras of classical type.

The main result of this paper is a complete classification of the representation type for the Schur superalgebras $S(m|n, d)$. These algebras are of particular interest because the category of $S(m|n, d)$-supermodules is equivalent to the category of polynomial representations of degree d for the supergroup $\mathrm{GL}(m|n)$. Also, as with the classical Schur algebra, there is a Schur–Weyl duality between $S(m|n, d)$ and the symmetric group Σ_d (see [5]). We shall show that in most cases $S(m|n, d)$ is not quasi-hereditary, contrary to a recent conjecture of Marko and Zubkov. Since the Schur superalgebras are not quasi-hereditary the aforementioned filtration techniques cannot be used. In order to obtain information about the basic algebra for certain Schur superalgebras we will compute the endomorphism algebras of direct sums of signed Young modules. This will entail knowing the structures of certain signed Young modules, which is of independent interest from the viewpoint of the representation theory of symmetric groups.

1.2 We review below the results for the ordinary Schur algebra $S(m, d)$, although the proof for $S(m|n, d)$ does not rely on these results. The representation type of $S(m, d)$ was determined in [10][1], [11], [13].

Theorem 1.2.1. *Let $S(m, d)$ be the Schur algebra over k where* $\mathrm{char}\, k = p \geqslant 0$.

(a) $S(m, d)$ *is semisimple if and only if one of the following holds*:
   (i) $p = 0$;
   (ii) $d < p$;
   (iii) $p = 2$, $m = 2$, $d = 3$.

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[1] The statement that $S(2, 11)$ for $p = 2$ has tame representation type was inadvertently omitted from the original statement of the theorem [10, Theorem 1.2(A)]. We thank Alison Parker for pointing this out to us.

(b) $S(m, d)$ *has finite representation type if and only if one of the following holds*:
    (i) $p \geqslant 2$, $m \geqslant 3$, $d < 2p$;
    (ii) $p \geqslant 2$, $m = 2$, $d < p^2$;
    (iii) $p = 2$, $m = 2$, $d = 5, 7$.

(c) $S(m, d)$ *has tame representation type if and only if one of the following holds*:
    (i) $p = 3$, $m = 3$, $d = 7, 8$;
    (ii) $p = 3$, $m = 2$, $d = 9, 10, 11$;
    (iii) $p = 2$, $m = 2$, $d = 4, 9, 11$.

(d) *In all other cases not listed above, $S(m, d)$ has wild representation type.*

The classification of the representation type of the q-Schur algebras $S_q(m, d)$ can be found in [14, Theorem 1.4(A)–(C)].

1.3 Throughout this paper we will assume that k is an algebraically closed field of characteristic $p \geqslant 0$. The following theorem summarizes our results on the Schur superalgebras.

Theorem 1.3.1. *Let $S(m|n, d)$ be the Schur superalgebra over k where* $\operatorname{char} k \neq 2$.

(a) $S(m|n, d)$ *is semisimple if and only if one of the following holds*:
    (i) $\operatorname{char} k = 0$;
    (ii) $d < p$;
    (iii) $m = n = 1$ *and* $p \nmid d$.

(b) $S(m|n, d)$ *has finite representation type but is not semisimple if and only if one of the following holds*:
    (i) $p \leqslant d < 2p$;
    (ii) $m = n = 1$ *and* $p | d$.

(c) *In all other cases not listed above, $S(m|n, d)$ has wild representation type.*

We should remark that in characteristic two $S(m|n, d)$ is equal to the ordinary Schur algebra $S(m + n, d)$, and its representation type is classified by Theorem 1.2.1.

2 Comparing module categories for superalgebras

2.1 Recall that k denotes our fixed ground field. All objects defined over k (algebras, modules, superalgebras, supermodules, etc.) are assumed to be finite-dimensional as k-vector spaces.

A *superspace* is a $\mathbb{Z}_2$-graded k-vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Given a homogeneous element v of a superspace V, we write $\bar{v} \in \mathbb{Z}_2$ for the degree of v. If V and W are superspaces, then $\operatorname{Hom}_k(V, W)$ is naturally a superspace with $\operatorname{Hom}_k(V, W)_{\bar{0}}$ (resp. $\operatorname{Hom}_k(V, W)_{\bar{1}}$) consisting of all linear maps which preserve (resp. reverse) grading.

A *superalgebra* is a $\mathbb{Z}_2$-graded associative k-algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ which satisfies $A_r A_s \subseteq A_{r+s}$ for all $r, s \in \mathbb{Z}_2$. An *A-supermodule* is a $\mathbb{Z}_2$-graded A-module

$M = M_{\bar{0}} \oplus M_{\bar{1}}$ which satisfies $A_r M_s \subseteq M_{r+s}$ for all $r, s \in \mathbb{Z}_2$. An A-supermodule homomorphism is a linear map $f : M \to N$ which satisfies $f(am) = (-1)^{\bar{f}\bar{a}} af(m)$ for all homogeneous $a \in A$ and all $m \in M$. The given condition makes sense only for homogeneous elements and it will be used later for such elements without comment. The general case is obtained by linearity. Note that if M and N are A-supermodules, then $\operatorname{Hom}_A(M, N)$ is naturally a superspace just as in the previous paragraph.

We can define the *parity change functor* Π on the category of A-supermodules as follows. On an object M we let $\Pi M = M$ as a vector space, but we switch the $\mathbb{Z}_2$-grading by setting $(\Pi M)_r = M_{r+\bar{1}}$ for $r \in \mathbb{Z}_2$. The action of A is defined via $a.m = (-1)^{\bar{a}} am$ for homogeneous $a \in A$ and $m \in M$. On a morphism f, Πf is the same linear map as f. We refer the reader to [4, Section 2] for further discussion of general results on superalgebras, supermodules, etc.

Throughout the remainder of this subsection we assume that $\operatorname{char} k \neq 2$. Given a superalgebra A there are three obvious module categories to consider:

 (i) the category A-smod of all A-supermodules and *all* (not necessarily graded) A-supermodule homomorphisms;

 (ii) the underlying even category of A-smod where we take the objects of A-smod but only the *even* (i.e. grading-preserving) homomorphisms;

(iii) the category A-mod of all A-modules.

Observe that an A-supermodule is indecomposable or irreducible regardless of whether we view it as an object of the category A-smod or its underlying even category. If M and N are indecomposable A-supermodules and $f : M \to N$ is an A-supermodule isomorphism, write $f = f_{\bar{0}} + f_{\bar{1}}$ with $f_r \in \operatorname{Hom}_A(M, N)_r$ for $r \in \mathbb{Z}_2$. By considering the $\mathbb{Z}_2$-grading it is straightforward to verify that $f_{\bar{0}} - f_{\bar{1}} : M \to N$ is also an isomorphism. Using this one can show that the map $M \oplus \Pi M \to N \oplus \Pi N$ given by

$$(m, n) \mapsto (f_{\bar{0}}(m) + f_{\bar{1}}(n), f_{\bar{1}}(m) + f_{\bar{0}}(n))$$

is in fact an *even* isomorphism of A-supermodules. Consequently, by the graded version of the Krull–Schmidt theorem, we see that M is isomorphic to either N or ΠN by an even isomorphism. The above discussion proves that the category A-smod is semisimple, finite, tame, or wild if and only if its underlying even category is semisimple, finite, tame, or wild, respectively. We use this observation without comment in what follows. The remainder of this section is devoted to studying the relationship between the representation type of these categories and the category A-mod.

Given a superspace V, let $\delta : V \to V$ be the linear involution defined by $\delta(v) = (-1)^{\bar{v}} v$, where $v \in V$ is homogeneous. In particular, the involution $\delta : A \to A$ is an algebra homomorphism and defines an action of $\mathbb{Z}_2$ on A. Consequently, we can define the smash product of $k\mathbb{Z}_2$ and A as follows. Set

$$B = k\mathbb{Z}_2 \otimes_k A$$

as a vector space. Identify $k\mathbb{Z}_2$ with $k[x]/(x^2 - 1)$. Then the product in B is given by

$$(x^r \otimes a)(x^s \otimes b) = x^{r+s} \otimes \delta^s(a)b, \tag{2.1.1}$$

where $r, s \in \{0, 1\}$ and $a, b \in A$. We identify $k\mathbb{Z}_2$ and A as subalgebras via $k\mathbb{Z}_2 \otimes 1$ and $1 \otimes A$, respectively. Note that since $\operatorname{char} k \neq 2$, x acts semisimply on any $k\mathbb{Z}_2$-module with eigenvalues ± 1. The following result is proven in [26, Theorem 1.4b].

Proposition 2.1.2. *The category of B-modules is isomorphic to the underlying even category of A-supermodules.*

Proof. Let M be an A-supermodule. We define an action of $k\mathbb{Z}_2$ on M via $xm = \delta(m)$. It is straightforward to verify that this makes M into a B-module. Conversely, if M is a B-module, then we can view it as an A-module by restriction. To obtain a $\mathbb{Z}_2$-grading we set $M_{\bar{0}}$ (resp. $M_{\bar{1}}$) to be the 1 (resp. -1) eigenspace of x acting on M. One can verify that this $\mathbb{Z}_2$-grading makes M into an A-supermodule. It is straightforward to verify that a morphism in one category restricts or extends to a morphism in the other category. $\square$

2.2 There is an involution $\zeta : B \to B$ given by $x^r \otimes a \mapsto (-1)^r x^r \otimes a$ for $r \in \{0, 1\}$ and $a \in A$. We can twist a B-module M by ζ to obtain a new B-module which we denote by M^ζ. Namely, $M^\zeta = M$ as a vector space but with the action of B given by $b.m = \zeta(b)m$ for all $b \in B$ and $m \in M$. Note that M^ζ is indecomposable if and only if M is indecomposable.

Since A is a subalgebra of B, we have a restriction functor $\operatorname{res}_A^B : B\text{-mod} \to A\text{-mod}$, and an induction functor $\operatorname{ind}_A^B : A\text{-mod} \to B\text{-mod}$ given by

$$\operatorname{ind}_A^B N = B \otimes_A N$$

for any A-module N. Note that

$$\operatorname{ind}_A^B N \cong 1 \otimes N \oplus x \otimes N \tag{2.2.1}$$

as A-modules. If M is a B-module, then we have

$$M \oplus M^\zeta \cong \operatorname{ind}_A^B \operatorname{res}_A^B M. \tag{2.2.2}$$

Since x acts semisimply on M and M^ζ, it suffices to define the isomorphism on eigenvectors $m \in M$, $m' \in M^\zeta$ of eigenvalue $\varepsilon, \varepsilon' \in \{\pm 1\}$, respectively. Then the map $M \oplus M^\zeta \to \operatorname{ind}_A^B \operatorname{res}_A^B M$ is given by

$$(m, m') \mapsto (1 + \varepsilon x) \otimes m + (1 - \varepsilon' x) \otimes m'.$$

To verify that the map is B-linear, one uses the observation that if $a \in A$ is an

eigenvector for the action of x with eigenvalue α and $m \in M$ is an eigenvector for the action of x with eigenvalue β, then am is an eigenvector with eigenvalue $\alpha\beta$.

Proposition 2.2.3. *Let M be an indecomposable B-module. Then M is either indecomposable as an A-module or the direct sum of two indecomposable A-modules. Furthermore, every indecomposable A-module N is a direct summand of $\operatorname{ind}_A^B N$ as A-modules.*

On the other hand, if M is an indecomposable B-module which contains N as a direct summand, then M is unique up to isomorphism and twisting by ζ.

Proof. Let M be an indecomposable B-module and let

$$M = M_1 \oplus \cdots \oplus M_k \tag{2.2.4}$$

be a decomposition of M into indecomposable A-modules. By using (2.2.2) and applying ind_A^B one has

$$M \oplus M^\zeta \cong \operatorname{ind}_A^B M_1 \oplus \cdots \oplus \operatorname{ind}_A^B M_k.$$

Since the left-hand side is a decomposition as indecomposable B-modules, by the Krull–Schmidt theorem it follows that (2.2.4) cannot have more than two direct summands.

Now let N be an indecomposable A-module. By (2.2.1) we see that N is a direct summand of $\operatorname{ind}_A^B N$. On the other hand, if M is an indecomposable B-module which contains N as a direct summand, then $\operatorname{ind}_A^B M \cong M \oplus M^\zeta$ contains $\operatorname{ind}_A^B N$. If $\operatorname{ind}_A^B N$ is indecomposable, then either $M \cong \operatorname{ind}_A^B N$ or $M \cong (\operatorname{ind}_A^B N)^\zeta$. If $\operatorname{ind}_A^B N$ is decomposable, then $\operatorname{ind}_A^B N \cong M \oplus M^\zeta$. This proves the desired result. $\square$

As is well known, if M is an irreducible A-supermodule then as an A-module M is either irreducible or the direct sum of two irreducibles. Combining the previous result with Proposition 2.2.3 shows that this behaviour extends to indecomposables.

2.3 We can now prove the main result of this section: the representation type of A as a superalgebra coincides with that of A as an algebra.

Theorem 2.3.1. *The category A-smod is semisimple, finite, tame, or wild if and only if the same is true for A-mod.*

Proof. By Proposition 2.1.2 it suffices to consider the category B-mod rather than A-smod.

Semisimple type. That A-smod is semisimple if and only if B-mod is semisimple is already known. See, for example, [4, Section 2] or [26, Theorem 1.4c].

Finite type. It is clear from Proposition 2.2.3 that A-mod has finitely many indecomposables if and only if B-mod does.

Tame type. Suppose that A-mod has tame representation type. Let t be an indeterminate and d a positive integer. Then there are indecomposable A-modules $D_{d,1}, \ldots, D_{d,r_d}$ and indecomposable $A-k[t]$-bimodules $E_{d,1}, \ldots, E_{d,s_d}$ which are free as right $k[t]$-modules so that any indecomposable A-module of dimension d either is isomorphic to $D_{d,i}$ for some i, or is isomorphic to $E_{d,i} \otimes_{k[t]} L$ for some i and some irreducible $k[t]$-module L. Observe that the proof of Proposition 2.2.3 applies verbatim to indecomposable $A-k[t]$-bimodules. Consequently, we can choose indecomposable B-modules $\tilde{D}_{d,1}, \ldots, \tilde{D}_{d,r_d}$ and indecomposable $B-k[t]$-bimodules $\tilde{E}_{d,1}, \ldots, \tilde{E}_{d,s_d}$ (which are free as $k[t]$-supermodules) where $\tilde{D}_{d,i}$ contains $D_{d,i}$ as a direct summand as A-modules and $\tilde{E}_{d,i}$ contains $E_{d,i}$ as a direct summand as $A-k[t]$-bimodules.

Let M be an indecomposable B-module and N an indecomposable direct summand of M as an A-module. If $N \cong D_{d,i}$ for some d and i, then by the uniqueness statement in Proposition 2.2.3 it follows that $M \cong \tilde{D}_{d,i}$ or $M \cong (\tilde{D}_{d,i})^\zeta$. On the other hand, say $N \cong E_{d,i} \otimes_{k[t]} L$ for some d, i, and irreducible $k[t]$-module L. Since $E_{d,i}$ is a direct summand of $\tilde{E}_{d,i}$, we have that $N \cong E_{d,i} \otimes_{k[t]} L$ is a direct summand of $\tilde{E}_{d,i} \otimes_{k[t]} L$. However, by [10, Lemma 2.2] we know $\tilde{E}_{d,i} \otimes_{k[t]} L$ is indecomposable. By uniqueness again we have $M \cong \tilde{E}_{d,i} \otimes_{k[t]} L$ or $M \cong (\tilde{E}_{d,i} \otimes_{k[t]} L)^\zeta = (\tilde{E}_{d,i})^\zeta \otimes_{k[t]} L$. Therefore, if A-mod is tame, then B-mod is tame.

Wild type. Suppose that A-mod has wild representation type. By [10, Theorem 2.2] and Proposition 2.2.3 it follows that B-mod has wild representation type. $\square$

3   Results on the Schur superalgebra

3.1   The Schur superalgebra. Fix non-negative integers m, n, and d, and fix a superspace V with fixed basis $v_1, \ldots, v_{m+n}$ and $\mathbb{Z}_2$-grading given by setting $\bar{v}_i = \bar{0}$ for $i = 1, \ldots, m$ and $\bar{v}_i = \bar{1}$ for $i = m+1, \ldots, m+n$. The tensor space $V^{\otimes d}$ is then naturally $\mathbb{Z}_2$-graded by setting the degree of $v_{i_1} \otimes \cdots \otimes v_{i_d}$ to be $\bar{v}_{i_1} + \cdots + \bar{v}_{i_d}$ for $1 \leqslant i_1, \ldots, i_d \leqslant m+n$. The symmetric group Σ_d acts on $V^{\otimes d}$ on the right via

$$v_{i_1} \otimes \cdots \otimes v_{i_j} \otimes v_{i_{j+1}} \otimes \cdots \otimes v_{i_d}(j\ j+1) := (-1)^{\bar{v}_{i_j} \bar{v}_{i_{j+1}}} v_{i_1} \otimes \cdots \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes \cdots \otimes v_{i_d}$$

for $1 \leqslant i_1, \ldots, i_d \leqslant m+n$ and each $1 \leqslant j < d$.

The Schur superalgebra is defined to be

$$S(m|n, d) = \text{End}_{\Sigma_d}(V^{\otimes d}),$$

with the $\mathbb{Z}_2$-grading given by setting $S(m|n, d)_{\bar{0}}$ (resp. $S(m|n, d)_{\bar{1}}$) to be the set of all degree-preserving (resp. degree-reversing) maps. Observe that when $p = 2$ the above action coincides with the usual permutation action and so $S(m|n, d) = S(m+n, d)$.

We also note that $S(m|n, d) \cong S(n|m, d)$ as superalgebras. Namely, suppose that $t : \{1, \ldots, m+n\} \to \{1, \ldots, m+n\}$ is given by $t(i) = i + n$ if $i = 1, \ldots, m$ and $t(i) = i - m$ if $i = m+1, \ldots, m+n$. Fix a superspace $\tilde{V}$ with fixed basis

$\tilde{v}_1, \ldots, \tilde{v}_{m+n}$ and $\mathbb{Z}_2$-grading given by setting $\bar{\bar{v}}_i = \bar{0}$ for $i = 1, \ldots, n$ and $\bar{\bar{v}}_i = \bar{1}$ for $i = n+1, \ldots, m+n$. Define a linear map $\tau : V^{\otimes d} \otimes \mathrm{sgn} \to \tilde{V}^{\otimes d}$ via

$$\tau(v_{i_1} \otimes \cdots \otimes v_{i_d} \otimes 1) = (-1)^{(d-1)\bar{v}_{i_1} + (d-2)\bar{v}_{i_2} + \cdots + \bar{v}_{i_{d-1}}} \tilde{v}_{t(i_1)} \otimes \cdots \otimes \tilde{v}_{t(i_d)},$$

where $v_{i_1}, \ldots, v_{i_d}$ are elements of our fixed homogeneous basis for V. It is straight-forward to verify on simple transpositions that in fact τ is a $k\Sigma_d$-module isomorphism (cf. the proof of [5, Theorem 6.4]). Consequently we have

$$S(m|n, d) = \mathrm{End}_{\Sigma_d}(V^{\otimes d}) \cong \mathrm{End}_{\Sigma_d}(V^{\otimes d} \otimes \mathrm{sgn}) \cong \mathrm{End}_{\Sigma_d}(\tilde{V}^{\otimes d}) = S(n|m, d),$$

and the isomorphisms preserve the $\mathbb{Z}_2$-grading.

We now introduce certain modules which will be crucial in later calculations. Given a tuple of integers $\gamma = (\gamma_1, \ldots, \gamma_t)$ we denote the integer $\gamma_1 + \cdots + \gamma_t$ by $|\gamma|$. Let

$$\Lambda(m|n, d) = \{(\lambda_1, \ldots, \lambda_m \,|\, \mu_1, \ldots, \mu_n) \in \mathbb{Z}^{m+n} \,|\, \lambda_i, \mu_i \geqslant 0 \text{ for all } i \text{ and } |\lambda| + |\mu| = d\}.$$

We write $(\lambda|\mu)$ for an element of $\Lambda(m|n, d)$.

In general, given a sequence $\gamma = (\gamma_1, \ldots, \gamma_t)$ of non-negative integers summing to d, let

$$\Sigma_d = \Sigma_{\gamma_1} \times \Sigma_{\gamma_2} \times \cdots \times \Sigma_{\gamma_t}$$

viewed as a subgroup of Σ_d in the natural way with Σ_{γ_1} acting as permutations of the first γ_1 letters, Σ_{γ_2} acting as permutations of the next γ_2 letters, and so on.

Given $(\lambda|\mu) \in \Lambda(m|n, d)$ we define the *signed permutation module* $M^{(\lambda|\mu)}$ as

$$M^{(\lambda|\mu)} = \mathrm{ind}_{\Sigma_\lambda \times \Sigma_\mu}^{\Sigma_d} k \boxtimes \mathrm{sgn},$$

where k denotes the trivial module for $\Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_m}$ and sgn denotes the one-dimensional sign representation for $\Sigma_{\mu_1} \times \cdots \times \Sigma_{\mu_n}$.

By definition, a Σ_d-module is a *signed Young module* if for some non-negative integers m and n and for some $(\lambda|\mu) \in \Lambda(m|n, d)$ it is a direct summand of $M^{(\lambda|\mu)}$. Signed permutation modules and signed Young modules were first considered by Donkin [9]. Finally we observe the well-known fact that

$$V^{\otimes d} \cong \bigoplus_{(\lambda|\mu) \in \Lambda(m|n, d)} M^{(\lambda|\mu)} \tag{3.1.1}$$

as $k\Sigma_d$-modules.

3.2 We begin by collecting a few preliminary results.

Proposition 3.2.1. *Let S be a finite-dimensional algebra containing an idempotent e, so that eSe is a subalgebra of S.*

(a) *Every indecomposable eSe-module is a summand of the restriction to eSe of some indecomposable S-module.*

(b) *If eSe has infinite representation type then so does S.*

(c) *If eSe has wild representation type then so does S.*

Proof. (a) Let N be an indecomposable eSe-module. Let $U = S \otimes_{eSe} N$. Then U is a non-zero S-module and we have

$$U \cong eU \oplus (1 - e)U$$

as eSe-modules. But $eU \cong N$. Now if U is written as a direct sum of indecomposable S-modules then an easy argument with the Krull–Schmidt theorem gives the result.

(b) This is clear from (a). If S has only finitely many indecomposable modules then (a) implies that eSe does as well.

(c) This statement follows from the argument of [10, Theorem 2.2]: the authors show that if A is tame and the statement of (a) holds for an arbitrary algebra A then so is each subalgebra B. $\square$

3.3   One can now apply the previous result to the situation of $S(m|n, d)$. This allows us to descend from the superalgebra $S(m'|n', d)$ to $S(m|n, d)$ for $m \leqslant m'$, $n \leqslant n'$.

Corollary 3.3.1. *Let $m \leqslant m'$ and $n \leqslant n'$. The following assertions hold*:

(a) *if $S(m|n, d)$ has infinite representation type then so does $S(m'|n', d)$;*

(b) *if $S(m|n, d)$ has wild representation type then so does $S(m'|n', d)$.*

Proof. Recall that $\Lambda(m|n, d)$ is the set of pairs $(\lambda|\mu)$ where λ and μ are compositions with $|\lambda| + |\mu| = d$ and λ has at most m parts, μ has at most n parts. Thus $\Lambda(m|n, d)$ is naturally a subset of $\Lambda(m'|n', d)$. Recall that

$$S(m'|n', d) \cong \mathrm{End}_{k\Sigma_d} \left(\bigoplus_{(\lambda|\mu) \in \Lambda(m'|n', d)} M^{(\lambda|\mu)} \right).$$

So if e denotes the projection onto the direct sum of all $M^{(\lambda|\mu)}$ with $(\lambda|\mu) \in \Lambda(m|n, d)$ then clearly $eS(m'|n', d)e \cong S(m|n, d)$, and we can apply Proposition 3.2.1 to prove the result.

3.4   The classical (and infinitesimal) Schur algebra $S(n, n)$ has a natural one-dimensional representation corresponding to the determinant for $\mathrm{GL}(n)$. This allows one to embed the module category for $S(n, d)$ into that of $S(n, d + n)$, which can be used to study representation type (e.g. [12, Theorem 2.2]). Unfortunately the analogue of the determinant representation for the supergroup $\mathrm{GL}(m|n)$, the one-dimensional Berezinian representation, is not polynomial and so it does not give a natural one-dimensional $S(m|n, m + n)$-module.

We instead use the existence of a one-dimensional representation for $S(2|1, p)$. Namely, let $L(0, 0|p)$ denote the one-dimensional supermodule which is the Frobenius twist of the one-dimensional GL(2) × GL(1)-module $L(0, 0) \boxtimes L(1)$ (see [5, Remark 4.6]). Since the polynomial representations of GL($m|n$) correspond with the representations of the supermonoid $M(m|n)$ one can follow the arguments given in [12, Propositions 2.1, 2.2] by replacing the determinant representation with $L(0, 0|p)$ to deduce the following result.

Proposition 3.4.1. *If the algebra $S(2|1, d)$ has wild representation type then so does $S(2|1, d + cp)$ for any $c \in \mathbb{N}$.*

3.5 We will assume familiarity with basic modular representation theory of the symmetric group. In the next section we will prove the main result of the paper. Surprisingly, we will use virtually no representation theory of the superalgebra. However, we will need to know that Young modules $Y^{(a,b,1^c)}$ are in fact signed Young modules when $c < p$ and $m = 2$, $n = 1$.

Proposition 3.5.1. *Let $1 \leqslant c < p$. Then the Young module $Y^{(a,b,1^c)}$ is a summand of the signed permutation module $M^{(a,b|c)}$.*

Proof. Since $c < p$ we have

$$M^{(1^c)} \cong k\Sigma_c \cong \operatorname{ind}_{\Sigma_{1^c}}^{\Sigma_c} k \cong \bigoplus_{\lambda \vdash c} (S^\lambda)^{\dim S^\lambda}. \qquad (3.5.2)$$

But the Specht module S^λ has dimension greater than one except when $\lambda = (1^c)$ or $\lambda = c$. Thus using (3.5.2) we obtain that

$$M^{(a,b,1^c)} \cong M^{(a,b,c)} \oplus M^{(a,b|c)} \bigoplus_{\lambda \vdash c,\, \lambda \neq (1^c),(c)} \operatorname{ind}_{\Sigma_a \times \Sigma_b \times \Sigma_c}^{\Sigma_d} (k \boxtimes k \boxtimes S^\lambda)^{d_\lambda}, \quad (3.5.3)$$

with each $d_\lambda > 1$. Since $Y^{(a,b,1^c)}$ occurs exactly once as a direct summand of $M^{(a,b,1^c)}$ and $\operatorname{Hom}_{\Sigma_d}(Y^{(a,b,1^c)}, M^{(a,b,c)}) = 0$, the result follows from (3.5.3). $\square$

3.6 The significance of Proposition 3.5.1 is that it gives us a Young module Y^λ as a summand of the tensor space $V^{\otimes d}$, even though λ has more than three parts and $m + n = 3$. We need information about one other special signed Young module for the $p = 3$ case.

Proposition 3.6.1. *Suppose that p does not divide a. Then*

$$M^{(a,0|p)} \cong S^{(a+1,1^{p-1})} \oplus S^{(a,1^p)}.$$

The two Specht modules in the decomposition are both irreducible and

$$S^{(a,1^p)} \cong D^{(a,2,1^{p-2})}.$$

Proof. The Specht module decomposition follows from Young's rule (see [20, p. 51]). The irreducibility of the 'hook' Specht modules when $p \nmid d$ is well-known work of Peel [27]. Finally, which irreducible module corresponds to $S^{(a,1^p)}$ can be calculated directly, or obtained as a special case of James [19, Theorem A]. $\quad\square$

4 Proof of the main result

In this section we prove Theorem 1.3.1. One of the main ideas will be to identify a subalgebra eSe with $k\mathcal{Q}/I$ for some quiver $\mathcal{Q}$. Then we can use Gabriel's theorem, since if the separated quiver is not a Dynkin or Euclidean diagram then the algebra is of wild type.

4.1 $S(1|1,d)$. If $n = 0$ or $m = 0$ the superalgebra $S(m|n,d)$ is isomorphic to the Schur algebra $S(m,d)$, and so the smallest non-trivial case for us is the superalgebra $S(1|1,d)$. This has been studied by Marko and Zubkov:

Proposition 4.1.1 [23, Proposition 2.1]. *Let $S = S(1|1,d)$. If p does not divide d then S is semisimple. If p divides d then S has $d + 1$ pairwise non-isomorphic one-dimensional modules, labelled by the weights $0, 1, \ldots, d$. The algebra is basic, connected, and has left regular representation given as follows.*

$$
{}_S S = \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \oplus \begin{matrix} 1 \\ 0 \\ \end{matrix} \begin{matrix} 2 \\ 1 \\ 2 \end{matrix} \oplus \begin{matrix} 2 \\ 1 \\ \end{matrix} \begin{matrix} 3 \\ \end{matrix} \oplus \cdots \oplus \begin{matrix} r-1 \\ r-2 \\ r-1 \end{matrix} \begin{matrix} r \\ r-1 \\ \end{matrix} \qquad (4.1.2)
$$

From here it is easy to obtain the quiver. In particular, for $m \geqslant 1$ we define an algebra $\tilde{\mathscr{A}}_m$ to be the algebra $k\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver with $m+1$ vertices

and $\mathscr{I}$ is the ideal generated by the relations

$$
\alpha_i \alpha_{i+1} = 0, \quad \beta_{i+1}\beta_i = 0, \quad \alpha_1 \beta_1 = \beta_m \alpha_m = 0,
$$
$$
\beta_i \alpha_i = \alpha_{i+1}\beta_{i+1} \quad \text{for } 1 \leqslant i \leqslant m - 1.
$$

The quiver for $\tilde{\mathscr{A}}_m$ is the same as the quiver for Erdmann's algebra $\mathscr{A}_{m+1}$ defined in [13, (3.1)], with one additional relation, namely $\beta_m \alpha_m = 0$. We will see in Section 5 that this extra relation is crucial because it makes the algebra have infinite global dimension.

Theorem 4.1.3. *When p divides d the algebra $S(1|1,d)$ is isomorphic to the algebra $\tilde{\mathscr{A}}_d$ and has finite representation type.*

Proof. The proof of the algebra isomorphism is essentially the same as in [13, Proposition 3.2], and so we do not give it here. The algebra $\tilde{\mathscr{A}}_m$ is a three nilpotent algebra, but the only indecomposable modules of radical length three are projective. Hence, the representation type of $\tilde{\mathscr{A}}_m$ is the same as the representation type of $\tilde{\mathscr{A}}_m$ modulo its radical squared (which is two nilpotent). The separated quiver is a disjoint union of two copies of the Dynkin diagram A_{m+1}, and hence by Gabriel's theorem the algebra is of finite type. $\square$

4.2 $S(m|n, d)$, $(d < 2p)$. We first consider the case where $p \leqslant d < 2p$. It turns out that this case is essentially the same as for the Schur algebra $S(2, d)$. Recall that the summands of the signed permutation modules $M^{(\alpha|\beta)}$ are called signed Young modules. We recall the following theorem of Donkin.

Theorem 4.2.1 [9, (2.3(6))]. *Let* $m, n \geqslant d$. *The isomorphism classes of indecomposable signed Young modules are labelled by the set*

$$\Lambda^{++}(m|n, d) := \{(\lambda \,|\, p\mu) \,|\, |\lambda| + p|\mu| = d\}$$

where λ *and* μ *are partitions.*

Following [9], we write $Y^{(\lambda|p\mu)}$ for the indecomposable signed Young module labelled by $(\lambda \,|\, p\mu) \in \Lambda^{++}(m|n, d)$.

   We wish to show that for $d < 2p$ the signed Young modules are exactly the ordinary Young and twisted Young modules. To begin we will show that the non-projective signed Young modules are all distinct.

   Given a partition $\lambda \vdash d$, let λ' denote the transpose partition of λ. For $\lambda \vdash d$ p-regular we let $m(\lambda)$ be the Mullineux conjugate of λ, so that $D^\lambda \otimes \mathrm{sgn} \cong D^{m(\lambda)}$. It is well known that the Young module $Y^{m(\lambda)'}$ is the projective cover of D^λ, a fact that we will use several times below.

Proposition 4.2.2. *If the Young module* Y^λ *is isomorphic to a twisted Young module* $Y^\mu \otimes \mathrm{sgn}$, *then* λ *is* p-*restricted, in which case* $\lambda = m(\mu')'$.

Proof. Applying the adjoint Schur functor to Y^λ and $Y^\mu \otimes \mathrm{sgn}$ (see [18, Theorems 3.4.2, 3.8.2]) tells us that $P(\lambda) \cong T(\mu')$. But the tilting modules are self-dual, so that $P(\lambda)$ is self-dual and hence has simple socle $L(\lambda)$. But $V(\mu')$ is a submodule of $T(\mu')$ and the socle of $V(\mu')$ is $L(m(\mu')')$, so that $\lambda = m(\mu')'$ is p-restricted. $\square$

   For $d < 2p$ it is clear that the p-singular partitions are exactly those of the form $(\mu_1 + p, \mu_2, \ldots)$ for μ a partition of $d - p$. Thus the number of signed Young modules for d is the total number of partitions of d plus the number of p-singular partitions, which by Proposition 4.2.2 is precisely the number of Young and twisted Young modules. We have shown the following result.

Proposition 4.2.3. *For* $p \leqslant d < 2p$ *the signed Young modules are exactly the set of*

Young modules Y^λ together with the non-projective twisted Young modules $Y^\mu \otimes \text{sgn}$ where μ is p-singular.

We also remark that the non-projective Young modules are actually irreducible and are in distinct blocks. Symmetric group blocks of defect 1 are well understood and we have the following. Let $\tau \vdash d - p$ and let B be the block of $k\Sigma_d$ with p-core τ. Let

$$\lambda_1 = (\tau_1 + p, \tau_2, \ldots, \tau_s) \rhd \lambda_2 \rhd \cdots \rhd \lambda_p = (\tau, 1^p)$$

be the p partitions of d with p-core τ.

Let $\mu = \tau' + (p)$. Then μ is p-singular and an easy calculation shows that $Y^\mu \otimes \text{sgn} \cong D^{\lambda_{p-1}}$, i.e. this is the non-projective twisted Young module in the block. So the signed Young modules in the block are

$$Y^{\lambda_1} \cong D^{\lambda_1}, \quad Y^{\lambda_2} \cong \begin{matrix} D^{\lambda_1} \\ D^{\lambda_2}, \\ D^{\lambda_1} \end{matrix} \quad Y^{\lambda_i} \cong \begin{matrix} D^{\lambda_{i-1}} \\ D^{\lambda_i} \quad D^{\lambda_{i-2}}, \\ D^{\lambda_{i-1}} \end{matrix}$$

$$Y^{\lambda_p} \cong \begin{matrix} D^{\lambda_{p-1}} \\ D^{\lambda_{p-2}}, \\ D^{\lambda_{p-1}} \end{matrix} \quad Y^\mu \otimes \text{sgn} \cong D^{\lambda_{p-1}}, \tag{4.2.4}$$

where $3 \leqslant i < p$. Thus the basic algebra for $S(m|n, d)$ is a direct sum of two-sided ideals which either are semisimple (corresponding to blocks of $k\Sigma_d$ of defect 0) or are isomorphic to

$$\text{End}_{k\Sigma_d}\left(\bigoplus_{i=1}^{p} Y^{\lambda_i} \oplus (Y^\mu \otimes \text{sgn}) \right). \tag{4.2.5}$$

The quiver and relations for the algebra in (4.2.5) correspond to $\tilde{\mathcal{A}}_p$, which we saw earlier is of finite type. So we can now prove the first half of Theorem 1.3.1.

Theorem 4.2.6. *Let $S(m|n, d)$ be the Schur superalgebra where $0 \leqslant d < 2p$.*

(a) *$S(m|n, d)$ is semisimple if and only if one of the following holds:*
   (i) *char $k = 0$;*
   (ii) *$d < p$;*
   (iii) *$m = n = 1$ and $p \nmid d$.*

(b) *$S(m|n, d)$ has finite representation type but is not semisimple if and only if one of the following holds:*
   (i) *$p \leqslant d < 2p$;*
   (ii) *$m = n = 1$ and $p | d$.*

Proof. (a) If $d < p$ or char $k = 0$ then $k\Sigma_d$ is semisimple, and hence so is the algebra $S(m|n, d) := \text{End}_{k\Sigma_d}(V^{\otimes d})$. Part (iii) follows by Proposition 4.1.1.

(b) The calculation above shows that when $\max(m, n) \geqslant 2$ and $p \leqslant d < 2p$ the algebra $S(m|n, d)$ has finite representation type (and is not semisimple). The same is true for $S(1|1, d)$ when $p|d$ by Proposition 4.1.1.   □

4.3 $S(2|1, d)$, $(d \geqslant 2p)$.

Now we begin to show that for $d \geqslant 2p$ the algebra $S(2|1, d)$ has wild type. By Proposition 3.4.1 it will suffice to prove this for $2p \leqslant d < 3p$. Recall that

$$S(2|1, d) \cong \text{End}_{k\Sigma_d}\left(\bigoplus_{(\lambda_1, \lambda_2|\mu_1) \in \Lambda(2|1, d)} M^{(\lambda_1, \lambda_2|\mu_1)} \right).$$

In each case we will find some collection $\{Y_i\}$ of indecomposable signed Young modules such that the algebra $\text{End}_{k\Sigma_d}(\oplus Y_i)$ is of wild type. But this will imply that $S(2|1, d)$ is of wild type by Proposition 3.2.1.

To show that the algebra $\text{End}_{k\Sigma_d}(\oplus Y_i)$ has wild type we will show, in each case, that it is isomorphic to an algebra $k\mathcal{Q}/I$ where $\mathcal{Q}$ is a quiver and $\mathcal{I}$ is an ideal contained in the ideal generated by paths of length at least two. In each case the separated quiver of $\mathcal{Q}$ will not be a union of Dynkin or extended Dynkin diagrams, which by Gabriel's theorem means that the algebra is of wild type. The main difficulty will be to make an appropriate choice of the $\{Y_i\}$.

Our proof will split into three cases, each following the same strategy. First we handle the case $2p \leqslant d < 3p - 2$. Then we handle $d = 3p - 2$ and $d = 3p - 1$ separately. That this happens is not surprising, as the blocks of defect 2 are slightly different in the three cases; see for instance [29, Table 1]. Essentially everything is understood for symmetric group blocks of defect 2. The decomposition numbers are all known by [28]. The structure of the projective indecomposable modules can be determined from [29]. Finally, Chuang and Tan [7] have determined the structure of the non-projective Young modules. Thus we just need to select signed Young modules judiciously to get a quiver of wild type. We will not give the details of determining the module structures we need; they can be found in the works cited above.

Recall that if λ is a p-regular partition of d, we write $m(\lambda)$ for the Mullineux conjugate of λ, so that $D^\lambda \otimes \text{sgn} \cong D^{m(\lambda)}$ and the Young module $Y^{m(\lambda)'}$ is the projective cover of D^λ.

4.4 $S(2|1, d)$, $(2p \leqslant d < 3p - 2)$. Let $d = 2p + t$ for $0 \leqslant t < p - 2$. Using the Chuang–Tan notation we have the following partitions in the principal block of Σ_d:

$$\lambda^{(0)} = (2p + t), \quad \lambda^{(1)} = (2p - 1, t + 1), \quad \lambda^{(2)} = (2p - 2, t + 1, 1), \quad \lambda^{(*)} = (p + t, p).$$

If we let $\sigma = (p + t, p - 1, 1)$ then σ is p-restricted and Y^σ is the projective cover of $D^{\lambda^{(1)}}$. Using i to denote $D^{\lambda^{(i)}}$ we have the following Loewy structures [7, Theorem 2.4]:

$$Y^{\lambda^{(0)}} \cong 0, \quad Y^{\lambda^{(1)}} \cong \begin{matrix} 0 \\ 1 \\ 0 \end{matrix}, \quad Y^{\lambda^{(2)}} \cong \begin{matrix} 1 \\ 0 \ 2 \\ 1 \end{matrix}, \quad Y^{\lambda^{(*)}} \cong \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} *,$$

$$Y^{\sigma} \cong \begin{matrix} & 1 & \\ * & 0 & 2 \\ 1 & \mu & 1 \\ * & 0 & 2 \\ & 1 & \end{matrix}.$$

$(4.4.1)$

Since all partitions above have third part 0 or 1 and since by Proposition 3.5.1 $Y^{(a,b,1)}$ is a summand of $M^{(a,b|1)}$, all modules in (4.4.1) are signed Young modules for $m = 2$, $n = 1$. Consider the quiver $\mathscr{L}$ given below:
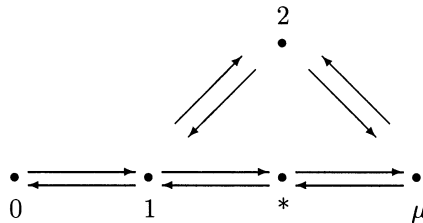

Let U be the direct sum of the five modules in (4.4.1). We have

Theorem 4.4.2. (a) *The algebra* $\mathrm{End}_{k\Sigma_{2p+t}}(U)$ *is isomorphic to the algebra* $k\mathscr{L}/I$ *where* I *is in the ideal generated by paths of length* 2. *In particular it is of wild representation type.*

(b) *The superalgebra* $S(2|1, d)$ *has wild representation type for* $2p \leqslant d < 3p - 2$.

Proof. (a) The signed Young modules making up U have the same structure as the modules E_i for $1 \leqslant i \leqslant 5$ in [13, Proposition 3.10], and the determination of the quiver proceeds in exactly the same way.

  (b) We obtain this immediately from (a) by calculating the separated quiver of $\mathscr{L}$. The separated quiver is two copies of the diagram
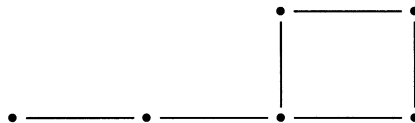
which is not a Dynkin or Euclidean diagram.   □

4.5 $S(2|1, 3p - 2)$. Let $d = 3p - 2$. Using the Chuang–Tan notation we have the following partitions in the principal block of Σ_d:

$$\lambda^{(0)} = (3p-2), \quad \lambda^{(1)} = (2p-1, p-1), \quad \lambda^{(2)} = (2p-2, p-2, 1^2),$$
$$\lambda^{(3)} = (2p-2, p-3, 1^3), \quad \lambda^{(*)} = (2p-2, p), \quad \delta = (2p-2, p-1, 1).$$

It is also an easy calculation that

$$m(\delta)' = (2p-3, p-1, 1^2),$$

so that $Y^{(2p-3, p-1, 1^2)}$ is the projective cover of D^δ. Let $\tau = (2p-3, p-1, 1^2)$. Using i to denote $D^{\lambda^{(i)}}$ we have the following structures for Young modules [7]:

$$Y^{\lambda^{(0)}} \cong 0, \quad Y^{\lambda^{(1)}} \cong \begin{smallmatrix}0\\1\\0\end{smallmatrix}, \quad Y^{\lambda^{(3)}} \cong \delta\begin{smallmatrix}2\\3\\2\end{smallmatrix}. \qquad (4.5.1)$$

All of the Young modules in (4.5.1) are either for two-part partitions or partitions of the form $(a, b, 1^c)$ for $c < p$, and so by Proposition 3.5.1 they are all signed Young modules. Now consider the following quiver which we denote by $\mathscr{Q}$.
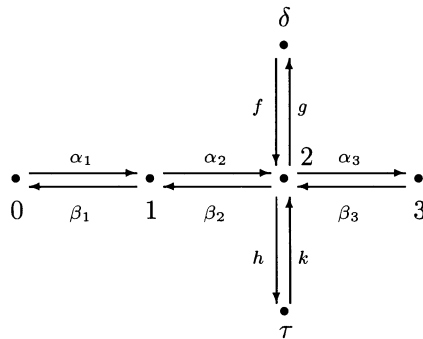
Let T be the direct sum of the six signed Young modules in (4.5.1). We have the following result.

Theorem 4.5.2. (a) *The algebra* $\Lambda = \mathrm{End}_{k\Sigma_{3p-2}}(T)$ *is isomorphic to the algebra* $k\mathscr{Q}/I$ *where $\mathscr{I}$ is the ideal generated by the relations*

$$\alpha_1\alpha_2 = \alpha_2\alpha_3 = \alpha_2 h = \beta_3\beta_2 = \beta_2\beta_1 = \beta_3 g = f\alpha_3 = 0,$$
$$hk = gf = \beta_1\alpha_2 + \alpha_3\beta_3.$$

(4.5.3)

(b) *The superalgebra $S(2|1, d)$ has wild representation type for $d = 3p - 2$.*

Proof. (a) Let $\{e_0, e_1, e_2, e_3, e_\delta, e_\tau\} \in \Lambda$ be the canonical projections onto the Young module summands of T. Choose non-zero maps α_i and β_i. (The module structures leave no choice for the kernels and images of these maps.) The image and kernels of the maps f, g, h, k can also be determined by the structure of the modules given in (4.5.1). In particular we have the following Loewy structures for the kernels (from which the images are easily determined).

$$\ker f \cong \begin{smallmatrix} * & 0 \\ & 1 \end{smallmatrix}, \quad \ker k \cong 2 \begin{smallmatrix} \tau \\ * & 0 \\ \delta \end{smallmatrix}, \quad \ker g \cong \begin{smallmatrix} 2 & 0 \\ \delta \end{smallmatrix}, \quad \ker h \cong \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}.$$

Knowing the kernels and images of the maps is sufficient to check that the zero relations in (4.5.3) are satisfied. We need only show we can choose the maps so that $hk = gf = \beta_1\alpha_2 + \alpha_3\beta_3$.

Notice that the image of $\beta_2\alpha_2$ is the simple module 1 and the image of $\alpha_3\beta_3$ is the simple module δ, while both hk and gh have image equal to the socle of $Y^{\lambda^{(2)}}$. Since $\mathrm{End}(Y^{\lambda^{(2)}})$ is three-dimensional, we can take as a basis $\{e_2, \alpha_3\beta_3, \beta_2\alpha_2\}$. Write

$$hk = a\beta_2\alpha_2 + b\alpha_3\beta_3, \quad gf = c\beta_2\alpha_2 + c\alpha_3\beta_3,$$

where the images ensure that a, b, c, d are all non-zero. Now make the substitutions

$$\beta_2' = a\beta_2, \quad \alpha_2' = c\alpha_2, \quad \beta_3' = b\beta_3, \quad \alpha_3' = d\alpha_4.$$

This will give the desired relation without effecting the earlier zero relations. The maps $\{\alpha_i, \beta_i, h, k, f, g\}$ generate rad Λ and are independent modulo I^2, and so there is an epimorphism $k\mathcal{Q}/I \to \Lambda$. Now one can simply check that the dimensions agree to see that it is an isomorphism.

(b) We obtain this immediately from (a) by calculating the separated quiver of $\mathcal{Q}$ and verifying that it is not a Dynkin or Euclidean diagram. □

In the special case where $p = 3$ (i.e. $S(2|1, 7)$), the partition $\lambda^{(3)}$ above does not exist. In this case to obtain the same quiver we replace $Y^{\lambda^{(3)}}$ by the irreducible module D^δ, which is a signed Young module by Proposition 3.6.1. The quiver and relations are easily seen to be identical.

4.6 $S(2|1, 3p - 1)$. Let $d = 3p - 1$. Using the Chuang–Tan notation we have the following partitions in the principal block of Σ_d:

$$\lambda^{(0)} = (3p - 1), \quad \lambda^{(1)} = (2p - 1, p - 1, 1), \quad \lambda^{(2)} = (2p - 1, p - 2, 1^2),$$
$$\lambda^{(3)} = (2p - 1, p - 3, 1^3), \quad \lambda^{(*)} = (2p - 1, p), \quad \rho = (2p - 2, p, 1),$$

where $\rho = m(\lambda^{(*)})'$, and hence Y^ρ is the projective cover of $D^{\lambda^{(*)}}$.

Using i to denote $D^{\lambda^{(i)}}$ we have the following structures for Young modules [7]:

$$
Y^{\lambda^{(0)}} \cong 0, \quad Y^{\lambda^{(*)}} \cong *, \quad Y^{\lambda^{(1)}} \cong \begin{matrix} 0 & * \\ 1 & \\ 0 & * \end{matrix}, \quad Y^{\lambda^{(2)}} \cong \begin{matrix} & 1 \\ 0 & \; 2 \\ & 1 \end{matrix},
$$

$$
Y^{\lambda^{(3)}} \cong \begin{matrix} & 2 \\ 1 & \; 3 \\ & 2 \end{matrix}, \quad Y^\rho \cong \begin{matrix} & * \\ & 1 \\ * & 0 & \rho \\ & 1 \\ & * \end{matrix}.
\tag{4.6.1}
$$

Now consider the following quiver called $\mathscr{H}$:

Let W be the direct sum of the six signed Young modules in (4.6.1). We have the following result.

Theorem 4.6.2. (a) *The algebra* $\mathrm{End}_{k\Sigma_{3p-1}}(W)$ *is isomorphic to the algebra* $k\mathscr{H}/I$ *where $\mathscr{I}$ is in the ideal generated by paths of length* 2. *In particular it is of wild representation type.*

(b) *The superalgebra* $S(2|1, d)$ *has wild representation type for* $d = 3p - 1$.

Proof. (a) We leave the calculations for the reader. They are similar to the proof of Theorem 4.5.2.

(b) Once again the separated quiver for $\mathscr{H}$ is neither Dynkin nor Euclidean and so $S(2|1, 3p - 1)$ is of wild representation type.   □

Just as in the case for $3p - 2$, the partition $\lambda^{(3)}$ does not exist when $p = 3$ (i.e. for

$S(2|1, 8)$). Again, replacing $Y^{\lambda^{(3)}}$ with the simple module $D^{(5,2,1)} \cong D^{\lambda^{(1)}}$ gives the same quiver. $D^{(5,2,1)}$ is a signed Young module by Proposition 3.6.1.

4.7 We can now complete the proof of Theorem 1.3.1 by proving the following result.

Theorem 4.7.1. *The algebra $S(m|n, d)$ has wild representation type when $\max(m, n) \geqslant 2$ and $d \geqslant 2p$.*

Proof. Since $S(m|n, d) \cong S(n|m, d)$ we may assume that $m \geqslant n$. By assumption $m \geqslant 2$ and $n \geqslant 1$. For $2p \leqslant d < 3p$ the algebra $S(2|1, d)$ has wild representation type from Theorems 4.4.2(b), 4.5.2(b), 4.6.2(b). Hence, by Proposition 3.4.1, $S(2|1, d)$ has wild type for $d \geqslant 2p$. One can now apply Corollary 3.3.1(b) to conclude that $S(m|n, d)$ has wild representation type for $d \geqslant 2p$. $\square$

5   Global dimension

5.1 In [23, Conjecture 1] it is conjectured that $S(m|n, d)$ is quasi-hereditary whenever d is coprime to p. In this section we use our previous calculations to show this is far from true. In fact, we will prove the following theorem.

Theorem 5.1.1. *For $d \geqslant p \geqslant 5$ and $m, n \geqslant d$ the superalgebra $S(m|n, d)$ has infinite global dimension.*

This result suggests the following question.

Question 5.1.2. Is it true that $S(m|n, d)$ either is semisimple or has infinite global dimension? In particular, is it true that non-semisimple Schur superalgebras are never quasi-hereditary?

After the completion of the present article we were informed by Marko and Zubkov that they obtained in [24] an affirmative answer to the above questions. Their approach is quite different from the one taken here. They study the representation theory of $S(m|n, d)$ and answer the questions by obtaining partial information about the Cartan matrix for $S(m|n, d)$.

5.2 Before proving Theorem 5.1.1 we first need to determine which block of $k\Sigma_d$ contains a given signed Young module. Throughout the remainder of this section we assume that $m, n \geqslant d$.

Recall the classification of irreducible $S(m|n, d)$-supermodules when $m, n \geqslant d$ given in [9, (2.3(3))] (for the statement when m, n, and d are arbitrary, see [5, Theorem 6.5]).

Theorem 5.2.1. *Let $m, n \geqslant d$. The irreducible supermodules of $S(m|n, d)$ are labeled by heighest weight by the set $\Lambda^{++}(m|n, d) = \{(\lambda \,|\, p\mu) \,|\, |\lambda| + p|\mu| = d\}$ where λ and μ are partitions.*

Write $L(\lambda \mid p\mu)$ for the irreducible $S(m|n,d)$-supermodule of highest weight $(\lambda \mid p\mu)$ and let $P(\lambda \mid p\mu)$ denote the projective cover of $L(\lambda \mid p\mu)$.

Just as with the ordinary Schur algebra, one can define a Schur functor

$$F : S(m|n,d)\text{-supermodules} \to k\Sigma_d\text{-modules}.$$

As with the ordinary Schur functor, F is exact and if L is an irreducible $S(m|n,d)$-supermodule then FL is either zero or an irreducible $k\Sigma_d$-module. All irreducible $k\Sigma_d$-modules appear in this way. In fact, by [5, (5.13)] we have

$$FL(\lambda \mid p\mu) \cong \begin{cases} D^{m(\lambda')} & \text{if } |\mu| = 0 \text{ and } \lambda' \text{ is } p\text{-regular,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.2}$$

Furthermore, by [9, p. 662] we have

$$FP(\lambda \mid p\mu) \cong Y^{(\lambda|p\mu)}. \tag{5.2.3}$$

Thus to determine which block contains $Y^{(\lambda|p\mu)}$ it will suffice to obtain information about the composition factors of $P(\lambda \mid p\mu)$.

Before proceeding, we require additional notation. Let

$$X = \{\xi = (\xi_1, \ldots, \xi_m \mid \xi_{m+1}, \ldots, \xi_{m+n}) \mid \xi_1, \ldots, \xi_{m+n} \in \mathbb{Z}\}.$$

We view $(\lambda \mid p\mu) \in \Lambda^{++}(m|n,d)$ as the element $(\lambda, 0, \ldots, 0 \mid p\mu, 0, \ldots, 0)$ of X in the natural way. For $i = 1, \ldots, m+n$, let $\varepsilon_i \in X$ be the element which has a 1 in the ith position and 0's elsewhere. Let $\vartheta \in X$ be the element

$$\vartheta = (-1, -2, \ldots, -m \mid m-1, m-2, \ldots, m-n+1, m-n).$$

Note that our definition of ϑ differs from the one in [22, (2.8)] by a multiple of the $GL(m|n)$ analogue of the determinant representation. This has no significant effect on our arguments.

Given $i = 1, \ldots, m+n$ we define $r_i : X \to \mathbb{Z}$ by

$$r_i(\xi) = \begin{cases} \xi_i + \vartheta_i & \text{if } i = 1, \ldots, m, \\ -(\xi_i + \vartheta_i) & \text{if } i = m+1, \ldots, m+n. \end{cases}$$

For $r \in \mathbb{Z}_p$ and $\xi \in X$, define

$$A_r(\xi) = |\{i = 1, \ldots, m+n \mid r_i(\xi + \varepsilon_i) \equiv r \ (\text{mod } p)\}|,$$
$$B_r(\xi) = |\{i = 1, \ldots, m+n \mid r_i(\xi) \equiv r \ (\text{mod } p)\}|.$$

For a partition λ we let $\tilde{\lambda}$ denote its p-core. Recall that the blocks of $k\Sigma_d$ are parametrized by the p-cores of the partitions of d; see [21, Section 2.7]. Also recall that

the category of $S(m|n, d)$-supermodules is equivalent to the category of polynomial representations of degree d for the supergroup $GL(m|n)$. Thus we can view the category of $S(m|n, d)$-supermodules as a full subcategory of the category of representations of $GL(m|n)$ and, in turn, as a full subcategory of the category of supermodules for the superalgebra $\mathrm{Dist}(GL(m|n))$ of distributions for $GL(m|n)$. See [5] for a full discussion of these matters.

Lemma 5.2.4. *Let $(\lambda \,|\, p\mu), (v \,|\, p\eta) \in \Lambda^{++}(m|n, d)$. If $L(\lambda \,|\, p\mu)$ and $L(v \,|\, p\eta)$ have the same central character as $\mathrm{Dist}(GL(m|n))$-supermodules, then $\tilde{\lambda} = \tilde{v}$.*

Proof. If $L(\lambda \,|\, p\mu)$ and $L(v \,|\, p\eta)$ have the same central character, then by [22, Lemma 3.3] we have

$$A_r(\lambda \,|\, p\mu) - B_r(\lambda \,|\, p\mu) = A_r(v \,|\, p\eta) - B_r(v \,|\, p\eta) \qquad (5.2.5)$$

for all $r \in \mathbb{Z}_p$. However, if $i = m + 1, \ldots, m + n$, then

$$r_i(\lambda \,|\, p\mu) \equiv -\vartheta_i \ (\mathrm{mod}\ p) \equiv r_i(v \,|\, p\eta) \ (\mathrm{mod}\ p),$$

and

$$r_i((\lambda \,|\, p\mu) + \varepsilon_i) \equiv -\vartheta_i - 1 \ (\mathrm{mod}\ p) \equiv r_i((v \,|\, p\eta) + \varepsilon_i) \ (\mathrm{mod}\ p).$$

Consequently, we see that (5.2.5) holds if and only if

$$A_r'(\lambda \,|\, p\mu) - B_r'(\lambda \,|\, p\mu) = A_r'(v \,|\, p\eta) - B_r'(v \,|\, p\eta) \qquad (5.2.6)$$

for all $r \in \mathbb{Z}_p$, where for $\xi \in X$ we define

$$A_r'(\xi) = |\{i = 1, \ldots, m \,|\, r_i(\xi + \varepsilon_i) \equiv r \ (\mathrm{mod}\ p)\}|,$$
$$B_r'(\xi) = |\{i = 1, \ldots, m \,|\, r_i(\xi) \equiv r \ (\mathrm{mod}\ p)\}|.$$

Given $(\lambda \,|\, p\mu)$ and $r \in \mathbb{Z}_p$, for brevity let us write A_r' for $A_r'(\lambda \,|\, p\mu)$ and B_r' for $B_r'(\lambda \,|\, p\mu)$. By definition, we have $A_r' = B_{r-1}'$ for all $r \in \mathbb{Z}_p$. We then have the following equations:

$$B_1' + B_2' + \cdots + B_p' = m, \quad B_{r-1}' - B_r' =: b_r \quad \text{for } r = 1, \ldots, p.$$

However, it is straightforward to check that the determinant for this system of linear equations is non-zero and so one can solve this system for the integers B_r'. That is, (5.2.6) holds if and only if

$$B_r'(\lambda \,|\, p\mu) = B_r'(v \,|\, p\eta) \qquad (5.2.7)$$

for all $r \in \mathbb{Z}_p$. We assume without loss that $|\lambda| = |\nu|$ since $B'_r(\lambda, 1^p \mid p\mu) = B'_r(\lambda \mid p\mu)$ and since the p-cores of $(\lambda, 1^p)$ and λ coincide.

Since for $i = 1, \ldots, m$ we have $r_i(\lambda \mid p\mu) = \lambda_i - i$, it follows that (5.2.7) holds if and only if there is an element $\sigma \in \Sigma_m$ so that

$$\lambda_i - i \equiv \nu_{\sigma(i)} - \sigma(i) \pmod{p} \tag{5.2.8}$$

for $i = 1, \ldots, m$. Finally, by the Nakayama rule [25, Theorem 5.1.1] we have that (5.2.8) holds if and only if $\tilde{\lambda} = \tilde{\nu}$. This proves the desired result. $\square$

Corollary 5.2.9. *The signed Young module $Y^{(\lambda \mid p\mu)}$ is in the $k\Sigma_d$ block with p-core $\tilde{\lambda}$.*

Proof. Let $L(\nu \mid p\eta)$ be a composition factor of $P(\lambda \mid p\mu)$. Then since both $L(\lambda \mid p\mu)$ and $L(\nu \mid p\eta)$ are composition factors of an indecomposable $\mathrm{Dist}(GL(m|n))$-supermodule, they must have the same central characters (cf. [22, Section 2.8]). By the previous lemma, we have $\tilde{\lambda} = \tilde{\nu}$. The result then follows by exactness of the Schur functor along with (5.2.2) and (5.2.3). $\square$

5.3   We can now prove Theorem 5.1.1. Since $p \geqslant 5$, by the work of Granville and Ono [17] there must be a partition $\tau \vdash d - p$ which is a p-core. Consequently, Σ_d has a p-block of defect 1. Let B denote this block.

Lemma 5.3.1. *There are $p + 1$ signed Young modules in B. They are the p distinct Young modules $Y^{(\lambda \mid \varnothing)} \cong Y^{\lambda}$ with $\tilde{\lambda} = \tau$ and one non-projective twisted Young module $Y^{(\tau \mid p)} \cong Y^{(\tau' + p)} \otimes \mathrm{sgn}$.*

Proof. By Corollary 5.2.9 the signed Young modules which occur in B are those of the form $Y^{(\lambda \mid p\mu)}$ with $\tilde{\lambda} = \tau$. Thus they are precisely the ones listed. Now, since twisted Young modules are signed Young modules, $Y^{(\tau' + p)} \otimes \mathrm{sgn}$ is a signed Young module in the block B, and it must be $Y^{(\tau \mid p)}$. $\square$

Now we have the direct sum decomposition

$$S(m|n, d) = \mathrm{End}_{k\Sigma_d}(V^{\otimes d}) \cong \mathrm{End}_{k\Sigma_d}\left(\bigoplus_{Y^{(\lambda \mid \mu)} \in B} a_{(\lambda \mid \mu)} Y^{(\lambda \mid \mu)}\right) \oplus \mathrm{End}_{k\Sigma_d}(U)$$

where $a_{(\lambda \mid \mu)}$ denotes the multiplicity of $Y^{(\lambda \mid \mu)}$ in $V^{\otimes d}$ and U has no summands in the block B. That is, we have direct sum decomposition of $S(m|n, d)$ into graded two-sided ideals, one of which is Morita equivalent to

$$\mathrm{End}_{\Sigma_d}\left(\bigoplus_{Y^{(\lambda \mid \mu)} \in B} Y^{(\lambda \mid \mu)}\right).$$

This is precisely the situation of (4.2.4), and the quiver is just $\tilde{\mathscr{A}}_p$. It is easy to see that this (super)algebra has infinite global dimension. The projective resolutions of the

simple modules are all periodic and do not terminate. Since a two-sided graded ideal has infinite global dimension, so does $S(m|n,d)$, proving Theorem 5.1.1.

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