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On the cohomology of Specht modules

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Abstract

We investigate the cohomology of the Specht module S^λ for the symmetric group Σ_d. We show if $0 \leqslant i \leqslant p-2$, then $\mathrm{H}^i(\Sigma_d, S^\lambda)$ is isomorphic to $\mathrm{H}^{s+i}(B, w_0 \cdot \lambda' - \delta)$ where $s = \frac{d(d-1)}{2}$, B is the Borel subgroup of the algebraic group $\mathrm{GL}_d(k)$ and $\delta = (1^d)$ is the weight of the determinant representation. We obtain similar isomorphisms of $\mathrm{Ext}^i_{\Sigma_d}(S^\lambda, M)$ with B-cohomology, which in turn yield isomorphisms of cohomology for Borel subgroups of $\mathrm{GL}_n(k)$ for varying $n \geqslant d$. In the case $i = 0$, and the case $i = 1$ for certain λ, we apply our result and known symmetric group results of James and Erdmann to obtain new information about B-cohomology. Finally we show that Specht module cohomology is closely related to cohomology for the Frobenius kernel B_1 for small primes.

1. Introduction

1.1. Let $G = \mathrm{GL}_n(k)$ be the general linear group and Σ_d be the symmetric group. In characteristic two very little is known about the inverse Schur functor and most of our results apply only in odd characteristic, so we assume throughout that k is an algebraically closed field of characteristic $p \geqslant 3$. For $n \geqslant d$, there is a well-known fundamental relationship between the representation theories for these two groups via the Schur functor. Under the Schur functor, injective

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polynomial G-modules are sent to Young modules, induced modules are sent to Specht modules, Weyl modules are sent to dual Specht modules and simple modules are sent to simple modules or zero. One natural question is how this correspondence can be used to relate the cohomology of these two groups.

Let λ be a partition of d and let Y^λ (respectively S^λ, S_λ) be the corresponding Young (respectively Specht, respectively dual Specht) module. Moreover, let $L(\lambda)$ be the corresponding simple $\mathrm{GL}_n(k)$-module. If λ is a p-restricted partition, let D_λ denote the corresponding simple $k\Sigma_d$-module. These modules are all described in [Mar]. It was shown in [BKM, Theorem 2.4], [DN, Corollary 4.2], [KN] that

$$\begin{gathered}\mathrm{H}^1\big(\Sigma_d, Y^\lambda\big)=0 \quad \text{if } p\geqslant 3,\\ \mathrm{H}^1\big(\Sigma_d, Y^\lambda\otimes \mathrm{sgn}\big)=0 \quad \text{if } p\geqslant 5,\\ \mathrm{H}^1(\Sigma_d, S_\lambda)=0 \quad \text{if } p\geqslant 5.\end{gathered}$$

In characteristic three, dual Specht modules can have nonzero cohomology, which was completely determined in [BKM]:

$$\mathrm{H}^1(\Sigma_d, S_\lambda)\cong \begin{cases} k & \text{if } \lambda=(1^3) \text{ or } (d-3, 1^3),\\ 0 & \text{otherwise.}\end{cases}$$

In [DEN, Theorem 5.2(ii)], it was shown that

$$\mathrm{H}^1(\Sigma_d, D_\lambda)\cong \mathrm{H}^1\big(\mathrm{GL}_n(k), L(\lambda)\big) \quad \text{if } p\geqslant 3. \tag{1.1.1}$$

In general the determination of $\mathrm{H}^1(\Sigma_d, S^\lambda)$ is an open question.

1.2. Results like (1.1.1) suggest there may be a reasonable interpretation of $\mathrm{H}^\bullet(\Sigma_d, S^\lambda)$ in terms of the cohomology for the algebraic group $\mathrm{GL}_n(k)$. The goal of this paper is to investigate the cohomology of the Specht modules $\mathrm{H}^\bullet(\Sigma_d, S^\lambda)$. Let B be the Borel subgroup of upper triangular matrices in G. We prove that for $n=d$:

$$\mathrm{H}^j\big(\Sigma_d, S^\lambda\big)\cong \mathrm{H}^{s+j}(B, w_0\cdot\lambda'-\delta) \tag{1.2.1}$$

where $\delta=(1^d)$, $s=\frac{d(d-1)}{2}$ and $0\leqslant j\leqslant p-2$. Here $\mathrm{H}^\bullet(B,M)$ denotes the rational cohomology of B with coefficients in M. In particular this result specializes to:

$$\mathrm{H}^1\big(\Sigma_d, S^\lambda\big)\cong \mathrm{H}^{1+s}(B, w_0\cdot\lambda'-\delta) \quad \text{if } p\geqslant 3.$$

These isomorphisms illustrate an interesting interplay between the cohomology of Specht modules for the symmetric group and the cohomology of one-dimensional (simple) B-modules. Historically, the B-cohomology with coefficients in these simple B-modules has been a central but difficult object of study. When λ is a dominant weight, one has $\mathrm{H}^\bullet(B,\lambda)\cong \mathrm{H}^\bullet(G, \mathrm{ind}_B^G\lambda)=0$ for $\bullet>0$ by Kempf's vanishing theorem. In general if λ is an arbitrary character there exists a spectral sequence [Jan, I.4.5]

$$E_2^{i,j}=\mathrm{H}^i\big(G, R^j\,\mathrm{ind}_B^G\lambda\big) \quad\Rightarrow\quad \mathrm{H}^{i+j}(B,\lambda). \tag{1.2.2}$$

This spectral sequence relates $\mathrm{H}^\bullet(B,\lambda)$ with the structure of the higher right derived functors of the line bundle cohomology $\mathcal{H}^\bullet(G/B,\mathcal{L}(\lambda)) = R^\bullet \operatorname{ind}_B^G \lambda$. The general structure of $\mathcal{H}^\bullet(G/B,\mathcal{L}(\lambda))$ still remains a complete mystery.

1.3. Notation and preliminaries

The basic references for this section are [Gr,J]. Recall k is an algebraically closed field of characteristic $p \geqslant 3$ and let $G = \mathrm{GL}_n(k)$. The Schur algebra $S(n,d)$ is the finite-dimensional associative k-algebra $\operatorname{End}_{k\Sigma_d}(V^{\otimes d})$ where V is the natural representation of G. The category $M(n,d)$ of polynomial G-modules of a fixed degree $d \geqslant 0$ is equivalent to the category of modules for $S(n,d)$ and we do not distinguish between the two categories. We denote by $\operatorname{Mod}(k\Sigma_d)$ (respectively $\operatorname{mod}(k\Sigma_d)$) the category of all (respectively all finite-dimensional) $k\Sigma_d$-modules.

Throughout the paper we assume that $n \geqslant d$. Let $e = \zeta_{(1,2,\ldots,d)(1,2,\ldots,d)}$ be the idempotent in $S(n,d)$ described in [Gr, (6.1)]. Then $eS(n,d)e \cong k\Sigma_d$. The Schur functor $\mathcal{F}$ is the covariant exact functor from $M(n,d)$ to $\operatorname{Mod}(k\Sigma_d)$ defined on objects by $\mathcal{F}(M) = eM$.

The simple $S(n,d)$-modules are in bijective correspondence with set of partitions of d. We will denote this set by $\Lambda = \Lambda^+(n,d)$ and the corresponding simple $S(n,d)$-module by $L(\lambda)$ for $\lambda \in \Lambda$. Note that one can also identify Λ as the set of dominant polynomial weights of G of degree d. For $\lambda \in \Lambda$, let $P(\lambda)$ be the projective cover of $L(\lambda)$ and $T(\lambda)$ be the corresponding tilting module. Moreover, let $H^0(\lambda) = \operatorname{ind}_B^G \lambda$ and let $V(\lambda)$ be the Weyl module when $\lambda \in \Lambda$. There exists a duality on $M(n,d)$ fixing simple modules called the transpose dual. This duality will be denoted by τ. The duality τ and the usual duality '$*$' in $\operatorname{mod}(k\Sigma_d)$ are compatible in the sense that $e(M^\tau) \cong (eM)^*$ for any finite-dimensional $M \in M(n,d)$.

A partition $(\lambda_1,\lambda_2,\ldots)$ is called *p-restricted* if $\lambda_i - \lambda_{i+1} \leqslant p-1$ for all i. As mentioned in Section 1.1, simple $k\Sigma_d$-modules can be indexed by the set Λ_{res} of p-restricted partitions. A partition λ is called *p-regular* if its *transpose* λ' is p-restricted. We denote the set of all p-regular partitions of d by Λ_{reg}. In [J], the simple $k\Sigma_d$-modules are labeled by the p-regular partitions and denoted by D^λ. We will use both parametrizations so note a result from [Gr, §6]:

$$D^\lambda \cong D_{\lambda'} \otimes \operatorname{sgn} \quad \text{for any } \lambda \in \Lambda_{\mathrm{reg}}. \tag{1.3.1}$$

The Specht, Young, and permutation modules for $k\Sigma_d$ corresponding to a partition $\lambda \in \Lambda$ are denoted by S^λ, Y^λ, and M^λ, respectively. These modules are all described in [Mar]. In particular M^λ is the module induced from the trivial module over the Young subgroup Σ_λ. One has following correspondences between $S(n,d)$-modules and $k\Sigma_d$-modules under $\mathcal{F}$:

$$\mathcal{F}\big(H^0(\lambda)\big) = S^\lambda, \qquad \mathcal{F}\big(V(\lambda)\big) = \big(S^\lambda\big)^*, \qquad \mathcal{F}\big(P(\lambda)\big) = Y^\lambda, \qquad \mathcal{F}\big(T(\lambda)\big) = Y^{\lambda'} \otimes \operatorname{sgn}.$$

2. Symmetric group cohomology versus B-cohomology

2.1. The Schur functor $\mathcal{F}$ admits a right adjoint functor, $\mathcal{G}$ defined by

$$\mathcal{G}(N) = \operatorname{Hom}_{k\Sigma_d}\big(eS(n,d), N\big).$$

Furthermore, $\mathcal{G}$ is a left inverse to $\mathcal{F}$. The functor $\mathcal{G}$ is left exact but not right exact and admits higher right derived functors

$$R^j\mathcal{G}(N) = \operatorname{Ext}^j_{k\Sigma_d}\big(eS(n,d), N\big).$$

The detailed study of $\mathcal{G}$ and its right derived functors was initiated in [DEN].

For $N \in \mathrm{Mod}(k\Sigma_d)$, let $h(N)$ be the smallest positive integer such that $R^j\mathcal{G}(N) = 0$ for $1 \leqslant j \leqslant h(N)$. If no such integer exists then $h(N) = 0$.

2.2. With the vanishing of higher right derived functors one can show that the cohomology for G and $k\Sigma_d$ stabilizes in a certain range. We say an $S(n,d)$-module has p-restricted head if $\forall\lambda \in \Lambda$, $\mathrm{Hom}_{S(n,d)}(M, L(\lambda)) \neq 0$ implies $\lambda \in \Lambda_{\mathrm{res}}$.

2.2.1. Proposition. *Let $n \geqslant d$, $M \in \mathrm{Mod}(S(n,d))$, and $N \in \mathrm{Mod}(k\Sigma_d)$. Then*

$$\mathrm{Ext}^i_{S(n,d)}\bigl(M, \mathcal{G}(N)\bigr) \cong \mathrm{Ext}^i_{k\Sigma_d}(eM, N)$$

for $0 \leqslant i \leqslant h(N)$. If M has p-restricted head then the isomorphism extends to degree $i = h(N) + 1$.

Proof. From [DEN, Theorem 2.2], there exists a first-quadrant Grothendieck spectral sequence, with E_2-page given by

$$E_2^{i,j} = \mathrm{Ext}^i_{S(n,d)}\bigl(M, R^j\mathcal{G}(N)\bigr) \quad \Rightarrow \quad \mathrm{Ext}^{i+j}_{k\Sigma_d}(eM, N). \tag{2.2.2}$$

By assumption $R^j\mathcal{G}(N) = 0$ for $1 \leqslant j \leqslant h(N)$, thus $E_2^{i,j} = 0$ if $j > 0$ and $1 \leqslant i + j \leqslant h(N)$. The differentials d_r have bidegree $(r, 1-r)$, so

$$E_2^{i,0} \cong \mathrm{Ext}^i_{S(n,d)}\bigl(M, \mathcal{G}(N)\bigr) \cong \mathrm{Ext}^i_{k\Sigma_d}(eM, N)$$

for $0 \leqslant i \leqslant h(N)$. The additional assumption on M ensures [DEN, Proposition 5.1(3)] that the $E_2^{0,h(N)+1}$ term vanishes, which allows us to extend the isomorphism to degree $h(N) + 1$. $\square$

2.3. We can now relate extensions between a Specht module and an arbitrary symmetric group module with extensions between modules for B. Recall that τ denotes the contravariant duality.

2.3.1. Theorem. *Let $N \in \mathrm{Mod}(k\Sigma_d)$, $\lambda \in \Lambda$ and $s = \frac{n(n-1)}{2}$ with $n \geqslant d$. Then*

$$\mathrm{Ext}^j_{k\Sigma_d}\bigl(S^\lambda, N\bigr) \cong \mathrm{Ext}^{j+s}_B\bigl(\mathcal{G}(N)^\tau, w_0 \cdot \lambda\bigr)$$

for $0 \leqslant j \leqslant h(N) + 1$.

Proof. In Proposition 2.2.1, set $M = H^0(\lambda)$, which always has p-restricted head. Then for $0 \leqslant j \leqslant h(N) + 1$.

$$\begin{aligned}
\mathrm{Ext}^j_{k\Sigma_d}\bigl(S^\lambda, N\bigr) &\cong \mathrm{Ext}^j_{S(n,d)}\bigl(H^0(\lambda), \mathcal{G}(N)\bigr) \\
&\cong \mathrm{Ext}^j_G\bigl(H^0(\lambda), \mathcal{G}(N)\bigr) \\
&\cong \mathrm{Ext}^j_G\bigl(\mathcal{G}(N)^\tau, V(\lambda)\bigr).
\end{aligned}$$

According to [Jan, II.4.2(10)], $V(\lambda) = R^s \operatorname{ind}_B^G\ w_0 \cdot \lambda$ where $s = |\Phi^+| = \frac{n(n-1)}{2}$. Therefore, for $0 \leqslant j \leqslant h(N) + 1$

$$\operatorname{Ext}^j_{k\Sigma_d}\big(S^\lambda, N\big) \cong \operatorname{Ext}^j_G\big(\mathcal{G}(N)^\tau, R^s \operatorname{ind}_B^G\ (w_0 \cdot \lambda)\big).$$

Now consider the following spectral sequence [Jan, I.4.5]

$$E_2^{i,j} = \operatorname{Ext}^i_G\big(\mathcal{G}(N)^\tau, R^j \operatorname{ind}_B^G\ (w_0 \cdot \lambda)\big) \quad \Rightarrow \quad \operatorname{Ext}^{i+j}_B\big(\mathcal{G}(N)^\tau, w_0 \cdot \lambda\big).$$

By Serre duality [Jan, II.4.2(9)],

$$\begin{aligned} R^i \operatorname{ind}_B^G(w_0 \cdot \lambda) &\cong \big[R^{s-i} \operatorname{ind}_B^G\big(-(w_0 \cdot \lambda + 2\rho)\big)\big]^* \\ &\cong \big[R^{s-i} \operatorname{ind}_B^G(-w_0\lambda)\big]^*. \end{aligned}$$

Since λ is a dominant weight, so is $-w_0\lambda$ and by Kempf's vanishing theorem,

$$R^{s-i} \operatorname{ind}_B^G(-w_0\lambda) = 0$$

when $s - i > 0$ (or $s > i$). Consequently, there is only one nonzero row in the spectral sequence. The spectral sequence thus collapses and provides an isomorphism for all $i > 0$:

$$\operatorname{Ext}^i_G\big(\mathcal{G}(N)^\tau, R^s \operatorname{ind}_B^G\ (w_0 \cdot \lambda)\big) \cong \operatorname{Ext}^{i+s}_B\big(\mathcal{G}(N)^\tau, w_0 \cdot \lambda\big). \tag{2.3.2}$$

Hence, for $0 \leqslant j \leqslant h(N)$, $\operatorname{Ext}^j_{k\Sigma_d}(S^\lambda, N) \cong \operatorname{Ext}^{j+s}_B(\mathcal{G}(N)^\tau, w_0 \cdot \lambda)$. □

2.4. In this section we recall lower bounds found in [KN] for $h(N)$ when N is a Young, signed Young or dual Specht module. We remark that Proposition 2.3 in [KN] has a minor error, the stability range should read $1 \leqslant j \leqslant t$ not $t + 1$, unless the additional assumption on M having p-restricted head is included. Let $\lambda \in \Lambda$. Then

$$h(N) \geqslant \begin{cases} p - 3 & \text{if } N = Y^\lambda,\ Y^\lambda \otimes \mathrm{sgn}, \text{ or } S_\lambda, \\ 2p - 4 & \text{if } N \cong k. \end{cases}$$

2.5. We can now plug various choices for N into Theorem 2.3.1 to obtain isomorphisms between symmetric group cohomology and B-cohomology.

2.5.1. Theorem. *Let $\lambda, \mu \in \Lambda$ and $s = \frac{n(n-1)}{2}$ with $n \geqslant d$.*

(a) *Assume further that $d = n$ and let $\delta = (1^d)$ be the weight of the determinant representation. Then for $0 \leqslant j \leqslant p - 2$ we have $\mathrm{H}^j(\Sigma_d, S^\lambda) \cong \mathrm{H}^{j+s}(B, w_0 \cdot \lambda' - \delta)$.*
(b) *For $0 \leqslant j \leqslant p - 2$, $\operatorname{Ext}^j_{k\Sigma_d}(S^\lambda, S_\mu) \cong \operatorname{Ext}^{j+s}_B(H^0(\mu), w_0 \cdot \lambda)$.*
(c) *For $0 \leqslant j \leqslant p - 2$, $\operatorname{Ext}^j_{k\Sigma_d}(S^\lambda, Y^\mu) \cong \operatorname{Ext}^{j+s}_B(I(\mu), w_0 \cdot \lambda)$.*
(d) *For $0 \leqslant j \leqslant p - 2$, $\operatorname{Ext}^j_{k\Sigma_d}(S^\lambda, Y^\mu \otimes \mathrm{sgn}) \cong \operatorname{Ext}^{j+s}_B(T(\mu'), w_0 \cdot \lambda)$.*
(e) *For $0 \leqslant j \leqslant 2p - 3$, $\operatorname{Ext}^j_{k\Sigma_d}(S^\lambda, k) \cong \operatorname{Ext}^{j+s}_B(H^0(d), w_0 \cdot \lambda)$. In particular we have*
   (i) *for $1 \leqslant j \leqslant p - 2$, $\operatorname{Ext}^j_G(H^0(d), V(\lambda)) \cong \operatorname{Ext}^{j+s}_B(H^0(d), w_0 \cdot \lambda) = 0$.*
   (ii) $\operatorname{Hom}_{k\Sigma_d}(S^\lambda, k) \cong \operatorname{Ext}^s_B(H^0(d), V(\lambda))$.

Proof. Parts (b)–(e) are immediate from Theorem 2.3.1, just taking various choices of N. To determine the $\mathcal{G}(N)$'s recall from [HN, 3.4.2, 3.8.2] that $\mathcal{G}(S_\lambda) \cong V(\lambda)$, $\mathcal{G}(Y^\lambda) \cong P(\lambda)$ and $\mathcal{G}(Y^\lambda \otimes \mathrm{sgn}) \cong T(\lambda')$.

In general for $n \geqslant d$ we have $\mathcal{G}(\mathrm{sgn}) \cong L(1^d)$, which is $\binom{n}{d}$-dimensional. In part (a) the additional assumption $n = d$ ensures $\mathcal{G}(\mathrm{sgn}) \cong \delta$ so for $0 \leqslant j \leqslant p-2$ we obtain:

$$\begin{aligned} \mathrm{H}^j\big(\Sigma_d, S^\lambda\big) &\cong \mathrm{Ext}^j_{k\Sigma_d}\big(k, S^\lambda\big) \quad \text{by definition,} \\ &\cong \mathrm{Ext}^j_{k\Sigma_d}\big(S^{\lambda'}, \mathrm{sgn}\big) \quad \text{because } S^\lambda \otimes \mathrm{sgn} \cong \big(S^{\lambda'}\big)^*, \\ &\cong \mathrm{Ext}^{j+s}_B(\delta, w_0 \cdot \lambda') \quad \text{by Theorem 2.3.1,} \\ &\cong \mathrm{Ext}^{j+s}_B(k, w_0 \cdot \lambda' - \delta) \quad \text{by tensoring both sides by } \delta^{-1}, \\ &\cong \mathrm{H}^{j+s}(B, w_0 \cdot \lambda' - \delta). \qquad \square \end{aligned} \tag{2.5.2}$$

Since $n \geqslant d$ can vary without changing the symmetric group side of the isomorphisms in Theorem 2.5.1, we can obtain results equating cohomology for Borel subgroups of different general linear groups. Let $B[t]$ denote the Borel subgroup of upper triangular matrices in $\mathrm{GL}_t(k)$ and set $s(t) = t(t-1)/2$. Let $w_{0,t}$ denote the unique element of Σ_t (the Weyl group of $\mathrm{GL}_t(k)$) with maximal length. Then equation (2.5.2) still holds, with $L(1^d)$ in place of δ and we obtain:

2.5.3. Corollary. *Let $\mu \vdash d$ and $n \geqslant d$. For $0 \leqslant j \leqslant p-2$, we have*

$$\mathrm{H}^{s(d)+j}\big(B[d], w_{0,d} \cdot \mu - \delta\big) \cong \mathrm{Ext}^{s(n)+j}_{B[n]}\big(V\big(1^d\big), w_{0,n} \cdot \mu\big).$$

Theorem 2.5.1(b)–(e) yield similar isomorphisms of B-cohomology.

3. Applications

In this section we give two applications of Theorem 2.5.1(a), where known results about symmetric group cohomology give new results about B-cohomology.

3.1. The first application uses work of James, who determined $\mathrm{H}^0(\Sigma_d, S^\lambda) = \mathrm{Hom}_{\Sigma_d}(k, S^\lambda)$ for all $\lambda \vdash d$. It is easy to see that $\mathrm{Hom}_{\Sigma_d}(k, M^\lambda) \cong k$ for all λ. The Specht module S^λ is given as a submodule of M^λ by the kernel intersection theorem [J, 17.18]. Using this, James determined combinatorially exactly when the image of this nonzero homomorphism was inside S^λ. For an integer t let $l_p(t)$ be the least nonnegative integer satisfying $t < p^{l_p(t)}$. James proved:

3.1.1. Theorem. [J, 24.4] *Over a field of characteristic p, S^λ has a submodule isomorphic to the trivial Σ_d-module k if and only if for all i, $\lambda_i \equiv -1 \bmod p^{l_p(\lambda_{i+1})}$.*

Now by using Theorem 3.1.1 together with Theorem 2.5.1(a), we obtain the following result, which determines the first possible nontrivial B-cohomology $\mathrm{H}^i(B, w_0 \cdot \lambda - \delta)$ for $\lambda \in \Lambda$. The cohomology groups $\mathrm{H}^i(B, w_0 \cdot \lambda - \delta) = 0$ for $0 \leqslant i \leqslant s-1$ with $\lambda \in \Lambda$ because of (2.3.2).

3.1.2. Corollary. *Let $G = \mathrm{GL}_d(k)$ and let B be the set of upper triangular matrices in G and $s = \frac{d(d-1)}{2}$. Let $\lambda \vdash d$ with $\lambda' = (\mu_1, \mu_2, \ldots, \mu_r)$. Then*

$$\mathrm{H}^s(B, w_0 \cdot \lambda - \delta) \cong \begin{cases} k & \text{if } \mu_i \equiv -1 \bmod p^{l_p(\mu_{i+1})}, i = 1, 2, \ldots, r, \\ 0 & \text{otherwise}. \end{cases}$$

3.2. In [E], Erdmann determined $\mathrm{Ext}^1_G(V(\lambda), V(\mu))$ whenever λ and μ are partitions of d which differ only in two consecutive rows. In particular she determined the Ext^1-groups for partitions with only two parts. We apply Erdmann's work to calculate $\mathrm{H}^1(\Sigma_d, S^\lambda)$ when λ has only two nonzero parts. First let us recall a stability result.

3.2.1. Lemma. [KN, Corollary 6.3(b)(iii)] *Let $0 \leqslant i \leqslant 2p - 3$. Then*

$$\mathrm{H}^i\big(\Sigma_d, S^\mu\big) \cong \mathrm{Ext}^i_{S(n,d)}\big(V(\mu), V(d)\big).$$

It is also straightforward (see [E, p. 458]) that for $\mu = (\mu_1, \mu_2)$ that

$$\mathrm{Ext}^i_{S(n,d)}\big(V(\mu), V(d)\big) \cong \mathrm{Ext}^i_{\mathrm{SL}_2(k)}\big(V(\mu_1 - \mu_2), V(d)\big).$$

For $i = 1$ these cohomology groups are calculated in [E]. From this we can deduce:

3.2.2. Theorem. *Let $\lambda = (\lambda_1, \lambda_2) \vdash d$ with $\lambda_2 \neq 0$. Then $\mathrm{H}^1(\Sigma_d, S^\lambda)$ is zero except when*:

(i) $\mathrm{Hom}_{\Sigma_d}(k, S^\lambda) \neq 0$ *or*
(ii) $\lambda_2 = p^a$, $a > 0$, *and* $\lambda_1 \neq p^a + p^b - 1$ *for any* $b \geqslant 1$

in which case $\mathrm{H}^1(\Sigma_d, S^\lambda)$ is one-dimensional.

Proof. From Lemma 3.2.1 and the remark following it we know

$$\mathrm{H}^1\big(\Sigma_d, S^\lambda\big) \cong \mathrm{Ext}^1_{\mathrm{SL}_2(k)}\big(V(\lambda_1 - \lambda_2), V(d)\big)$$

and the SL_2 extensions were determined by Erdmann. To translate to Erdmann's notation in [E] we remark that her r is our $\lambda_1 - \lambda_2$, her s is our d, and her d is our λ_2. Then it is just a matter of determining when $\lambda_2 \in \Psi(\lambda_1 - \lambda_2)$. This is a straightforward but a somewhat tedious calculation which entails considering possible p-adic expansions of the numbers λ_2 and $\lambda_1 - \lambda_2$ and gives the result. □

Now for two part partitions one can combine these results to prove the following result for B-cohomology.

3.2.3. Corollary. *Let $G = \mathrm{GL}_d(k)$ and B be the set of upper triangular matrices in G. Let $s = \frac{d(d-1)}{2}$ and $\lambda \vdash d$ with $\lambda_1 \leqslant 2$. Then, setting $\lambda' = (\mu_1, \mu_2)$, we have*

$$\mathrm{H}^{s+1}(B, w_0 \cdot \lambda - \delta) \cong \begin{cases} k & \text{if } \mu_1 \equiv -1 \bmod p^{l_p(\mu_2)} \\ & \text{or } \mu_2 = p^a,\ a > 0, \text{ and } \mu_1 \neq p^a + p^b \text{ for any } b \geqslant 1, \\ 0 & \text{otherwise}. \end{cases}$$

It is interesting to note that when λ has two parts, $\operatorname{Hom}_{\Sigma_d}(k, S^\lambda) \neq 0$ implies that $\operatorname{Ext}^1_{\Sigma_d}(k, S^\lambda) \neq 0$. We know of no examples, other than $\lambda = (d)$, where $\operatorname{Hom}_{\Sigma_d}(k, S^\lambda) \neq 0$ but $\operatorname{Ext}^1_{\Sigma_d}(k, S^\lambda) = 0$. H.H. Andersen has indicated to us that a more general result for arbitrary partitions holds here by using the universal coefficient theorem.

4. Connections with Frobenius kernels

4.1. The category of polynomial representations for $G = \mathrm{GL}_n(k)$ is equivalent to the direct sum of the module categories for $S(n,d)$ for $d \geqslant 0$, which in turn is equivalent to the rational representation theory of the reductive monoid $M = M_n(k)$ of $n \times n$ matrices. Another approach to study representations of G involves the use of the group schemes G_rT associated to the rth Frobenius kernel of G (regarded as a group scheme). Doty, Peters, and the second author [DNP] have created a module category for a monoid scheme M_rD which combines facets of the two aforementioned theories where D is the set of diagonal matrices in M. The monoid scheme M_rD is a natural object of study if viewed as an object in the following commutative diagram of k-functors:

$$\begin{array}{ccccc} T & \longrightarrow & G_rT & \longrightarrow & G \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & M_rD & \longrightarrow & M \end{array}$$

One can define finite-dimensional algebras $S(n,d)_r$ which we call *infinitesimal* Schur algebras whose direct sum of module categories for $d \geqslant 0$ is equivalent to the rational representation theory of M_rD. Note that $S(n,d)_r$ is a subalgebra of $S(n,d)$ for all $r \geqslant 1$.

4.2. For $n \geqslant d$, the idempotent $e = \zeta_{(1,2,\dots,d)(1,2,\dots,d)}$ defined in Section 1.3 lives in $S(n,d)_r$, and $eS(n,d)_re \cong k\Sigma_d$ [DNP, Proposition 5.5]. For $N \in \operatorname{Mod}(k\Sigma_d)$, recall that one has $\mathcal{G}(N) = \operatorname{Hom}_{k\Sigma_d}(V^{\otimes d}, N)$. In this case we are regarding $\mathcal{G}(N)$ as a $S(n,d)_r$-module. Using the construction given in [DEN, Theorem 2.2], for $M \in \operatorname{Mod}(S(n,d)_r)$ and $N \in \operatorname{Mod}(k\Sigma_d)$, we have a first quadrant spectral sequence:

$$E_2^{i,j} = \operatorname{Ext}^i_{S(n,d)_r}\big(M, R^j\mathcal{G}(N)\big) \quad \Rightarrow \quad \operatorname{Ext}^{i+j}_{k\Sigma_d}(eM, N). \tag{4.2.1}$$

For the rest of the paper assume that $p > 3$. Let $n = d$ and let δ denote the determinant representation. Since $p > 3$ we know $\mathcal{G}(\mathrm{sgn}) \cong \delta$ [DEN, Theorem 5.2] and $R^1\mathcal{G}(\mathrm{sgn}) = 0$ [HN, Proposition 3.3.1]. Thus setting $N = \mathrm{sgn}$ in the five term exact sequence associated to the spectral sequence (4.2.1), one obtains the following isomorphisms:

$$\operatorname{Hom}_{M_rD}(M, \delta) \cong \operatorname{Hom}_{S(d,d)_r}(M, \delta) \cong \operatorname{Hom}_{k\Sigma_d}(eM, \mathrm{sgn}), \tag{4.2.2}$$

$$\operatorname{Ext}^1_{M_rD}(M, \delta) \cong \operatorname{Ext}^1_{S(n,d)_r}(M, \delta) \cong \operatorname{Ext}^1_{k\Sigma_d}(eM, \mathrm{sgn}). \tag{4.2.3}$$

For any $Q_1, Q_2 \in \operatorname{Mod}(M_rD)$, one has [DNP, (6.2.1)]

$$\operatorname{Hom}_{G_rT}(Q_1, Q_2) \cong \operatorname{Hom}_{M_rD}(Q_1, Q_2). \tag{4.2.4}$$

Jantzen showed that this also holds for Ext^1 (see [N, §5] and [DNP, Remark 6.2]):

$$\mathrm{Ext}^1_{G_rT}(Q_1, Q_2) \cong \mathrm{Ext}^1_{M_rD}(Q_1, Q_2). \tag{4.2.5}$$

By setting $M = H^0(\lambda)$ and using these isomorphisms, one can conclude that for $i = 0, 1$,

$$\mathrm{Ext}^i_{G_rT}\big(\delta, V(\lambda)\big) \cong \mathrm{H}^i\big(\Sigma_d, S^{\lambda'}\big). \tag{4.2.6}$$

Observe that Theorem 3.1.1 and (4.2.6) allows us to compute $\mathrm{Hom}_{G_rT}(\delta, V(\lambda))$ for all $\lambda \vdash d$.

4.3. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the functors from $\mathrm{Mod}(B)$ to $\mathrm{Mod}(G)$ defined by

$$\mathcal{F}_1(M) = \big(\mathrm{ind}_B^G M\big)^{G_1}, \tag{4.3.1}$$

$$\mathcal{F}_2(M) = \mathrm{ind}_{B/B_1}^{G/G_1}\big(M^{B_1}\big). \tag{4.3.2}$$

The functors $\mathcal{F}_1$ and $\mathcal{F}_2$ are isomorphic (see [Jan, I Proposition 6.12(a)]). From [Jan, I Proposition 6.12(b)], there are two spectral sequences which converge to the same abutment:

$$E_2^{i,j} = \mathrm{Ext}^i_{G_r}\big(\delta, R^j \mathrm{ind}_B^G w_0 \cdot \lambda\big) \quad \Rightarrow \quad R^{i+j}\mathcal{F}_1\big(w_0 \cdot \lambda \otimes \delta^*\big), \tag{4.3.3}$$

$$E_2^{i,j} = R^i \mathrm{ind}_B^G \mathrm{Ext}^j_{B_r}(\delta, w_0 \cdot \lambda) \quad \Rightarrow \quad R^{i+j}\mathcal{F}_2\big(w_0 \cdot \lambda \otimes \delta^*\big). \tag{4.3.4}$$

From our work in Section 2.3, we can conclude that the first spectral sequence collapses and yields the following isomorphisms:

$$\mathrm{Ext}^i_{G_r}\big(\delta, V(\lambda)\big) \cong \mathrm{Ext}^i_{G_r}\big(\delta, R^s \mathrm{ind}_B^G(w_0 \cdot \lambda)\big) \cong R^{i+s}\mathcal{F}_1\big(w_0 \cdot \lambda \otimes \delta^*\big). \tag{4.3.5}$$

By combining these spectral sequences and using the fact that the fixed point functor for T is exact one can construct the following spectral sequences.

4.3.6. Theorem. *Let $\lambda \vdash d$ and $s = \frac{d(d-1)}{2}$. One has the following first quadrant spectral sequences*:

(a) $E_2^{i,j} = R^i \mathrm{ind}_B^G \mathrm{Ext}^j_{B_r}(\delta, w_0 \cdot \lambda) \Rightarrow \mathrm{Ext}^{i+j+s}_{G_r}(\delta, V(\lambda))$,

(b) $E_2^{i,j} = [R^i \mathrm{ind}_B^G \mathrm{Ext}^j_{B_r}(\delta, w_0 \cdot \lambda)]^T \Rightarrow \mathrm{Ext}^{i+j+s}_{G_rT}(\delta, V(\lambda))$.

We can now set $r = 1$. From the spectral sequences given in the preceding theorem and the isomorphism given in (4.2.6), the cohomology of $\mathrm{H}^1(\Sigma_d, S^{\lambda'})$ can be computed by knowing $\mathrm{H}^\bullet(B_1, w_0 \cdot \lambda - \delta)$. For $\mathrm{GL}_d(k)$, the B_1-cohomology groups $\mathrm{H}^\bullet(B_1, \mu)$ are known for all one-dimensional B_1-modules μ only when $p > d$ [AJ, §2.9(2)], whereas the B-cohomology of one-dimensional simple modules is not even known for large primes. In general the B_1-cohomology of simple modules is not presently computed when $p \leqslant d$, and our analysis demonstrates that this calculation should play an important role in understanding the Specht module cohomology.

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