Research Statement for David J. Hemmer

1. Introduction

My research involves the modular representation theory and cohomology of the symmetric group and related objects, including algebraic groups, Frobenius kernels, Schur algebras and superalgebras, and Iwahori-Hecke algebras. Frequently we use connections between these different objects to obtain results.

The complex character theory of the symmetric group Σ_d has been well studied for almost a century. In particular there are beautiful combinatorial formulas for the irreducible characters and a complete description of what happens when one induces and restricts to other symmetric groups. When the field has finite characteristic the situation is entirely different. Not even the dimensions of the simple modules are known and the available branching theorems, while deep and powerful, are far from complete. Connections between the symmetric and general linear groups, understood in characteristic zero in the early twentieth century, are still being explored and producing important new results.

This is an exhilarating time to be working in this field. The Lusztig conjecture is likely the most significant open problem in the representation theory of algebraic groups. In a remarkable recent paper [PS05], Parshall and Scott gave an equivalent statement of the celebrated Lusztig conjecture, for the case of the general linear group $GL_n(k)$, entirely within the representation theory of the symmetric group. Their paper generalized our recent work in [Hem05], and revealed a connection between that work and the Lusztig conjecture.

Meanwhile progress continues on using the connection between Σ_d and $GL_n(k)$ via the Schur functor to study cohomology. In joint work with Nakano [HN04] we applied these techniques to prove the surprising result that multiplicities in a Specht filtration of a Σ_d module are well-defined unless the characteristic is two or three. More recently [Hem06b] we used similar techniques to develop the first criterion, stated entirely in terms of symmetric group cohomology, which is sufficient to guarantee a Σ_d-module has a Specht filtration. Both of these are versions of classical results for algebraic groups which are new for symmetric groups.

2. Notation and Preliminaries

2.1. **Modules for $S(n,d)$ and $k\Sigma_d$.** Henceforth let k be an algebraically closed field of characteristic p. For any $n, d > 0$ there is a finite dimensional algebra called the Schur algebra, $S(n,d)$. Its module category is equivalent to the collection of $GL_n(k)$ modules which are polynomial of homogeneous degree d. Thus we often consider $S(n,d)$ modules as $GL_n(k)$ modules, via this equivalence.

In this section we introduce some important modules for $S(n,d)$ and $k\Sigma_d$. These modules will be labelled by partitions of d. Recall that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ is a *partition* of d with s parts, denoted $\lambda \vdash d$, if

$$\sum_{i=1}^{s} \lambda_i = d \text{ and } \lambda_i \geq \lambda_{i+1} > 0.$$

We say λ is *p-restricted* if $\lambda_i - \lambda_{i+1} < p$ for all i. Say λ is *p-regular* if no part of λ repeats p or more times. We let $\Lambda^+(n,d)$ be the partitions of d with $\leq n$ parts.

Simple $S(n,d)$-modules are indexed by $\Lambda^+(n,d)$, and are denoted $L(\lambda)$. The module $L(\lambda)$ is the socle of the *induced module* $\nabla(\lambda)$ and the head of the *Weyl module* $\Delta(\lambda)$.

When $n \geq d$ there is an exact functor, $\mathcal{F}$, from mod-$S(n,d)$ to mod-$k\Sigma_d$, called the Schur functor. It has a one-sided inverse $\mathcal{G}$. Then $\mathcal{F}(L(\lambda))$ is nonzero exactly when λ is p-restricted. In this case $\mathcal{F}(L(\lambda))$ is a simple $k\Sigma_d$-module, denoted D_λ. The set $\{D_\lambda \mid \lambda \text{ is p-restricted}\}$ forms a complete set of nonisomorphic simple $k\Sigma_d$-modules.

In characteristic zero the simple Σ_d-modules are indexed by partitions of d and are called Specht modules, denoted S^λ. These modules are defined over the integers, so over any field. Over k the module S^λ may not be irreducible. However the irreducible $k\Sigma_d$ modules can be found as socles of Specht modules; when λ is p-restricted then $D_\lambda = \mathrm{soc}(S^\lambda)$. Irreducible $k\Sigma_d$-modules are also labelled by p-regular partitions; if λ is p-regular then $S^\lambda/\mathrm{rad}(S^\lambda) \cong D^\lambda$. The two labellings are related by

$$D_\lambda \otimes \mathrm{sgn} \cong D^{\lambda'}$$

where λ' is the transpose of λ. The D_λ are all self-dual, but the Specht modules in general are not. We let S_λ denote the dual of S^λ.

2.2. **Algebraic groups and the Lusztig conjecture.** Let G be a reductive algebraic group. Let $X = X(T)$ be the set of weights with the usual partial order $\leq$. Let $\Phi \subset X$ be the root system, Φ^+ the set of positive roots, and h be the Coxeter number.

A central problem is to determine the formal character of an irreducible module $L(\lambda)$. The Lusztig conjecture gives a formula, the Lusztig character formula (LCF), for this character when for $p \geq h$ and $\lambda \in W_p.0$ is in the so-called Jantzen region.

When $p \geq 2h-3$, the Lusztig conjecture, together with translation functors and the Steinberg tensor product theorem, would give the formal characters for all the simple modules. There is a well-known equivalent condition for when the LCF holds [CPS93]:

Theorem 2.1. [CPS93] *Let Γ be a finite ideal in $(X^+, \leq)$. The LCF holds for $\Gamma \cap W_p.0$ if and only if for each dominant weight λ and for certain $s \in W_p$ we have:*

$$\mathrm{Ext}^1_G(L(\lambda), L(\lambda s)) \neq 0$$

whenever $\lambda < \lambda s$.

Essentially if there is a path from the zero weight to λ, each step moving to an adjacent weight, and for each step there is a nonsplit extension between the two simple modules, then the LCF holds for $L(\lambda)$. As soon as the path ends, i.e when Ext^1 is zero between the two adjacent simple modules, then the LCF will not hold for the larger weight. Restating the Lusztig conjecture in terms of extensions between simple modules is a key step in framing the conjecture as a problem about symmetric groups.

3. Symmetric groups and the Lusztig conjecture

Parshall and Scott recently generalized our result from [Hem05] on the Ext^1-quiver for the Schur algebra, and closely related the situation to determining the region for which the LCF holds. They further gave an equivalent statement to the Lusztig conjecture for GL_n stated in terms for maps between Specht modules and submodules of Young modules. This is exciting in its own right, but perhaps even more so since it is in terms of Specht modules S^λ where λ has less than p parts. Several recent results about the symmetric group are known *only* in this setting, see for instance [KS99], or the symmetric group results in [Hem05].

3.1. **First row removal and Ext^1-quivers.** James showed that if $\lambda, \mu \vdash d$ have the same first row $\lambda_1 = \mu_1$, then the decomposition number $[\Delta(\lambda) : L(\mu)]$ is equal to the decomposition number $[\Delta(\overline{\lambda}) : L(\overline{\mu})]$ where $\overline{\lambda}$ denotes the partition of $d - m$ obtained by removing λ_1 from λ. We recently proved that first row removal induces an *injection* on Ext^1 between simple modules:

Theorem 3.1. [Hem05] *Let $\lambda, \mu \in \Lambda^+(n, d)$ with $\lambda_1 = \mu_1 = m$. Then there is an injection*

$$\mathrm{Ext}^1_{S(n,d)}(L(\lambda), L(\mu)) \hookrightarrow \mathrm{Ext}^1_{S(n-1,d-m)}(L(\overline{\lambda}), L(\overline{\mu})).$$

Parshall and Scott applied Theorem 2.1 and gave a close connection between the regions where our injection from Theorem 3.1 is an isomorphism with weights for which the LCF holds.

We also proved Theorem 3.1 for the symmetric group and simple modules D^λ and D^μ, but only in the case λ and μ have less than p parts. Thus we can ask:

Problem 3.2. *In* [Hem05] *we proved a symmetric group analogue to Theorem 3.1 for a certain collection of simple modules and conjectured it to hold in general. Can we say anything relating the set of partitions λ, μ where*

$$\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\mu) \cong \mathrm{Ext}^1_{\Sigma_{d-m}}(D_{\overline{\lambda}}, D_{\overline{\mu}})$$

with regions for which the LCF is valid?

4. Fixed-point functors and row removal

This section describes some recent work on symmetric group cohomology, specifically the question of extensions between simple modules. Although the work was originally motivated by an attempt to prove that simple $k\Sigma_d$-modules in odd characteristic do not admit self-extensions, we anticipate it may be applicable to problems discussed in the previous sections as well.

4.1. **Extensions between simple modules.** Determining nonsplit extensions between simple modules for symmetric groups and Schur algebras is a subject of active research, see e.g. [EM94, FM03, Hem01, MR99]. A particular area of interest is the study of *self-extensions*, i.e. determining $\mathrm{Ext}^1_A(S, S)$. For simple $S(n, d)$-modules it has long been known [Jan03, II,2.12] that

$$\mathrm{Ext}^1_{S(n,d)}(L(\lambda), L(\lambda)) = 0. \tag{4.1}$$

The statement corresponding to (4.1) for the symmetric group is conjectured to hold when $p \geq 3$ (it is false for $p = 2$):

Conjecture 4.1 (Kleshchev, Martin)**.** *Let $p \geq 3$. Then:*

$$\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\lambda) = 0.$$

Conjecture 4.1 has been verified in only a few cases.

4.2. **Row-removal theorems.** We recently introduced a new approach, motivated by work of James, to attacking Conjecture 4.1 by induction on d using row-removal. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \vdash d$ let

$$\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_s) \vdash d - \lambda_1$$

denote λ with its first row removed. In 1981 James proved the following theorem for decomposition numbers of $k\Sigma_d$:

Theorem 4.2. [Jam81] *Let $\lambda, \mu \vdash d$ with $\lambda_1 = \mu_1 = m$ and let λ be p-restricted. Then*

$$[S^\mu : D_\lambda] = [S^{\overline{\mu}} : D_{\overline{\lambda}}].$$

In a certain sense theorems about decomposition numbers are really theorems about $\mathrm{Hom} = \mathrm{Ext}^0$, so we believe it is natural to look for theorems relating higher Ext^i with row and column removal. We have made the following conjecture:

Conjecture 4.3. [Hem05] *Let $p \geq 3$ and let λ and μ be p-restricted partitions of d with $\lambda_1 = \mu_1 = m$. Then there is an injection:*

$$\mathrm{Ext}^1_{k\Sigma_d}(D_\lambda, D_\mu) \hookrightarrow \mathrm{Ext}^1_{k\Sigma_{d-m}}(D_{\overline{\lambda}}, D_{\overline{\mu}}).$$

We remark that Conjecture 4.3 immediately implies Conjecture 4.1 by induction on d. Our approach to attacking Conjecture 4.3 has been to consider James' work in [Jam81] and see if it can be applied.

By closely studying James' situation we showed:

Theorem 4.4. [Hem05] *Let $\lambda, \mu \vdash d$ with $\lambda_1 = \mu_1 = m$. Then there is an injection:*

$$0 \to \mathrm{Ext}^1_{S(n,d)}(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{S(n-1,d-m)}(L(\overline{\lambda}), L(\overline{\mu})).$$

Theorem 4.4 was the first result of its kind, and we would like to extend it to the symmetric group, where the vanishing of self-extensions remains open.

We would like to prove Conjecture 4.3. Consider Σ_m and Σ_{d-m} as commuting subgroups of Σ_d permuting $\{1, 2, \ldots, m\}$ and $\{m+1, m+2, \ldots, d\}$ respectively. For $U \in \text{mod-}k\Sigma_d$, the fixed points of U under the action of Σ_m are clearly invariant under the action of Σ_{d-m}. So we can define a fixed-point functor:

$$\mathcal{F}_m : \text{mod-}k\Sigma_d \to \text{mod-}k\Sigma_{d-m}$$

by

$$\mathcal{F}_m(U) = U^{\Sigma_m} \cong \mathrm{Hom}_{\Sigma_m}(k, U) \cong \mathrm{Hom}_{\Sigma_d}(M^{(m,1^{d-m})}, U).$$

In [Hem05] we determined $\mathcal{F}_m$ on dual Specht modules and on the simple modules D_λ when $m < p$. This was enough information to prove:

Theorem 4.5. *Let $\lambda_1 = \mu_1 = m < p$. Then there is an injection:*

$$\mathrm{Ext}^1_{k\Sigma_d}(D_\lambda, D_\mu) \hookrightarrow \mathrm{Ext}^1_{k\Sigma_{d-m}}(D_{\overline{\lambda}}, D_{\overline{\mu}}).$$

The proof of Theorem 4.5 proceeds by analyzing $\mathcal{F}_m$ and using a result of Kleshchev and Sheth [KS99], which is also only valid when $\lambda_1 < p$. Theorems 4.4 and 4.5 were the first to relate the combinatorial notion of row removal with cohomology. We are hopeful that a better understanding of these functors will lead to many more results.

4.3. **Problems.** Our work in [Hem05] suggests several directions for research.

Problem 4.6. *Can we say more about how the functor $\mathcal{F}_m$ behaves when $m \geq p$, i.e. when $\mathcal{F}_m$ is not exact?*

So far we understand it fairly well on dual Specht modules, it gives the dual of skew Specht modules. However understanding it on simple modules and Specht modules is much harder. We hope techniques like those used in [DEN04] to study the Schur and inverse Schur functors can be applied to studying this situation. In [PS05, Remark 5.5], Parshall and Scott raise the possibility of proving the symmetric group version of the Lusztig conjecture by inductively reducing to $\Sigma_p \times \Sigma_{d-p}$. This raises the following:

Problem 4.7. *Can we apply the functor $\mathcal{F}_m$ to give an inductive proof of symmetric group version of the Lusztig conjecture. Following the Parshall-Scott suggestion would lead to considering the case $m = p$. This is the smallest m where $\mathcal{F}_m$ is not exact, so the situation may not be intractable.*

Problem 4.8. *What is the structure of $\mathcal{F}_m(S^\lambda)$ as a $k\Sigma_{d-m}$-module?*

Since even its dimension is unknown, this may be a very difficult problem. Based on some evidence we have conjectured:

Conjecture 4.9. [Hem05] *$\mathcal{F}_m(S^\lambda)$ has a Specht filtration.*

Conjecture 4.9, if true, would be particularly interesting to us since it connects our study of Specht filtrations, discussed in the next section, with our work on fixed-point functors and self-extensions.

5. Specht filtrations for $k\Sigma_d$-modules

An $S(n,d)$-module is said to have a *good filtration* if it has a filtration with successive quotients isomorphic to induced modules. Similarly we will say a module has a *Weyl filtration*, a *Specht filtration*, or a *dual Specht filtration*. A fundamental result (see e.g. [Jan03, II,4.16]) is that when $M \in \text{mod-}S(n,d)$ has a good filtration, the number of times $\nabla(\lambda)$ appears is independent of the choice of filtration and given by a nice formula. There is also a cohomological criterion for M to have a good filtration. The theory of good filtrations is very well-studied, see e.g.[Don87, Kop84, Mat90, Rin91].

The functor $\mathcal{F}$ is exact and maps $\nabla(\lambda)$ to S^λ, so it is natural to look for a theory of "Specht filtrations" for $k\Sigma_d$-modules. But there are well-known examples [Mar93, p.126] where the same module has two different Specht filtrations with different sets of multiplicities. Thus it has long been believed that a theory of Specht filtrations for symmetric groups was impossible. Recently however, Nakano and I discovered that these modules only occur in characteristics two and three. We showed:

Theorem 5.1. [HN04] *Let $p > 3$ and let $M \in \text{mod-}k\Sigma_d$. Then Mhas a Specht filtration if and only if :*

$$\mathrm{Ext}^1_{S(n,d)}(\mathcal{G}(M \otimes \mathrm{sgn}), \nabla(\lambda)) = 0 \ \ for\ all\ \lambda. \tag{5.1}$$

If M has a Specht filtration, then the multiplicities are independent of the choice of filtration, and are given by:

$$[M : S^\lambda] = \dim_k \mathrm{Hom}_{k\Sigma_d}(\mathcal{G}(M \otimes \mathrm{sgn}), \nabla(\lambda)). \tag{5.2}$$

where $\mathcal{G}$ is the adjoint Schur functor.

Theorem 5.1 raises the possibility that a theory of Specht filtrations can be developed for the symmetric group. Some problems we are studying include:

Problem 5.2. *Can we classify $k\Sigma_d$-modules with both Specht and dual Specht filtrations?*

The indecomposable modules for $S(n, d)$ with both good and Weyl filtrations are the set of tilting modules $\{T(\lambda) \mid \lambda \in \Lambda(n, d)\}$. These modules are all self-dual and play an important role in the theory. For $k\Sigma_d$ we know the answer is more complicated. Young and twisted Young modules are included, as well as any irreducible (and hence self-dual) Specht modules. But there are other interesting examples. Recently we discovered modules which have both Specht and dual Specht filtrations but are not self-dual [Hem06b]. Paget and Wildon have constructed modules which are self-dual, have Specht filtrations, and are not signed Young modules.

We believe restricting to indecomposable self-dual modules will make a classification possible. As a starting point we make a conjecture which would imply there are only finitely many of these modules up to isomorphism:

Conjecture 5.3. [Hem06b] *Let $N \in$ mod $k\Sigma_d$ be indecomposable and self-dual. Suppose N has a Specht (and hence also a dual Specht) filtration. Then N is a trivial source module.*

The most obvious choice for N in Conjecture 5.3 is a self-dual Specht module. It is easy to see that in odd characteristic, S^λ is self-dual if and only if it is irreducible. The irreducible Specht modules have only very recently been completely classified by Fayers [Fay05]. This allowed us to prove Conjecture 5.3 for this case. In fact we showed even more:

Theorem 5.4. [Hem06a] *For $p > 2$ any irreducible Specht module is actually a signed Young module. In particular it has trivial source.*

The collection of $S(n, d)$-modules of the form $\mathcal{G}(M)$, where M is a signed Young module are known as listing modules [Don01]. We believe the (larger) collection of $S(n, d)$-modules of the form $\mathcal{G}(M)$ where M has both a Specht and dual Specht filtration, but is not necessarily indecomposable and self dual, will also be an interesting class of $S(n, d)$-modules to study.

Problem 5.5. *Is there a criterion, like the one given in Equation 5.1, for when a $k\Sigma_d$-module M has a Specht filtration, except with the criterion in terms of $k\Sigma_d$-cohomology of M instead of $S(n, d)$-cohomology of $\mathcal{G}(M \otimes \mathrm{sgn})$?*

So far we have made progress on this question only in blocks of small defect, where the answer is yes. Interestingly, in this case the criterion requires vanishing of higher Ext^i, where i depends on p, rather than just vanishing of Ext^1.

In the general case we can prove a criterion that is sufficient but not necessary:

Theorem 5.6. [Hem06b] *Suppose $p > 3$ and let $M \in \mathrm{mod}\text{-}\Sigma_d$. Then:*

(i) *If* $\mathrm{Ext}^1_{k\Sigma_d}(M, S^\lambda) = 0\ \forall \lambda \vdash d$ *then* M *has a dual Specht filtration.*
(ii) *If* $\mathrm{Ext}^1_{k\Sigma_d}(S^\lambda, M) = 0\ \forall \lambda \vdash d$ *then* M *has a dual Specht filtration.*
(iii) *If* $\mathrm{Ext}^1_{k\Sigma_d}(M, S_\lambda) = 0\ \forall \lambda \vdash d$ *then* M *has a Specht filtration.*
(iv) *If* $\mathrm{Ext}^1_{k\Sigma_d}(S_\lambda, M) = 0\ \forall \lambda \vdash d$ *then* M *has a Specht filtration.*

In [HN04] we gave a new construction of Young modules modelled on a similar construction of tilting modules. As an application of this construction we obtained a result which equates certain decomposition numbers $d_{\lambda\mu}$ for $k\Sigma_d$ with the dimension of $\mathrm{Ext}^1_{k\Sigma_d}(S^\lambda, S^\mu)$. This leads us to ask:

Problem 5.7. *To what extent does information about* $\mathrm{Ext}^1_{k\Sigma_d}(S^\lambda, S^\mu)$ *give us information about decomposition numbers for* $k\Sigma_d$*?*

Finally we would like to know:

Problem 5.8. *Can our filtration results (which are actually for Hecke algebras of type A) be extended to cyclotomic Hecke algebras, or other algebras which have natural analogues to Specht modules for the symmetric group (e.g. cellular algebras)?*

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