THE UNIVERSITY OF CHICAGO

EXTENSIONS OF HOOK AND COMPLETELY SPLITTABLE MODULES FOR THE SYMMETRIC GROUP

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ABSTRACT

In this paper we study $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$, where D^λ and D^μ are completely splittable $K\Sigma_d-$ modules. We prove that it is always at most one-dimensional, and give a simple characterization of when it is nonzero. We proceed by first studying $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$ where λ and μ are hooks, i.e. of the form $(d-k, 1^k)$, and then use this information and similar techniques to study extensions of completely splittable modules. Our study of these hook modules also gives a proof that the category of modules for $K\Sigma_{mp}$ with composition factors only of the form D^λ for λ a hook is equivalent to the module category for the principal block of the Schur algebra $S(p,p)$. This category is related to a family of quasi-hereditary algebras of finite type studied by Erdmann [5], and the result leads to several interesting open questions.

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CHAPTER 1
INTRODUCTION

We give here a brief review of the basic facts about the representation theory of the symmetric group, which we denote by Σ_d, and its connection with the representation theory of the general linear group $GL_n(K)$. For a comprehensive treatise of the symmetric group theory we refer the reader to [8] and [9]. For the classic work on the relationship between the representation theory of $GL_n(K)$ and Σ_d, specifically via the Schur functor, see [7].

1.1 Partitions

We say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a *partition* of d with k parts if $\lambda_i \geq \lambda_{i+1} > 0$ and $\sum_{i=1}^{k} \lambda_k = d$. We let $\Lambda^+(n, d)$ be the set of partitions of d into at most n parts, and let $\Lambda^+(d)$ be the set of all partitions of d.

Associated to a partition λ is its *Young diagram* $[\lambda]$, defined as:

$$[\lambda] = \{(i, j) \mid i, j \in \mathbb{N}, 1 \leq j \leq \lambda_i\}.$$

We will represent Young diagrams with boxes and with the positive y-axis pointing down, as illustrated in Figure 1.1.

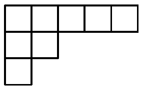
$\lambda = (5, 2, 1)$ $\qquad$ $[\lambda] =$ 

Figure 1.1: The Young diagram for $\lambda = (5, 2, 1)$

If $(i, j) \in [\lambda]$, then (i, j) is called a *node* of $[\lambda]$. The *kth row* of $[\lambda]$ consists of nodes $(k, j) \in [\lambda]$ and the *kth column* consists of nodes $(i, k) \in [\lambda]$.

A node (i, λ_i) is called a *removable node* if $\lambda_i > \lambda_{i+1}$. A node $(i, \lambda_i + 1)$ is called an *addable node* if $i = 1$ or if $i > 1$ and $\lambda_{i-1} > \lambda_i$.

The removable (resp. addable) nodes are those which can be removed from (resp. added to) the diagram $[\lambda]$ to give a valid Young diagram with $d - 1$ (resp. $d + 1$) nodes.

We will often not distinguish between λ and its Young diagram $[\lambda]$. For example, we will freely discuss the nodes of λ, the first row of λ, etc.

For a node $A = (i, j)$, we define its *p-residue* as res $A := j - i \bmod p$. The *residue content* of λ is defined as:

$$\mathrm{cont}(\lambda) = (c_0, c_1, \ldots c_{p-1})$$

where c_k is the number of nodes of $[\lambda]$ of residue i. We write $\lambda \sim \mu$ when $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$. This defines an equivalence relation on $\Lambda^+(d)$ which will play a vital role in the block theory of $K\Sigma_d$.

A partition λ is *p-singular* if

$$\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1}$$

for some i. Otherwise we say λ is *p-regular*. For example $\lambda = (5, 4, 4, 4, 2)$ is 5-regular but not 3-regular. The set of p-regular partitions in $\Lambda^+(n, d)$ is denoted $\Lambda^p(n, d)$ and the set of all p-regular partitions of d is denoted $\Lambda^p(d)$.

For a partition λ, the *conjugate*, or transpose, partition λ' is defined so that $[\lambda']$ is just $[\lambda]$ with the rows and columns reversed.

We say λ is *column p-regular* if λ' is p-regular, and we denote the set of column p-regular partitions in $\Lambda^+(n, d)$ by $\Lambda_p(n, d)$. The set of all column p-regular patitions of d is denoted $\Lambda_p(d)$. These partitions are also referred to in the literature as *p-restricted*.

The set $\Lambda^+(d)$ is totally ordered by the lexicographic order. However there is a more useful partial order which is compatible with lexicographic order, called the dominance order, which we now define. Say that λ *dominates* μ, and write $\lambda \trianglerighteq \mu$, if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i \qquad \forall j \geq 1.$$

A λ-*tableau* is a bijection from $[\lambda]$ to $\{1, 2, \ldots d\}$. We can represent a tableau t by labeling the nodes of $[\lambda]$ with the corresponding numbers $\{1, 2, \ldots d\}$. Clearly Σ_d acts on the set of all λ-tableaux. Fix a λ-tableau t. Let $R(t) \leq \Sigma_d$ denote subgroup of permutations which preserve the rows of t setwise. Note that:

$$R(t) \cong \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \cdots \times \Sigma_{\lambda_k}. \tag{1.1}$$

Similarly let $C(t)$ denote the subgroup of permutations which preserve the columns of t setwise.

Define an equivalence relation on the set of λ-tableaux by $t_1 \sim t_2$ if $t_2 = \pi t_1$ for some $\pi \in R(t_1)$. It then makes sense to define the *tabloid* $\{t\}$ as the equivalence class of the tableau t under the action of $R(t)$. Finally, define a tableau t to be *standard* if its rows and columns are increasing. Figure 1.2 shows a standard $(5, 2, 1)$-tableau. We shall need these concepts to construct Σ_d- modules in the next section.

$$t = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 7 \\ \hline 4 & 6 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$$

Figure 1.2: A standard $(5, 2, 1)$-tableau

1.2 $K\Sigma_d-$ modules

Fix an algebraically closed field K of characteristic $p > 0$. We will discuss here the construction of the irreducible $K\Sigma_d-$ modules. First we construct the irreducible $\mathbb{C}\Sigma_d-$ modules.

For $\lambda \vdash d$, consider the action of Σ_d on the set of all λ-tabloids $\{t\}$. The stabilizer of $\{t\}$ is isomorphic to $R(t)$, so it is clear from (1.1) that this action is the same as the action of Σ_d on cosets of the *Young subgroup* Σ_λ, defined by:

$$\Sigma_\lambda = \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \cdots \times \Sigma_{\lambda_k} \leq \Sigma_d.$$

This $\mathbb{Z}\Sigma_d$ permutation module has a natural basis given by λ-tabloids, and we will denote the module by $M_{\mathbb{Z}}^\lambda = \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_d} 1$. The module $M_{\mathbb{Z}}^\lambda$ is endowed with a natural bilinear form $<,>$ obtained by making the basis of λ-tabloids orthonormal.

For a tableau t, define the column symmetrizer as:

$$C_t = \sum_{\sigma \in C(t)} \mathrm{sgn}(\sigma)\sigma.$$

Then for each λ-tableau t, there is corresponding *polytabloid* $e_t \in M_{\mathbb{Z}}^\lambda$ defined as:

$$e_t = C_t\{t\}.$$

For each $\lambda \vdash d$ there is a *Specht module* S^λ. It is a $\mathbb{Z}\Sigma_d-$ module which is free as a $\mathbb{Z}-$ module and has a basis $\{e_t \mid t \text{ is standard.}\}$ Thus the dimension of the $K\Sigma_d-$ module $S^\lambda := K \otimes_{\mathbb{Z}} S_{\mathbb{Z}}^\lambda$ is independent of K. For more details on the construction of Specht modules and determining their bases, see [8].

In characteristic zero, the S^λ are all irreducible and $\{S^\lambda \mid \lambda \vdash d\}$ is a complete set of nonisomorphic, irreducible $\mathbb{C}\Sigma_d-$ modules. If $p > 0$ the Specht modules are not generally irreducible. However the simple $K\Sigma_d-$ modules can be obtained from the Specht modules. James proved that when $\lambda \in \Lambda^p(d)$, the module S^λ has a simple head, denoted D^λ. He further proved that $\{D^\lambda \mid \lambda \in \Lambda^p(d)\}$ is a complete set of nonisomorphic irreducible $K\Sigma_d-$ modules.

James actually proved that for $\lambda \in \Lambda^p(d)$, the radical of S^λ is equal to $S^\lambda \cap S^{\lambda\perp}$, where $S^{\lambda\perp}$ is defined with respect to the bilinear form $<,>$ on M^λ restricted to S^λ.

Alternatively, for $\lambda \in \Lambda_p(d)$ the Specht module S^λ has a simple socle, denoted D_λ. The set $\{D_\lambda \mid \lambda \in \Lambda_p(d)\}$ is also a complete set of nonisomorphic simple $K\Sigma_d-$ modules.

These two parametrizations of the simple $K\Sigma_d-$ modules are related. It is not difficult to determine that:

$$S^\lambda \otimes \mathrm{sgn} \cong (S^{\lambda'})^*. \tag{1.2}$$

In characteristic zero the module S^λ is simple and self-dual. Thus the bijection on simple modules given by tensoring with the signature representation is described by taking the transpose of the corresponding partition.

In finite characteristic the simple modules are indexed by $\Lambda^p(d)$. But for $\lambda \in \Lambda^p(d)$, the conjugate partition λ' may not be in $\Lambda^p(d)$, so the permutation of simple modules induced by tensoring with the signature representation is necessarily more complicated. There is an involution on $\Lambda^p(d)$ called the *Mullineaux map*, and denoted $m(\lambda)$. It is known that this map satisfies:

$$D^{m(\lambda)} \cong D^\lambda \otimes \mathrm{sgn}. \tag{1.3}$$

The two parametrizations of simple modules are then related by:

$$\begin{array}{ccccc} D_\lambda & \cong & D^{\lambda'} \otimes \mathrm{sgn} & \cong & D^{m(\lambda')} \\ D^\lambda & \cong & D_{\lambda'} \otimes \mathrm{sgn} & \cong & D_{(m(\lambda))'} \end{array}. \tag{1.4}$$

Much remains unknown about the modules D^λ in positive characteristic. For example, the dimension of the Specht modules are determined by a nice formula, but the dimensions of the simple modules D^λ are unknown in most cases.

Also the decomposition numbers, which describe the composition factors of S^λ, are unknown. However it is known that the decomposition matrix is lower unitriangular:

Theorem 1.2.1. *[8, Corollary 12.2] If $\lambda \in \Lambda^p(d)$, then S^λ has a unique top composition factor $D^\lambda \cong S^\lambda/\mathrm{rad}S^\lambda$. If D^μ is a composition factor of $\mathrm{rad}S^\lambda$, then $\mu \rhd \lambda$. If λ is p-singular, then all composition factors of S^λ are of the form D^μ for $\mu \rhd \lambda$.*

It is also known that the modules D^λ are self-dual. In particular then, we know that:

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) \cong \mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\mu, \mathrm{D}^\lambda), \tag{1.5}$$

a fact we shall use several times later.

1.3 Blocks of $K\Sigma_d$ and α-induction.

Although the decomposition matrix for $K\Sigma_d$ is not known, the partition of the complex irreducible and Brauer characters into blocks is well known. The next result is called the "Nakayama Conjecture," although it has been a theorem for many years:

Theorem 1.3.1. *[9, 6.1.21,2.7.41] D^λ and D^μ belong to the same p-block of $K\Sigma_d$ if and only if $\lambda \sim \mu$. Similarly S^λ and S^μ belong to the same block if and only if $\lambda \sim \mu$.*

Theorem 1.3.1 allows us to define, for each residue mod p, blockwise induction and restriction functors. Suppose U is a $K\Sigma_d-$ module in a block corresponding to the residue content $(c_0, c_1, \ldots, c_{p-1})$. Define $\mathrm{Ind}^\alpha U$ to be the summand of $\mathrm{Ind}U$ in the block of $K\Sigma_{d+1}$ corresponding to the residue content $(c_0, c_1, \ldots, c_\alpha + 1, c_{\alpha+1}, \ldots, c_{p-1})$. Similarly define $\mathrm{Res}_\alpha U$ to be the summand of $\mathrm{Res}U$ in the block of $K\Sigma_{d-1}$ corresponding to the residue content $(c_0, c_1, \ldots, c_\alpha - 1, c_{\alpha+1}, \ldots, c_{p-1})$. These functors extend additively to functors:

$$\mathrm{Ind}^\alpha : K\Sigma_d - \mathrm{mod} \Rightarrow K\Sigma_{d+1} - \mathrm{mod}$$

$$\mathrm{Res}^\alpha : K\Sigma_d - \mathrm{mod} \Rightarrow K\Sigma_{d-1} - \mathrm{mod}$$

where the functors are defined on maps in the obvious way. These functors are exact, and are known as *Robinson's α-induction and α-restriction functors.* It is known that no other blocks occur in the modules $\mathrm{Ind}U$ or $\mathrm{Res}U$, so:

$$\mathrm{Ind}U = \bigoplus_{\alpha\in\mathbb{Z}/p\mathbb{Z}} \mathrm{Ind}^{\alpha}U \quad \text{and} \quad \mathrm{Res}U = \bigoplus_{\alpha\in\mathbb{Z}/p\mathbb{Z}} \mathrm{Res}_{\alpha}U.$$

See [9, 6.3.16] for more details on α-induction and restriction.

We will often use a special case of the Eckmann-Shapiro Lemma:

Lemma 1.3.2 (Eckmann-Shapiro). *Let U be a $K\Sigma_d-$ module and V be a $K\Sigma_{d-1}-$ module. Then:*

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{U}, \mathrm{Ind}^{\Sigma_\mathrm{d}}\mathrm{V}) \cong \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-1}}(\mathrm{Res}_{\Sigma_{\mathrm{d}-1}}\mathrm{U}, \mathrm{V}).$$

It is not difficult to prove that Lemma 1.3.2 holds if the functors Ind and Res are replaced by Ind^{α} and Res_{α}.

1.4 Hooks, rim hooks and p-cores

Nakayama's conjecture says S^{λ} and S^{μ} are in the same p-block if and only if λ and μ have the same residue content. There is an equivalent and often useful characterization of this, which we describe in this section.

A partition or Young diagram of the form $(d-r, 1^r)$ is called a hook. Choose any partition λ. For a node $(i,j) \in \lambda$, we denote by H_{ij} the (i,j) *hook* of λ, which consists of the node (i,j) together with all nodes to the right of it in row i and all nodes below it in column j. The node (i, λ_i) is called the *hand* of the hook while the node (λ'_j, j) is the *foot*. The number h_{ij} of nodes in H_{ij} is:

$$h_{ij} = \lambda_i + \lambda'_j + 1 - i - j$$

and is called the *(i,j) hook length*.

The *rim* of λ is the set of nodes $(i,j) \in [\lambda]$ such that $(i+1, j+1) \notin [\lambda]$. To the hook H_{ij} there is a corresponding *rim hook* of the same length. It consists of nodes on the rim of λ between and including the hand and foot of H_{ij}. Note that the (i,j) rim hook has the same number of nodes as H_{ij}. Figure 1.3 shows $(3,2,1)$ with the nodes in the rim hook corresponding to H_{11} circled. Notice that removing a rim hook leaves a diagram of a valid partition.

$$\begin{array}{ccc} \times & \otimes & \otimes \\ \otimes & \otimes & \\ \otimes & & \end{array}$$

Figure 1.3: The $(1,1)$ rim hook of $(3,2,1)$

A *rim p-hook* is a rim hook of length p. A diagram which does not contain any p-hook, and hence no rim p-hook, is called a *p-core*. It is true though not obvious [9, 2.7.16] that each partition λ has a uniquely determined p-core, denoted $\tilde{\lambda}$. One can obtain $\tilde{\lambda}$ by successive removal of rim p-hooks from λ. In the next section we will see part of the proof that the p-core is well-defined, i.e. that $\tilde{\lambda}$ is independent of the order in which rim p-hooks are removed.

The following theorem explains our interest in p-cores:

Theorem 1.4.1. *[9, 2.7.41] Suppose that λ and μ are partitions of d. Then $\tilde{\lambda} = \tilde{\mu}$ if and only if* $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$.

Theorem 1.4.1 gives an equivalent statement of the Nakayama Conjecture, namely that D^λ and D^μ are in the same block if and only if $\tilde{\lambda} = \tilde{\mu}$ if and only if $\lambda \sim \mu$.

The number of rim p-hooks which must be removed from λ to obtain $\tilde{\lambda}$ is called the *p-weight* of λ. This is illustrated in Figure 1.4, where for $\lambda = (5,3,3,2,1)$ we

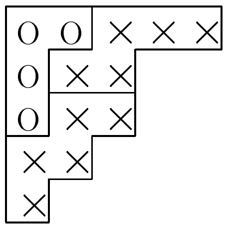


Figure 1.4: The 5-core of $(5,3,3,2,1)$

demonstrate that $\tilde{\lambda} = (2,1,1)$ and that the 5-weight of λ is two. The figure illustrates the removal of two rim 5-hooks, marked with $\times$'s, leaving the 5-core $(2,1,1)$.

1.5 Representing partitions on an abacus

The description of the p-blocks of $K\Sigma_d$ in terms of p-cores makes it unsurprising that adding and removing rim p-hooks to diagrams is a common and useful process. For example, if S^λ is in the principal block $B_0(K\Sigma_d)$ and we remove a rim p-hook from λ to obtain a new partition $\bar{\lambda}$, then $S^{\bar{\lambda}}$ is in $B_0(K\Sigma_{d-p})$. We shall need this fact in Section 4.3.

James devised a convenient means to keep track of such operations on an abacus, and we describe this construction now.

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we define the *first-column hook lengths* of λ by:

$$h_{i1} = \lambda_i - i + k.$$

Given $\{h_{i1}\}$, it is easy to reconstruct λ uniquely.

The set of first-column hook lengths is a special case of the more general notion of *β-numbers*. Fix any $r \geq k$ and define λ_t to be zero for $t > k$. Then the set

$$\{\lambda_1 - 1 + r, \lambda_2 - 2 + r, \ldots, \lambda_r - r + r\} \tag{1.6}$$

is called a *set of β numbers* for λ.

Note that setting $r = k$ recovers the first-column hook lengths as a set of β-numbers. It will be convenient to record β-numbers on an abacus with p vertical runners. We label the possible bead positions as $0, 1, \ldots$, as shown in Figure 1.5.

$$\begin{array}{ccccc} 0 & 1 & \cdots & \text{p-2} & \text{p-1} \\ \text{p} & \text{p+1} & \cdots & \text{2p-2} & \text{2p-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \end{array}$$

Figure 1.5: Labeling of bead positions on the abacus

We number the runners $0, 1, \ldots, p-1$ from left to right. To represent a set of β-numbers on the abacus, simply put a bead at the locations corresponding to the β-numbers. For example, let $\lambda = (8, 5, 3, 2, 2)$ so $k = 5$. Choose $r = 9$. From (1.6) we obtain the set of β-numbers {16,12,9,7,6,3,2,1,0}. For $p = 5$ the abacus representation of this sequence is displayed in Figure 1.6 with the runners labeled.

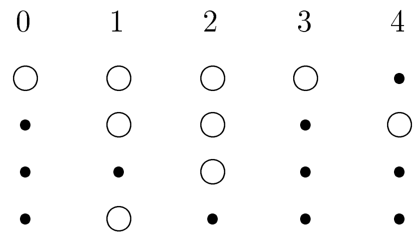


Figure 1.6: β numbers {0,1,2,3,6,7,9,12,16}

Given a bead configuration representing a set of β-numbers of λ, it is simple to recover the first-column hook lengths, and thus determine the corresponding partition. Let c be the smallest abacus position which is unoccupied. Let $\{b_1, b_2, \ldots, b_k\}$ be the locations of the beads in positions $> c$. Then $\{b_i - c\}_{i=1}^{k}$ are the first-column hook lengths of λ.

For example, in Figure 1.6 the first empty position is at bead position four, so $c = 4$. The occupied positions greater than four are $\{6, 7, 9, 12, 16\}$. Subtracting four from these numbers gives $\{2, 3, 5, 8, 12\}$. These are indeed the first-column hook lengths for λ. Thus the number, k, of beads after the first unoccupied position is the number of parts in λ.

We will need to know the effect various moves on the abacus have on the corresponding partition. Most importantly:

Lemma 1.5.1. *[9, 2.7.13] Removing a rim p-hook from λ is equivalent to sliding a bead one space up on the abacus.*

This has an immediate and important corollary:

Corollary 1.5.2. *The p-core $\tilde{\lambda}$ is well-defined.*

The bead configuration corresponding to the p-core occurs when all beads are pushed to the top of the runner. This is clearly independent of the order the beads are moved, i.e. the order in which rim p-hooks are removed. Note that we can immediately determine from the abacus representation if a partition is a p-core. For example we know from Figure 1.6 that $\lambda = (8, 5, 3, 2, 2)$ is not a 5-core. There are two rim 5-hooks which can be removed, corresponding to the beads on runners one and four which can slide up one position.

Adding an addable node to λ corresponds to sliding a bead one position to the right, and removing a removable node corresponds to sliding a bead one position to the left. We can say even more though:

Lemma 1.5.3. *[9, 2.7.38] Provided the abacus has a multiple of p beads (i.e. the set of β-numbers has cardinality divisible by p), a bead on the ith runner corresponds to a row of $[\lambda]$ whose last node has p-residue i.*

Thus sliding a bead from runner i to runner $i + 1$ (which may be runner 0 if $i = p - 1$) corresponds to adding a node of residue $i + 1$. Sliding it from runner i to runner $i - 1$ corresponds to removing a node of residue i.

1.6 The Schur functor

In this section we describe some basic representation theory of the general linear group $GL_n(K)$ and its relationship to that of the symmetric group Σ_d. The basic reference for this section is [7].

Let E be the n-dimensional K-vector space on which $GL_n(K)$ acts naturally. We will assume $n \geq d$. Then Σ_d acts by place permutations on $E^{\otimes d}$, and $GL_n(K)$ acts diagonally on $E^{\otimes d}$, and these two actions commute. One of several equivalent ways to define the Schur algebra $S(n, d)$ is:

$$S(n, d) := \mathrm{End}_{K\Sigma_d}(E^{\otimes d}).$$

It turns out that the category of $S(n,d)-$ modules is equivalent to the category of $GL_n(K)-$ modules which are polynomial of homogeneous degree d. This equivalence is described in detail in [7].

The simple $S(n,d)-$ modules are indexed by partitions λ of d, and the corresponding simple module is denoted $L(\lambda)$.

The representation theory of Σ_d and that of $GL_n(K)$ are related by the *Schur functor*

$$\mathcal{F} : S(n,d) - \text{mod} \Longrightarrow K\Sigma_d - \text{mod}.$$

$\mathcal{F}$ is exact and covariant such that:

$$\mathcal{F}(L(\lambda)) = \begin{cases} D_\lambda & \text{if } \lambda \in \Lambda_p(n,d) \\ 0 & \text{otherwise} \end{cases} . \tag{1.7}$$

We will use this functor in Chapter 4 to obtain results about extensions of $GL_n(K)-$ modules from our results about $K\Sigma_d-$ modules.

CHAPTER 2
BRANCHING RULES

2.1 The Classical Branching Theorem

Branching theorems describe how Σ_d- modules behave on restriction to Σ_{d-1} and on induction to Σ_{d+1}. The complex case is considered well understood, and the main result is known as the classical branching theorem:

Theorem 2.1.1 (Classical branching theorem).

$$(i)\ \mathrm{Ind}_{\mathbb{C}\Sigma_d}^{\mathbb{C}\Sigma_{d+1}} S^{\lambda} \cong \bigoplus_{\text{B addable}} S^{\lambda^B}$$

$$(ii)\ \mathrm{Res}_{\mathbb{C}\Sigma_{d-1}}^{\mathbb{C}\Sigma_d} S^{\lambda} \cong \bigoplus_{\text{A removable}} S^{\lambda_A}$$

The proof of Theorem 2.1.1 actually gives a filtration by Specht modules of the induced and restricted Specht modules which is valid over any field, so we can say even more.

Theorem 2.1.2. *[11, 3.1] Let A_1, A_2, ..., A_s be the removable nodes of λ counted from top to bottom. Then there exists a filtration:*

$$0 = S_0 \subset S_1 \subset \cdots \subset S_s = \mathrm{Res} S^{\lambda}$$

of the $K\Sigma_{d-1}-$ module $\mathrm{Res} S^{\lambda}$ *such that $S_i/S_{i-1} \cong S^{\lambda_{A_i}}$, $i = 1, 2, \ldots, s$.*

The filtration above is often useful, but it is not a complete solution to the branching problem. For example, the structure of $\mathrm{Res} S^{\lambda}$ is still largely unknown. Indeed, not even the socle of this module is not known [11, p.1005].

Theorem 2.1.2 has the expected "dual" result:

Theorem 2.1.3. *Let B_1, B_2, ..., B_l be the addable nodes of λ counted from bottom to top. Then there exists a filtration:*

$$0 = S_0 \subset S_1 \subset \cdots \subset S_l = \mathrm{Ind}S^\lambda$$

of the $K\Sigma_{d+1}-$ module $\mathrm{Ind}S^\lambda$ such that $S_i/S_{i-1} \cong S^{\lambda^{B_i}}$, $i = 1, 2, \ldots, l$.

2.2 Kleshchev's branching theorems

As we mentioned earlier, the branching problem for the modular irreducibles D^λ is extremely complicated. In the complex case, $\mathrm{Res}S^\lambda$ is multiplicity-free with an easily described decomposition given by Theorem 2.1.1. In the modular case, $\mathrm{Res}D^\lambda$ is far from multiplicity-free; indeed the multiplicities of simple modules in $\mathrm{Res}D^\lambda$ can take on any non-negative integral values! (See [11, 3.3] for example.) The difficulty of the modular branching problem is evident from the fact that solving it is equivalent to determining all the decomposition numbers. For an explanation of this equivalence and a thorough survey of the branching problem see [11].

Remarkable progress has been made recently, most notably by Kleshchev, on the modular branching problem. We collect here the branching theorems we shall need later. But first we need some definitions. Fix an arbitrary partition $\lambda \vdash d$.

We say a node $B = (i, j)$ is *above* a node $A = (s, t)$ if $i < s$. A removable node A of λ is *normal* if for every addable node B above A with $\mathrm{res}B = \mathrm{res}A$, there exists a removable node $C(B)$ strictly between A and B with $\mathrm{res}C(B) = \mathrm{res}A$, and $C(B) \neq C(B')$ for $B \neq B'$. A removable node is called *good* if it is the lowest among the normal nodes of a fixed residue. An addable node B is called *good addable* if it is good as a removable node of λ^B.

We give an example of these concepts in Figure 2.1, where we have labeled all the addable and removable nodes with p-residues zero or one.

The top removable node is normal, and hence good, as it is the only removable node of residue one. The next two removables from the top are not normal, but the

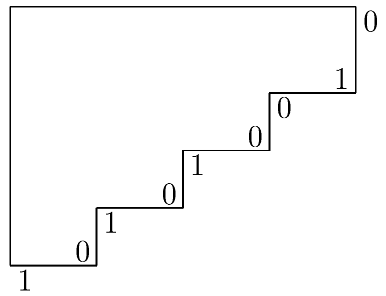


Figure 2.1: Good and normal nodes

lowest removable is normal, and hence good, as it is the lowest (and only) normal node of residue zero.

We now collect the branching results we need. These and several others can be found in Kleshchev's excellent survey paper [11]. Theorem 2.2.1 will be used repeatedly in Chapters 3 and 4.

Theorem 2.2.1. *Let* $\lambda \in \Lambda^p(d)$*.*

1. *Let* $A_1, \ldots A_r$ *be all the good nodes of* λ*. Then:*

$$\mathrm{Res} D^{\lambda} \cong I_1 \oplus I_2 \oplus \cdots \oplus I_r$$

where

$$\mathrm{soc} I_j \cong \mathrm{head} I_j \cong D^{\lambda_{A_j}}$$

and the modules I_j *are indecomposable, self-dual and all belong to distinct blocks. A similar result holds for* $\mathrm{Ind} D^{\lambda}$*. In particular:*

$$\begin{aligned} \#\{\text{indecomposable summands of } \mathrm{Res} D^{\lambda}\} &= \#\{\text{block components of } \mathrm{Res} D^{\lambda}\} \\ &= \#\{\text{good nodes of } \lambda\} \end{aligned}$$

and

$$\begin{aligned}\#\{\text{indecomposable summands of } \mathrm{Ind} D^\lambda\} &= \#\{\text{block components of } \mathrm{Ind} D^\lambda\} \\ &= \#\{\text{good addable nodes of } \lambda\}.\end{aligned}$$

2. $\mathrm{Ind}^{\Sigma_{d+1}} D^\lambda$ *is completely reducible if and only if the number of normal nodes of* λ *is one less than the number of good addable nodes for* λ*, in which case:*

$$\mathrm{Ind}^{\Sigma_{d+1}} D^\lambda \cong \bigoplus_{\text{B good addable}} D^{\lambda^B}.$$

3. $\mathrm{Res}_{\Sigma_{d-1}} D^\lambda$ *is completely reducible if and only if all the normal nodes of* λ *have distinct residues (equivalently, all normal nodes are good), in which case*

$$\mathrm{Res}_{\Sigma_{d-1}} D^\lambda \cong \bigoplus_{\text{A good}} D^{\lambda_A}.$$

4. *If* λ *has no good addable node of residue* α *then*

$$\mathrm{Ind}^\alpha D^\lambda = 0.$$

If λ *has no good node of residue* α *then*

$$\mathrm{Res}_\alpha D^\lambda = 0.$$

CHAPTER 3
HOOK REPRESENTATIONS

In this chapter we begin our study of $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$. There is considerable interest in determining $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$, as this would be a first step toward determining the indecomposable projective modules corresponding to each simple module.

Usually information about Ext^1 is recorded in a directed graph known as the Ext^1-quiver. The vertices are the simple modules and the number of arrows from S to T is the dimension of $\mathrm{Ext}^1(\mathrm{S}, \mathrm{T})$. By (1.5) we can consider the Ext^1- quiver of $K\Sigma_d$ as an undirected graph.

Erdmann, Martin and Russell have determined the Ext^1- quiver for principal blocks of small defect, see [6, 16, 17] for example, but little is known in general. Our interest is motivated by the following conjecture:

Conjecture 3.0.2 (Kleshchev-Martin). *Let $p > 2$ and $\lambda \in \Lambda^p(d)$. Then*

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\lambda) = 0.$$

The results for blocks of small defect and the following theorem provide evidence for this conjecture. For example:

Theorem 3.0.3. *[14] Let λ have less than p parts. Then* $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\lambda) = 0$.

In this chapter we give further evidence by verifying the conjecture for simple modules corresponding to hook partitions. We also determine $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$ for arbitrary hook partitions λ and μ.

3.1 Preliminaries

We will make extensive use of the following easy corollary of Theorem 2.1.3:

Lemma 3.1.1. *[14, 2.1] Let $\lambda \vdash d$ and let A be a removable node of λ with $\alpha = \mathrm{res}A$. If there are no addable nodes C (for λ) above A with $\mathrm{res}C = \alpha$ then there exists a short exact sequence:*

$$0 \to N \to \mathrm{Ind}^{\alpha} S^{\lambda_A} \to S^{\lambda} \to 0,$$

such that N has a Specht filtration with quotients $S^{\lambda_A^B}$, where B runs over the addable nodes of λ_A below A with $\mathrm{res}B = \alpha$.

We will also need to know the structure of the Specht modules $S^{(d-r,1^r)}$, which was determined thirty years ago by Peel [18]:

Theorem 3.1.2. *Let $\lambda = (d-r, 1^r)$ be a p-regular hook with $d-r > 1$ and $r > 1$ (so S^{λ} is not one-dimensional). Then:*

$$S^{\lambda} = \begin{cases} D^{\lambda} & \text{if } p \nmid d \\ \\ \begin{matrix} D^{\lambda} \\ D^{(d-r+1,1^{r-1})} \end{matrix} & \text{if } p \mid d \end{cases} .$$

3.2 Main Result

We are ready to state our main result about hook partitions:

Theorem 3.2.1. *Let $\lambda = (d-r, 1^r)$ and $\mu = (d-s, 1^s)$ be p-regular. Then:*

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^{\lambda}, \mathrm{D}^{\mu}) \cong \begin{cases} K & \text{if } p \mid d,\ |r-s| = 1 \\ 0 & \text{otherwise} \end{cases} .$$

Proof. We first note that for $\lambda = \mu$ the result holds by Theorem 3.0.3 except in the case $\lambda = (d-p+1, 1^{p-1})$, since all other p-regular hook partitions have at most $p-1$ parts. Also, unless $p \mid d$, the hook representations lie in distinct blocks by the Nakayama conjecture, so $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = 0$ for hooks $\lambda \neq \mu$. Thus we need to verify the theorem for $\lambda \neq \mu$ when $p \mid d$ and for any d when $\lambda = \mu = (d-p+1, 1^{p-1})$.

We prove the theorem by induction on d. The base case is well known. For instance Martin [16] has determined the Ext^1- quiver for $d \leq 2p$, and the theorem holds there, so we will assume $d > 2p$ and that the theorem holds for Σ_k, $k < d$.

We handle self extensions (i.e. $\lambda = \mu$) first, so let $\lambda = (d-p+1, 1^{p-1})$. We must show $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\lambda) = 0$. Since $d > p$, all p-regular hooks have two removable nodes. Throughout this proof we let A denote the higher removable node and B denote the lower. If the partition being considered is not clear from context, we will write A_λ to specify the higher removable node of λ. We will consider three separate cases, depending on whether p divides d, $d-1$, or neither.

Case 1: $d \not\equiv 0, 1 \bmod p$.

In this case res $B = 1$, res $A > 1$, and both A and B are normal and good. Since $p \nmid d$ and $p \nmid d-1$, we know by Theorem 3.1.2 that $S^\lambda \cong D^\lambda$, $S^{\lambda_A} \cong D^{\lambda_A}$ and $S^{\lambda_B} \cong D^{\lambda_B}$. Theorem 2.2.1(i) (or the classical branching rule) gives:

$$\mathrm{Res} D^\lambda \cong D^{\lambda_A} \oplus D^{\lambda_B} \tag{3.1}$$

where λ_A and λ_B are in different p-blocks. Since the addable nodes of λ_A below A have residue 0, and $\mathrm{res} A \neq 0$, the module N in Lemma 3.1.1 is 0, so we get by Lemma 3.1.1:

$$0 \to \mathrm{Ind}^{\mathrm{res} A} S^{\lambda_A} \to D^\lambda \to 0.$$

Thus:

$$\begin{aligned}
\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda,\mathrm{D}^\lambda) &= \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{Ind}^{\mathrm{resA}}\mathrm{D}^{\lambda_\mathrm{A}},\mathrm{D}^\lambda)\\
&= \mathrm{Ext}^1_{\mathrm{K\Sigma_{d-1}}}(\mathrm{D}^{\lambda_\mathrm{A}},\mathrm{Res}_{\mathrm{resA}}\mathrm{D}^\lambda)\ \text{ by Shapiro's Lemma}\\
&= \mathrm{Ext}^1_{\mathrm{K\Sigma_{d-1}}}(\mathrm{D}^{\lambda_\mathrm{A}},\mathrm{D}^{\lambda_\mathrm{A}})\ \text{ by Equation 3.1}\\
&= 0\ \text{ by inductive hypothesis.}
\end{aligned}$$

Case 2: $d \equiv 1 \bmod p$.

We have $\lambda_B = (d-p+1, 1^{p-2})$. Notice that λ_B has its top removable node normal of p-residue 1 and its bottom removable node of p-residue 2 is not normal. The lower two (of the three) addable nodes for λ_B are good addable nodes so by Theorem 2.2.1(i) we have:

$$\mathrm{Ind}D^{\lambda_B} = D^\lambda \oplus D^{(d-p+1,2,1^{p-3})} \tag{3.2}$$

By Theorem 3.1.2, $D^\lambda \cong S^\lambda$ so the classical branching theorem yields:

$$0 \to S^{\lambda_A} \to \mathrm{Res}D^\lambda \to S^{\lambda_B} \to 0.$$

Theorem 3.1.2 tells us that:

$$S^{\lambda_A} \cong \begin{array}{c} D^{\lambda_A} \\ D^{\lambda_B} \end{array}, \quad S^{\lambda_B} \cong \begin{array}{c} D^{\lambda_B} \\ D^{\mu} \end{array} \tag{3.3}$$

where $\mu = (d-p+2, 1^{p-3})$.

Since $D^\lambda \cong S^\lambda$, $\mathrm{Res}D^\lambda$ has a filtration with factors S^{λ_A} and S^{λ_B} by Theorem 2.1.2. But A and B both have residue one so Theorem 2.2.1(i) tells us $\mathrm{Res}D^\lambda = \mathrm{Res}_1 D^\lambda$ is indecomposable and self-dual. Putting this information together with (3.3) gives the structure of $\mathrm{Res}D^\lambda$:

$$M := \mathrm{Res}D^\lambda \cong \begin{array}{c} D^{\lambda_B} \\ D^{\lambda_A} D^{\mu} \\ D^{\lambda_B} \end{array}. \tag{3.4}$$

We will see that the module M defined above as a restriction can also be viewed as an induced module from Σ_{d-2}. Define a partition of $d-2$ by:

$$\tau = (d-p, 1^{p-2}).$$

Since λ_B has a unique good node, and removing it gives τ, by Theorem 2.2.1(ii) we get

$$D^\tau \cong S^\tau \cong \mathrm{Res}_{\Sigma_{d-2}} D^{\lambda_B}. \tag{3.5}$$

In the same manner that we determined the structure of M, (applying the classical and Kleshchev branching theorems to $\mathrm{Ind} D^\tau \cong \mathrm{Ind} S^\tau$), we obtain:

$$\mathrm{Ind}^{\Sigma_{d-1}} D^\tau \cong S^{(d-p,2,1^{p-3})} \oplus M \tag{3.6}$$

where the two summands are in different blocks. Thus:

$$M \cong \mathrm{Ind}^1 D^\tau. \tag{3.7}$$

Combining these results gives:

$$\begin{aligned}
\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\lambda) &= \mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{Ind}^1\mathrm{D}^{\lambda_\mathrm{B}}, \mathrm{D}^\lambda) \text{ by (3.2)} \\
&= \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-1}}(\mathrm{D}^{\lambda_\mathrm{B}}, \mathrm{Res}_1\mathrm{D}^\lambda) \text{ by Schapiro} \\
&= \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-1}}(\mathrm{D}^{\lambda_\mathrm{B}}, \mathrm{M}) \text{ by (3.4)} \\
&= \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-1}}(\mathrm{D}^{\lambda_\mathrm{B}}, \mathrm{Ind}^1\mathrm{D}^\tau) \text{ by (3.7)} \\
&= \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-2}}(\mathrm{Res}_1\mathrm{D}^{\lambda_\mathrm{B}}, \mathrm{D}^\tau) \text{ by Schapiro} \\
&= \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-2}}(\mathrm{D}^\tau, \mathrm{D}^\tau) \text{ by (3.5)} \\
&= 0 \text{ by inductive hypothesis.}
\end{aligned}$$

Case 3: $d \equiv 0 \bmod p$.

We begin with a preliminary lemma:

Lemma 3.2.2.

$$S^{(d-p,1^p)} = \begin{array}{c} D^{\phi} \\ D^{\lambda} \end{array}$$

where ϕ is some p-regular partition of d that is not a hook.

Proof. By a theorem of Peel [8, 24.1], $S^{(d-p,1^p)}$ has the correct composition factors. The Specht modules are indecomposable for $p \neq 2$, so it is enough to prove $\mathrm{soc} S^{(d-p,1^p)} \cong D^{\lambda}$.

Peel proved in [18, Thm. 1] that:

$$\mathrm{Res}_{\Sigma_{d-1}} \mathrm{soc} S^{(d-p,1^p)} \cong S^{(d-p,1^{p-1})} \cong D^{\lambda_A}.$$

By Theorem 2.2.1(ii), $\mathrm{Res} D^{\lambda} \cong D^{\lambda_A}$. Also, since no other node of λ_A is good addable, Theorem 2.2.1(ii) implies $\mathrm{Res} D^{\phi} \not\cong D^{\lambda_A}$. Thus the structure of $S^{(d-p,1^p)}$ is as desired. □

In this case $\mathrm{res} A = 0$, and λ has two addable nodes below A of residue 0. Thus the module N has composition factors $S^{(d-p,1^p)}$ and $S^{(d-p,2,1^{p-2})}$. The module $S^{(d-p,2,1^{p-2})}$ has a simple head isomorphic to $D^{(d-p,2,1^{p-2})}$. By Lemma 3.2.2, $S^{(d-p,1^p)}$ has a simple head that is not isomorphic to D^{λ}.

In particular $\mathrm{Hom}_{\Sigma_d}(N, D^{\lambda}) = 0$, so the exact sequence in Lemma 3.1.1 yields:

$$0 \to \mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{S}^{\lambda}, \mathrm{D}^{\lambda}) \to \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-1}}(\mathrm{S}^{\lambda_\mathrm{A}}, \mathrm{Res}_0 \mathrm{D}^{\lambda}). \tag{3.8}$$

But $p \nmid d-1$ so $S^{\lambda_A} \cong D^{\lambda_A}$ and $\mathrm{Res} D^{\lambda} = D^{\lambda_A}$ since A is the only normal node of λ. Thus

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-1}}(\mathrm{S}^{\lambda_\mathrm{A}}, \mathrm{Res}_0 \mathrm{D}^{\lambda}) \cong \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-1}}(\mathrm{D}^{\lambda_\mathrm{A}}, \mathrm{D}^{\lambda_\mathrm{A}}) = 0$$

by the inductive hypothesis. So (3.8) gives

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{S}^{\lambda}, \mathrm{D}^{\lambda}) = 0. \tag{3.9}$$

Finally we apply $\mathrm{Hom}_{\Sigma_d}(-, D^{\lambda})$ to the short exact sequence:

$$0 \to D^{(d-p+2,1^{p-2})} \to S^\lambda \to D^\lambda \to 0$$

to obtain

$$0 \to \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\lambda) \to \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{S}^\lambda, \mathrm{D}^\lambda)$$

so $\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\lambda) = 0$, completing case three.

We now proceed to determine $\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$ for hooks $\lambda \neq \mu$. We will not need to use induction here.

We call hooks $(a, 1^b)$ and $(a+1, 1^{b-1})$ *adjacent.* Unless $p \mid d$, the p-regular hooks have different residue contents, and so the corresponding simple modules are in distinct blocks. When $p \mid d$ we know adjacent hooks have nonsplit extensions by each other, as they occur in the Specht modules. So to complete the proof of Theorem 4.1.1 it is enough to show that for p-regular hooks $\lambda \rhd \mu$, with $p \mid d$, we have:

$$\dim_K \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) \leq 1,$$

and is 0 unless λ and μ are adjacent.

We first handle the case where $\lambda \rhd \mu$ are nonadjacent and $p \mid d$. We write $\overset{a}{\rhd}$ to denote adjacent hooks. In this case we have:

$$\lambda_B \overset{a}{\rhd} \lambda_A \rhd \mu_B \overset{a}{\rhd} \mu_A.$$

Also, since λ and μ are both p-regular, we know $\mathrm{res}A_\lambda \neq \mathrm{res}A_\mu$. Let C be the bottom addable node of λ_A. Then Lemma 3.1.1 gives:

$$0 \to N \to \mathrm{Ind}^{\mathrm{res}A_\lambda} S^{\lambda_A} \to S^\lambda \to 0 \tag{3.10}$$

where N has composition factors $S^{(\lambda_A)^C}$ and perhaps $S^{(-,2,1^-)}$. Both these are p-regular partitions, and neither is equal to μ. Thus $\mathrm{Hom}_{\Sigma_d}(N, D^\mu) = 0$ so applying $\mathrm{Hom}_{\Sigma_d}(-, D^\mu)$ to (3.10) gives:

$$0 \to \mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{S}^\lambda, \mathrm{D}^\mu) \to \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-1}}(\mathrm{S}^{\lambda_\mathrm{A}}, \mathrm{Res}_{\mathrm{resA}_\lambda}\mathrm{D}^\mu).$$

But A_μ is the only normal node of μ and since $\mathrm{res}A_\lambda \neq \mathrm{res}A_\mu$, Theorem 2.2.1(ii) implies $\mathrm{Res}_{\mathrm{res}A_\lambda} D^\mu = 0$. Thus $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{S}^\lambda, \mathrm{D}^\mu) = 0$ which implies $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = 0$ by applying $\mathrm{Hom}_{\Sigma_d}(-, D^\mu)$ to

$$0 \to D^\psi \to S^\lambda \to D^\lambda \to 0.$$

Finally we handle the case of adjacent hooks. We have

$$S^\lambda = \begin{array}{c} D^\lambda \\ D^\psi \end{array}, \quad S^\mu = \begin{array}{c} D^\mu \\ D^\lambda \end{array}, \quad \text{and } \psi \rhd \lambda \rhd \mu.$$

For both λ and μ, $\mathrm{res}A = \mathrm{res}B - 1$, A is normal and B is not. Thus Theorem 2.2.1(ii) gives

$$\mathrm{Res}D^\lambda = D^{\lambda_A}, \ \ \mathrm{Res}D^\mu = D^{\mu_A}.$$

Also observe that $\mathrm{res}A_\lambda = \mathrm{res}A_\mu + 1$, so μ has no normal nodes of residue $\mathrm{res}A_\lambda$.

Consider

$$0 \to N \to \mathrm{Ind}^{\mathrm{res}A_\lambda} S^{\lambda_A} \to S^\lambda \to 0$$

from Lemma 3.1.1. N has as composition factors S^μ and perhaps $S^{(-,2,1^-)}$ if $\mathrm{res}A_\lambda = 0$. Hence $\dim_K \mathrm{Hom}_{K\Sigma_d}(N, D^\mu) \leq 1$. We have:

$$0 \to \mathrm{Hom}_{K\Sigma_d}(N, D^\mu) \to \mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{S}^\lambda, \mathrm{D}^\mu) \to \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-1}}(\mathrm{S}^{\lambda_\mathrm{A}}, \mathrm{Res}_{\mathrm{resA}_\lambda}\mathrm{D}^\mu) \quad (3.11)$$

But $\mathrm{Res}_{\mathrm{res}A_\lambda} D^\mu$ is 0 since μ has no normal nodes of residue $\mathrm{res}A_\lambda$. Thus the last term in (3.11) is 0 so $\dim \mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{S}^\lambda, \mathrm{D}^\mu) \leq 1$. But applying $\mathrm{Hom}_{K\Sigma_d}(-, D^\mu)$ to

$$0 \to D^\psi \to S^\lambda \to D^\lambda \to 0$$

gives

$$0 \to \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) \to \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{S}^\lambda, \mathrm{D}^\mu) \to \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\psi, \mathrm{D}^\mu). \tag{3.12}$$

But ψ and μ are not adjacent so the last term in (3.12) is 0 by the proof of the nonadjacent case above. Hence $\dim_K \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) \leq 1$ as desired. □

3.3 Further Remarks on Hooks

This chapter was inspired by [14] where Kleshchev and Sheth made extensive use of Lemma 3.1.1 to relate $\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$ to the second Loewy layer of S^λ when λ and μ are incomparable partitions with at most $p-1$ parts. In the case where λ and μ are hooks, λ and μ are comparable, so $\mathrm{Hom}_{K\Sigma_d}(N, D^\lambda)$ will not always be zero. Further results may be available in other cases where the module N is not too complicated so that $\mathrm{Hom}_{K\Sigma_d}(N, D^\lambda)$ can be determined and Lemma 3.1.1 applied.

Also in [14] is the following theorem:

Theorem 3.3.1. *[14, 2.10] Let $p > 2$ and let λ and μ be partitions of d with at most $p-1$ nonzero parts. If λ does not strictly dominate μ then:*

$$\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) \cong \mathrm{Hom}_{\mathrm{K\Sigma_d}}(\mathrm{radS}^\lambda, \mathrm{D}^\mu).$$

Recall that the modules D^λ are always self-dual, so we have:

$$\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) \cong \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\mu, \mathrm{D}^\lambda)$$

for any p-regular λ and μ. For λ and μ different hook partitions, either $\lambda \not\triangleright \mu$ or $\mu \not\triangleright \lambda$. Thus Theorems 3.3.1 and 3.1.2 give a much shorter proof of this result for the case $\lambda \neq \mu$. We are grateful to a referee for this remark.

Theorem 3.2.1 was proven for $d = p$ in [3]. The entire Ext^1- quiver was already known for Σ_p, but the proof in [3] is very different, using spectral sequences. The module M we encounter in case 2 above occurs as the middle term in (5.7.2) of [3], also as an induced module, this time from a Young subgroup. However when $d = p$ the

Young subgroups of Σ_d have p' order, so the module M is projective. In particular, it is the projective cover of D^{λ_B}, so $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^{\lambda_\mathrm{B}}, \mathrm{D}^{\lambda_\mathrm{A}})$ can be deduced immediately.

In our example the module M is not projective, but we will see that in the appropriate module category it will be. We will return to these modules M in Chapter 5, where we will prove they are the projectives in the category of $\Sigma_{mp}-$ modules all of whose composition factors are hook modules.

CHAPTER 4
COMPLETELY SPLITTABLE MODULES

4.1 Statement of results

In this chapter we apply techniques similar to those in the previous chapter to determine $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$ for an entirely different class of $K\Sigma_d-$ modules called completely splittables. We will see however that they are not entirely unrelated to hook modules. We begin with the definition:

Definition: An irreducible $K\Sigma_d-$ module D^λ is called *completely splittable* if and only if the restriction $D^\lambda \downarrow_{\Sigma_\mu}$ to any Young subgroup $\Sigma_\mu < \Sigma_d$ is semisimple. We will also say λ is completely splittable.

In [13], Kleshchev and Nakano obtained several results about the cohomology of completely splittable modules, suggesting that obtaining the Ext^1- quiver for these modules should be possible. In this chapter we apply Kleshchev's branching theorems to prove:

Theorem 4.1.1. *Let D^λ and D^μ be completely splittable and in the same p-block of $K\Sigma_d$. Then*

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = \begin{cases} K & \text{if } h_{11}(\lambda) \geq p,\ h_{11}(\mu) \geq p \\ & h_{21}(\lambda) < p,\ h_{21}(\mu) < p \\ & \text{and } \mid h(\lambda) - h(\mu) \mid = 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

where $h(\lambda)$ is the height of λ and

$$h_{ij}(\lambda_1, \lambda_2, \ldots, \lambda_k) = \lambda_i + \lambda'_j + 1 - i - j$$

is the (i, j) hook length.

This theorem plus the work of [13] yields a corresponding result for $GL_n(K)$ when $n \geq d$. Let $L(\lambda)$ denote the simple, polynomial $GL_n(K)-$ module with highest weight λ. Recall that m denotes the Mullineaux map on p-regular partitions defined by:

$$D^{\lambda} \otimes \mathrm{sgn} \cong D^{m(\lambda)}.$$

Then:

Corollary 4.1.2. *Let D^{λ} and D^{μ} be completely splittable. Then:*

$$\mathrm{Ext}^1_{\mathrm{GL_n}}(\mathrm{L}(\mathrm{m}(\lambda)'), \mathrm{L}(\mathrm{m}(\mu)')) = \begin{cases} K & \textit{if } h_{11}(\lambda) \geq p,\ h_{11}(\mu) \geq p \\ & h_{21}(\lambda) < p,\ h_{21}(\mu) < p \\ & \textit{and } \mid h(\lambda) - h(\mu) \mid = 1 \\ 0 & \textit{otherwise} \end{cases}$$

The Ext^1 in Corollary 4.1.2 is taken in the category $M(n, d)$ of polynomial $GL_n(K)$ modules of homogeneous degree d.

4.2 Background about completely splittable modules

We now recall the main results of [12] on completely splittable modules. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ define:

$$h(\lambda) := k \quad \text{and} \quad \chi(\lambda) := \lambda_1 - \lambda_k + h(\lambda).$$

Then:

Theorem 4.2.1. *[12] D^{λ} is completely splittable if and only if $\chi(\lambda) \leq p$.*

We remark than $\chi(\lambda) = h_{1,\lambda_k}$ is the length of the hook with foot at (k, λ_k). We further remark that Theorem 4.2.1 holds as well for $p = 2$, but since there are no nontrivial completely splittable modules in this case, we can assume $p > 2$ for this chapter.

It is clear from the definition that restricting a completely splittable module to Σ_{d-1} will give a direct sum of completely splittable $\Sigma_{d-1}-$ modules, since Σ_{d-1} is isomorphic to the Young subgroup Σ_μ for $\mu = (d-1, 1)$. Kleshchev determined this decomposition:

Theorem 4.2.2. *[12] Let D^λ be completely splittable. Then:*

$$\mathrm{Res}_{\Sigma_{d-1}} D^\lambda \cong \bigoplus D^{\lambda_A}$$

where the sum is over all removable nodes A with $\chi(\lambda_A) \leq p$.

We will make extensive use of induced modules, so the branching theorems will be of particular importance. We elaborate further on them here. First, we state a simple corollary of Theorem 2.1.3:

Lemma 4.2.3. $S^\lambda \cong \mathrm{Ind}^{\mathrm{res}A} S^{\lambda_A}$ *if and only if A is the unique addable node of λ_A of residue* $\mathrm{res}A$.

Theorem 2.2.1 has obvious restatements in terms of α-induction and α-restriction. For example:

Lemma 4.2.4. *Let B be a good addable node for λ. Then:*

$$\mathrm{soc}(\mathrm{Ind}^{\mathrm{res}B} D^\lambda) \cong \mathrm{head}(\mathrm{Ind}^{\mathrm{res}B} D^\lambda) \cong D^{\lambda^B}.$$

If there is no good addable node of λ of residue α then $\mathrm{Ind}^\alpha D^\lambda = 0$.

Because there is at most one good addable node of each residue, Lemma 4.2.4 makes sense.

If the Specht module S^λ is α-induced from Σ_{d-1} as in Lemma 4.2.3 and if D^λ is completely splittable, then D^λ is also α-induced from Σ_{d-1}:

Lemma 4.2.5. *Let $\chi(\lambda) \leq p$. If $S^\lambda \cong \mathrm{Ind}^{\mathrm{res}A} S^{\lambda_A}$, then $D^\lambda \cong \mathrm{Ind}^{\mathrm{res}A} D^{\lambda_A}$.*

Proof. Since $\chi(\lambda) \leq p$, λ_A must be p-regular. Also, there are no removable or addable nodes of λ of residue res$A$ except A and perhaps the bottom addable node. Thus A is good addable for λ_A. We apply $\mathrm{Ind}^{\mathrm{res}A}$ to:

$$0 \longrightarrow R \longrightarrow S^{\lambda_A} \longrightarrow D^{\lambda_A} \longrightarrow 0$$

to give:

$$0 \longrightarrow \mathrm{Ind}R \longrightarrow S^{\lambda} \longrightarrow \mathrm{Ind}D^{\lambda_A} \longrightarrow 0.$$

But $[S^\lambda : D^\lambda] = 1$ so Lemma 4.2.4 applied to λ_A implies the result. ∎

4.3 Minimal modules

We wish to determine $\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$ for completely splittable modules D^λ and D^μ by using induction and the Eckmann-Shapiro lemma. It is not surprising then that it is useful to determine which completely splittable modules arise by α-inducing a completely splittable $\Sigma_{d-1}-$ module from Σ_{d-1} to Σ_d. This motivates the following definition:

Definition: A completely splittable module D^λ is *minimal* if there does not exist $\mu \vdash d-1$, with D^μ completely splittable, such that $D^\lambda \cong \mathrm{Ind}^\alpha D^\mu$ for some α. If D^λ is minimal, we also say λ is minimal.

Notice that if D^λ and D^μ are completely splittable and D^λ is not minimal then there is some removable node A of λ such that D^{λ_A} is completely splittable and $D^\lambda \cong \mathrm{Ind}^{\mathrm{res}A} D^{\lambda_A}$. In this case:

$$\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) \cong \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{Ind}\mathrm{D}^{\lambda_\mathrm{A}}, \mathrm{D}^\mu) \cong \mathrm{Ext}^1_{\mathrm{K\Sigma_{d-1}}}(\mathrm{D}^{\lambda_\mathrm{A}}, \mathrm{Res}_{\mathrm{resA}}\mathrm{D}^\mu). \quad (4.1)$$

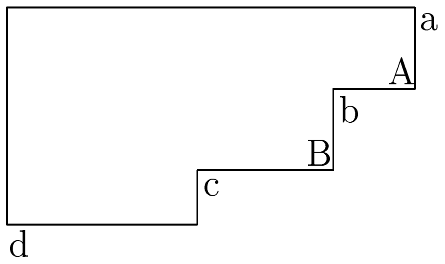
$\mathrm{Res}_{\mathrm{resA}} D^\mu$ is either 0 or completely splittable by Theorem 4.2.2, so (4.1) and (1.5) imply that unless both partitions are minimal, we can determine $\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$

from knowledge of the Ext^1- quiver for completely splittable modules for Σ_{d-1}. Thus our next goal will be to classify the minimal modules and study their extensions.

We begin classifying the minimal modules with the following lemma:

Lemma 4.3.1. *If D^λ is minimal then λ is either of the form $(\lambda_1^{a_1})$ or $(\lambda_1^{a_1}, \lambda_2^{a_2})$.*

Proof. Suppose $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_t^{a_t})$ is minimal and $t > 2$. Then the diagram of λ looks like:

$\lambda =$ 

where A and B are the top two of the at least three removable nodes and a, b and c are the top three of the at least four addable nodes. Note that the residues of a, b, c, A, and B are all distinct because $\chi(\lambda) \leq p$. Thus at least one of A or B, (whichever has residue not equal to res$d$) will be the unique addable node of its residue when it is removed from λ. Suppose, for example, that resA $\neq$ resd. Since $\chi(\lambda_A) \leq \chi(\lambda_B) = \chi(\lambda) \leq p$, we know D^{λ_A} is completely splittable. Then Lemmas 4.2.3 and 4.2.5 imply:

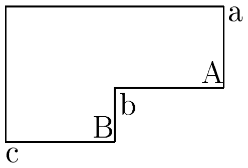
$$D^\lambda \cong \mathrm{Ind} D^{\lambda_A}$$

so λ is not minimal, contradicting our assumption. The same argument works with D^{λ_B} if resB $\neq$ resd. Thus the assumption that $t > 2$ contradicts the assumption that D^λ is minimal. $\square$

We can now completely classify the minimal modules:

Theorem 4.3.2. *A completely splittable $K\Sigma_d-$ module D^λ is minimal if and only if $p \mid d$ and D^λ is in the principal block $B_0(K\Sigma_d)$.*

Proof. Let D^λ be completely splittable and minimal. By Lemma 4.3.1, we can assume λ has at most two distinct parts. Suppose first that $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2})$ for $\lambda_1 \neq \lambda_2$. We label the addable and removable nodes of λ as follows:

$\lambda =$ 

We calculate the following congruences mod-p:

$$\begin{aligned} \text{res } A &\equiv \lambda_1 - a_1 \\ \text{res } B &\equiv \lambda_2 - a_1 - a_2 \\ \text{res } a &\equiv \lambda_1 \\ \text{res } b &\equiv \lambda_2 - a_1 \\ \text{res } c &\equiv -a_1 - a_2 \\ \chi(\lambda) &= \lambda_1 - \lambda_2 + a_1 + a_2. \end{aligned} \tag{4.2}$$

Notice that Lemmas 4.2.3 and 4.2.5 imply $D^\lambda \cong \mathrm{Ind} D^{\lambda_A}$ unless $\mathrm{res}A = \mathrm{res}c$. But λ is minimal so we can assume $\mathrm{res}A = \mathrm{res}c$. Because $\chi(\lambda) \leq p$ we know $\mathrm{res}A \neq \mathrm{res}B$, and hence $\mathrm{res}B \neq \mathrm{res}c$. Using Lemmas 4.2.3 and 4.2.5 again, we conclude that $\mathrm{res}B = \mathrm{res}a$ (else $D^\lambda \cong \mathrm{Ind}^{\mathrm{res}B} D^{\lambda_B}$).

Setting $\mathrm{res}B = \mathrm{res}a$ and $\mathrm{res}A = \mathrm{res}c$, a simple calculation using (4.2) shows that $\chi(\lambda) = p$ and $\mathrm{resb} = 0$.

But $\mathrm{res}b = 0$ if and only if $\lambda_2 \equiv a_1$. And $\mathrm{res}A = \mathrm{res}c$ if and only if $\lambda_1 \equiv -a_2$. So D^λ being minimal implies $\lambda_2 \equiv a_1$ and $\lambda_1 \equiv -a_1$ which yields:

$$d = a_1\lambda_1 + a_2\lambda_2 \equiv -a_1a_2 + a_1a_2 = 0$$

so $p \mid d$.

Before proceeding further, we verify that $\chi(\lambda) = p$ and $\mathrm{res}b = 0$ are sufficient conditions for the partition $(\lambda_1^{a_1}, \lambda_2^{a_2}) \vdash mp$ to be minimal:

Lemma 4.3.3. *Let* $(\lambda_1^{a_1}, \lambda_2^{a_2}) \vdash mp$ *satisfy* $\chi(\lambda) = p$ *and* $\mathrm{res}b = 0$ (*i.e.* $\lambda_2 \equiv a_1$). *Then* λ *is minimal.*

Proof. First suppose $m = 1$ so λ is a hook. Then $D^{\lambda_A} \cong S^{\lambda_A}$ and $D^{\lambda_B} \cong S^{\lambda_B}$ (because λ_A and λ_B partition $p-1$, and $K\Sigma_{p-1}$ is semisimple). Then λ_A has two addable nodes of residue res$A$, so the classical branching theorem implies $\mathrm{Ind}^{\mathrm{resA}} D^{\lambda_A} \cong \mathrm{Ind}^{\mathrm{resA}} S^{\lambda_A}$ is not simple. Similarly $\mathrm{Ind}^{\mathrm{res}B} D^{\lambda_B}$ is not simple, so λ must be minimal.

Next suppose $m > 1$. We are assuming $\lambda_2 \equiv a_1$ and we know $a_1 < p$ since λ is p-regular. Hence, if $\lambda_2 = 1$ then $a_1 = 1$ and $d = \lambda_1 + a_2 = \chi(\lambda) = p$, contradicting $m > 1$. Thus $\lambda_2 > 1$ so $\chi(\lambda_B) = p + 1$ and λ_B is not completely splittable. So to prove that λ is minimal we must verify that $D^\lambda \not\cong \mathrm{Ind}^{\mathrm{res}A} D^{\lambda_A}$.

If $\lambda_1 - \lambda_2 = 1$ then λ_A has one normal node, the top removable, and one good addable node, A. Thus $\mathrm{Ind}^{\mathrm{res}A} D^{\lambda_A} \cong \mathrm{Ind} D^{\lambda_A}$ is not simple by Lemmas 4.2.4 and 2.2.1(iii).

If $\lambda_1 - \lambda_2 > 1$ then λ_A has two normal nodes, the top two removables, and two good addable nodes, A and b. The node b has residue $0 \neq \mathrm{res}A$ and $\mathrm{Ind}^0 D^{\lambda_A}$ is simple by Lemmas 4.2.3 and 4.2.5. But $\mathrm{Ind} D^{\lambda_A}$ is not semisimple by Theorem 2.2.1(iii), so $\mathrm{Ind}^{\mathrm{res}A} D^{\lambda_A}$ is not simple. This completes the proof that λ is minimal. □

Consider the rim hook with foot at B. It has p nodes since $\chi(\lambda) = p$. Removing it preserves a_1 and a_2 and decreases λ_1 and λ_2 by one. Thus, by (4.2) it preserves $\chi = p$ and res$b = 0$, so preserves minimality by Lemma 4.3.3. Of course, it also preserves membership in B_0, by Theorem 1.4.1. Thus $\lambda \in B_0$ by induction.

Now suppose λ is rectangular, i.e. $\lambda = (\lambda_1^{a_1})$ with $a_1 < p$, and let A be the one removable node of λ. Unless $p \mid \lambda_1$, the partition λ_A will have a unique addable node of residue res$A$ and, as in the previous case, $D^\lambda \cong \mathrm{Ind} D^{\lambda_A}$. So the minimality of λ implies $p \mid \lambda_1$ so $p \mid d$ and $\lambda \in B_0$. An argument as in the proof of Lemma 4.3.3 confirms that $\lambda = (cp^{a_1})$ is indeed minimal.

To complete the proof of Theorem 4.3.2 we must show that a completely splittable module $D^\lambda \in B_0(K\Sigma_{mp})$ must be minimal. This too follows by induction. The base case $m = 1$ is just $\lambda = (p - r, 1^r)$ for $0 \leq r \leq p - 2$. In this case D^λ is minimal by Lemma 4.3.3. Suppose then that $m > 1$ and D^λ is completely splittable and in $B_0(K\Sigma_{mp})$. Since $\chi(\lambda) \leq p$ and the rim of λ must have at least p nodes (else λ is a p-core, so not in the principal block), there must be a rim p-hook with head at $(1, \lambda_1)$.

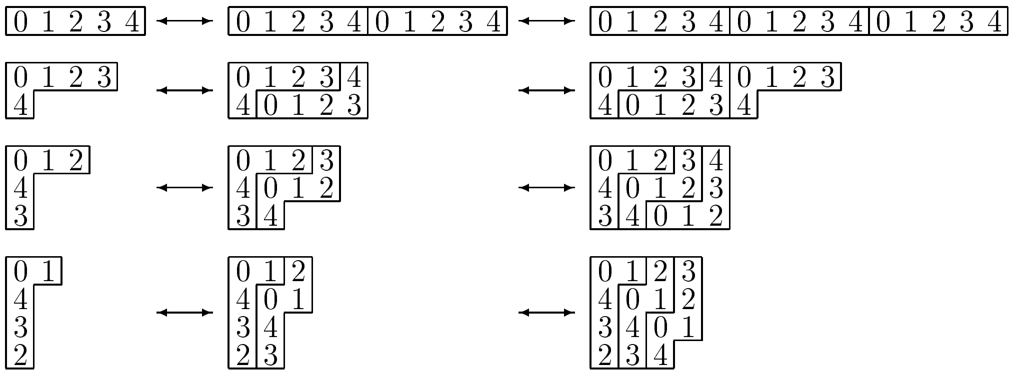


Figure 4.1: Minimal partitions for $p = 5$ and $n = 5, 10, 15$

Strip off this rim hook to get $\overline{\lambda}$. Clearly $\chi(\overline{\lambda}) \leq p$ so $D^{\overline{\lambda}}$ is completely splittable and in $B_0(K\Sigma_{(m-1)p})$, so is minimal by induction.

We leave it to the reader to verify that there is only one way to add a hook to a minimal partition such that the result is completely splittable and such that the head of the hook ends up in the first row. In particular the foot of the hook will be in the last row. (See for example Figure 4.1). Doing so preserves $\chi(\lambda) = p$ and $\mathrm{res} b = 0$, and hence produces another minimal partition. So we conclude that λ must be minimal, completing the proof of Theorem 4.3.2. $\square$

We now determine how many minimal modules there are, and some information we will need to determine the extensions between them.

Lemma 4.3.4. *Let $d = pm$. There are $p - 1$ minimal partitions of d. They have $1, 2, \ldots p - 1$ parts. The top removable node of the one with i parts has residue $p - i$.*

Proof. The result is clear for $m = 1$, the partitions are just the p-regular hooks and there are $p-1$ of them. Now observe that removing the rim p-hook with head $(1, \lambda_1)$ or adding a rim p-hook with foot at $(k, \lambda_k + 1)$ gives a bijection between minimal partitions of mp and $(m+1)p$ that preserves the number of parts and the residue of the top removable node. The bijection for $p = 5$ and $n = 5,\ 10,\ 15$ is illustrated in Figure 4.1, with the p-residues labelled. $\square$

We will need a few more technical results about minimal partitions before we can begin determining the Ext^1- quiver.

Lemma 4.3.5. *Let $\lambda \rhd \mu$ be minimal partitions of $d = mp$ with $m > 1$. Then $\mu \not\trianglerighteq \lambda_A^B$ for any removable node A and addable node B of λ.*

Proof. First we recall the well known fact [9, 1.4.10] that partitions $\rho \rhd \tau$ are neighbors in the dominance order if and only if τ is obtained from ρ by moving single node down to the next available location. There are few cases to check, since minimal partitions have at most two removable nodes and at most three addable nodes. First note that $m > 1$ implies λ_A^B is never minimal. Let $\overset{a}{\trianglerighteq}$ denote partitions which are neighbors in the dominance order. Suppose we let A be the top removable node of λ. Then if we remove A there are two addable nodes distinct from A, which we denote B and B'. Figure 4.2 illustrates λ, λ_A^B and $\lambda_A^{B'}$.

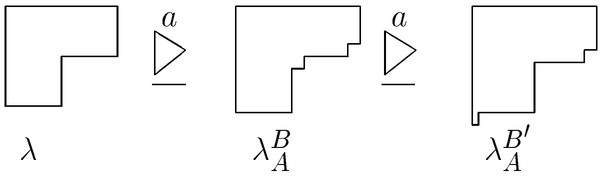


Figure 4.2: Neighbors in dominance order

Since the three partitions drawn in Figure 4.2 are all adjacent in the dominance order, and only λ is minimal, there can not be a minimal partition μ with $\lambda \rhd \mu \not\trianglerighteq \lambda_A^B$. The other case, where A is the lower removable node of λ, is similar. $\square$

The final observation about minimal partitions we need is clear from Figure 4.1:

Lemma 4.3.6. *The minimal partitions are totally ordered by $\trianglerighteq$ and $\lambda \rhd \mu$ if and only if λ has fewer parts than μ.*

4.4 Ext^1 for minimal modules

Recall from (4.1) that if we can determine $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$ for minimal modules D^λ and D^μ, we should be able to determine it for all completely splittable modules. Thus we begin our consideration of extensions between minimal modules by proving:

Theorem 4.4.1. *Let $\lambda \neq \mu$ be minimal partitions of mp and $m > 1$. Then*

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{mp}}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = 0.$$

Proof. By Lemma 4.3.6 and Equation 1.5, we can assume without loss of generality that $\lambda \rhd \mu$. Let A be the top removable node of λ. By Lemma 4.3.4, the top removable node of μ has residue different from resA. But, since $m > 1$, removing the bottom removable node of μ (if it has two), leaves a partition with $\chi = p+1$. Thus by Lemma 4.2.2,

$$\mathrm{Res}_{\mathrm{resA}} D^\mu = 0. \tag{4.3}$$

We assumed λ is minimal so $D^\lambda \ncong \mathrm{Ind}^{\mathrm{res}A} D^{\lambda_A}$. However, A is the only good addable node for λ_A, so by Lemma 4.2.4, $\mathrm{Ind}^{\mathrm{res}A} D^{\lambda_A}$ has the following structure:

$$\mathrm{Ind}^{\mathrm{resA}} D^{\lambda_A} \cong \mathrm{Ind} D^{\lambda_A} \cong \begin{matrix} D^\lambda \\ H \\ D^\lambda \end{matrix}. \tag{4.4}$$

Equation 4.3 plus the Eckmann-Shapiro lemma proves:

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{mp}}}(\mathrm{Ind}\mathrm{D}^{\lambda_\mathrm{A}}, \mathrm{D}^\mu) = 0. \tag{4.5}$$

Consider the exact sequence arising from (4.4):

$$0 \longrightarrow D^\lambda \longrightarrow \mathrm{Ind} D^{\lambda_A} \longrightarrow \begin{matrix} D^\lambda \\ H \end{matrix} \longrightarrow 0.$$

Apply $\mathrm{Hom}_{\Sigma mp}(-, D^\mu)$ to this to obtain a long exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{mp}}}\left(\begin{matrix} D^\lambda \\ H \end{matrix}, \mathrm{D}^\mu\right) \longrightarrow \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{mp}}}(\mathrm{Ind}\mathrm{D}^{\lambda_\mathrm{A}}, \mathrm{D}^\mu) \longrightarrow$$

then use (4.5) to conclude that:

$$\mathrm{Ext}^1_{\mathrm{K\Sigma_{mp}}}(\begin{array}{c} D^\lambda \\ H \end{array}, \mathrm{D}^\mu) = 0. \tag{4.6}$$

Similarly we apply $\mathrm{Hom}_{K\Sigma_{mp}}(-, D^\mu)$ to

$$0 \longrightarrow H \longrightarrow \begin{array}{c} D^\lambda \\ H \end{array} \longrightarrow D^\lambda \longrightarrow 0$$

to obtain a long exact sequence and conclude, by (4.6), that:

$$\mathrm{Ext}^1_{\mathrm{K\Sigma_{mp}}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) \cong \mathrm{Hom}_{\mathrm{K\Sigma_{mp}}}(\mathrm{H}, \mathrm{D}^\mu). \tag{4.7}$$

But H is a subquotient of $\mathrm{Ind}^{\mathrm{resA}} D^{\lambda_A}$ so it is a subquotient of $\mathrm{Ind}^{\mathrm{resA}} S^{\lambda_A}$. However $\mathrm{Ind}^{\mathrm{resA}} S^{\lambda_A}$ has a filtration with factors S^λ and $S^{\lambda_A^B}$ by Lemma 2.1.3. (B is the lowest addable node of λ.) Since a Specht module S^τ has a composition factor D^ρ only if $\rho \trianglerighteq \tau$, Lemma 4.3.5 implies there are no composition factors D^μ in H. Thus $\mathrm{Hom}_{\Sigma_{mp}}(H, D^\mu) = 0$, and the proof is complete by (4.7). □

Often a completely splittable module D^λ can be obtained by α-induction from more than one completely splittable $K\Sigma_{d-1}-$ module. The next lemma determines when this happens:

Lemma 4.4.2. *Let D^λ be completely splittable, and A be a removable node of λ. Then D^{λ_A} is completely splittable and $D^\lambda \cong \mathrm{Ind}^{\mathrm{resA}} D^{\lambda_A}$ unless:*

(i) $\chi(\lambda) = p$ and A is the bottom removable node of λ or:

(ii) resA *is equal to the residue of the lowest addable node of λ.*

Proof. Suppose we are not in case (i) or (ii). Since we are not in case (i), A is not the bottom removable node of λ, so $\chi(\lambda_A) \le p$, and D^{λ_A} is completely splittable. Since we are not in case (ii), A is the unique addable node for λ_A with residue resA. So Lemmas 4.2.3 and 4.2.5 prove $D^\lambda \cong \mathrm{Ind} D^{\lambda_A}$.

□

We remark that if A satisfies (i) or (ii) from Lemma 4.4.2, then a calculation similar to the proof of Lemma 4.3.3 shows that $D^\lambda \ncong \mathrm{Ind}^{\mathrm{resA}} D^{\lambda_A}$. So Lemma 4.4.2

tells us exactly the ways we can obtain D^λ by α-inducing a completely splittable $\Sigma_{d-1}-$ module.

Since we plan to reduce the calculation of $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$ to the case where λ and μ are minimal, the next definition is natural:

Definition: Given a completely splittable module D^λ, the *minimal core* of λ, denoted $\hat{\lambda}$, is obtained by successively removing nodes from λ which do not satisfy Lemma 4.4.2 (i) or (ii) until no such nodes can be removed. The remark following Lemma 4.4.2 implies that $\hat{\lambda}$ is indeed minimal.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be completely splittable. It is easy to see from a few examples that $\hat{\lambda}$ is well defined and easy to determine. Starting with λ, simply remove removable nodes, never removing any of residue equal to the residue of the bottom addable node, and never allowing χ to be $> p$.

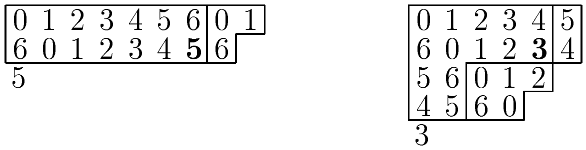
More explicitly, suppose the bottom addable node of λ has residue α. Let (u, v) be the node of residue α farthest to the right. This node is uniquely determined because $h(\lambda) \leq p-1$, so no column has two nodes of the same residue. If there is no such node, $\hat{\lambda}$ is empty. If $u = k$ then $\hat{\lambda} = (v^k)$. If $u < k$ then $\hat{\lambda}$ is the partition with rim consisting of the nodes: $(1, v), (2, v), \cdots, (u, v), (u, v-1), (u, v-2), \cdots, (u, v-p+k), (u+1, v-p+k), \cdots (k, v-p+k), (k, v-p+k-1), \cdots (k, 1)$. In particular, note that $h(\lambda) = h(\hat{\lambda})$ when $\hat{\lambda} \neq \emptyset$.

Two examples are given in Figure 4.3, with the node (u, v) in bold, and the minimal core drawn inside the partition. The 7-residues of all nodes of λ and the bottom addable node of λ are labeled.

We now use the minimal cores and (4.1) to formalize the reduction of determining Ext^1 between two minimal modules:

Theorem 4.4.3. *Let D^λ and D^μ be completely splittable and in the same block. Then $\hat{\lambda}$ and $\hat{\mu}$ partition the same number $mp \leq d$, and:*

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{mp}}}(\mathrm{D}^{\hat{\lambda}}, \mathrm{D}^{\hat{\mu}}) \cong \mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu).$$



$\lambda = (9, 8)$ $\qquad$ $\lambda = (6, 6, 5, 4)$

$\hat{\lambda} = (7, 7)$ $\qquad$ $\hat{\lambda} = (5, 5, 2, 2)$

Figure 4.3: Two examples of minimal 7-cores

Proof. We first remark that D^λ and D^μ are in the same block so, by Theorem 4.3.2, either λ and μ are both minimal or neither is. So we will assume λ and μ are not minimal, since otherwise $\hat{\lambda} = \lambda$ and $\hat{\mu} = \mu$ and the theorem is trivial.

To prove this claim we will use the abacus notation for partitions and the characterization of blocks by p-cores. See Section 1.5 for details. We make a series of observations about the abacus representations of λ and μ.

(i). Since λ has less than p parts, we will represent it on an abacus with p beads. We label the runners $0, 1, \ldots, p-1$ from left to right.

(ii). Notice that $\chi(\lambda)$ can be read off the abacus display. It is equal to $s - t + 1$ where t is the position of the first bead after the first gap and s is the position of the last bead.

(iii). Since $\chi(\lambda) \le p$, by (ii), the core of λ (also represented with p beads) has at most two beads on any runner and must have a bead on runner 0.

(iv). λ is not minimal, so D^λ is not in the principal block of $K\Sigma_{mp}$. Thus λ does not have an empty p-core, so some runner has two beads. Since $\chi(\lambda) \le p$, if runner i has no beads, then runners to the right of i have at most one bead by (ii).

(v). By (ii) and (iv) there must be a bead in position 0.

Of course, (i).-(v). hold for μ as well. Also $\lambda \sim \mu$, so the abacus displays for λ and μ have the same number of beads on each runner.

Now suppose the lowest addable node for λ has residue i. Then in the abacus display for λ, the first gap occurs on runner i so (ii) implies that runner i has at most one bead. Similarly, if the lowest addable node of μ has residue j then runner j has at most one bead.

Consider the abacus diagram for the p-core $\tilde{\lambda} = \tilde{\mu}$. Note that (i) and (iv) imply runner $p-1$ can not have two beads. We consider separately the case where it has no beads and where it has one bead.

Case 1 is if runner $p-1$ has no beads. Let runner $k < p-1$ be the leftmost runner with two beads. Then (even if k=0), both λ and μ must have a removable node of residue k. Furthermore, k is not equal to i or j since runners i and j have at most one bead.

Case 2 is when runner $p-1$ has one bead. There must be some runner with no beads, and by (iv) all runners to the right of it have at most one bead. Thus we can choose a runner k such that runner k has one bead and runner $k-1$ has no beads. Again both λ and μ must have a removable node of residue k. Since runner $k-1$ has no beads, the partitions have no addable nodes of residue k so k is not equal to i or j. The two cases are illustrated below:

$$\begin{array}{ccccccc} \multicolumn{7}{c}{\text{Case 1}} \\ & & & & k & & p\text{-}1 \\ \circ & \circ & \circ & \circ & \circ & \circ & \bullet \\ \bullet & \bullet & \bullet & \bullet & \circ & \bullet & \bullet \end{array} \qquad\qquad \begin{array}{cccc} \multicolumn{4}{c}{\text{Case 2}} \\ & & & k \\ \circ & \circ & \bullet & \circ \\ ? & ? & \bullet & \bullet \end{array}$$

In both cases λ and μ must have a removable node of residue k. Suppose that $\chi(\lambda) = p$ and the node of residue k is the bottom removable node for λ. Then removing this node will increase $\chi(\lambda)$ from p to $p+1$. Then λ will have an abacus configuration:

$$\begin{array}{cc} & k \\ \bullet & \circ \\ \circ & ? \end{array}$$

which is impossible. In Case 1 above, runner k has another bead so $\chi(\lambda) > p$, contradicting λ being completely splittable. In Case 2 above, runner $k-1$ has no beads so this configuration cannot occur. The same applies for μ.

We have found removable nodes A_λ and A_μ, both with residue k. We have shown that k is not equal to the residue of the bottom addable node of λ or μ. Also, $\chi(\lambda_{A_\lambda}) \leq p$ and $\chi(\mu_{A_\mu}) \leq p$ so these nodes are not the lowest removable nodes for λ and μ. Thus A_λ is the unique addable node of λ_{A_λ} of residue $\text{res} A_\lambda$, and similarly for A_μ.

Lemma 4.4.2 implies:

$$D^\lambda \cong \mathrm{Ind}^k D^{\lambda_{A_\lambda}} \text{ and } D^\mu \cong \mathrm{Ind}^k D^{\mu_{A_\mu}}. \tag{4.8}$$

Also,

$$\mathrm{Res}_k D^\lambda \cong D^{\lambda_{A_\lambda}} \text{ and } \mathrm{Res}_k D^\mu \cong D^{\mu_{A_\mu}}. \tag{4.9}$$

Equations 4.8 and 4.9 give us:

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}-1}}(\mathrm{D}^{\lambda_{\mathrm{A}_\lambda}}, \mathrm{D}^{\mu_{\mathrm{A}_\mu}}). \tag{4.10}$$

But clearly $\hat{\lambda} = \widehat{\lambda_{A_\lambda}}$ and $\hat{\mu} = \widehat{\mu_{A_\mu}}$ so the theorem follows from (4.10) by induction. □

Before we state our main theorem, we need to deal with self extensions and with the base case of the induction. Self extensions have been handled previously:

Lemma 4.4.4. $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\lambda) = 0$ *for* D^λ *completely splittable.*

Proof. This is true by Theorem 3.0.3 for partitions with $< p$ parts, which clearly holds for λ since $\chi(\lambda) \leq p$ and λ is p-regular. □

We have determined $\mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{mp}}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = 0$ for λ and μ minimal partitions of mp with $m > 1$. The minimal partitions for $d = p$ are exactly the p-regular hooks. So we need to handle $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{p}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$ for λ and μ hook partitions of p. But this is just the principal block of $K\Sigma_p$, and it is well known (see e.g. [16] or Theorem 4.4.5) that:

Lemma 4.4.5. *Let* $\lambda = (p-r, 1^r)$ *and* $\mu = (p-s, 1^s)$ *for* $0 \leq r \leq p-2$ *be* p*-regular. Then:*

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{p}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = \begin{cases} K & \text{if } |r-s| = 1 \\ 0 & \text{otherwise} \end{cases}.$$

We call the two hooks *adjacent* when $|r - s| = 1$. Thus for completely splittable modules, $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu)$ will be zero unless $\hat{\lambda}$ and $\hat{\mu}$ are adjacent hook partitions of p.

Everything is now in place to prove Theorem 4.1.1. We know $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = 0$ unless $\hat{\lambda}$ and $\hat{\mu}$ are adjacent hook partitions of p. Let the bottom addable node of λ have residue α. It is not difficult to see from our description of the minimal core that

$\hat{\lambda} = \emptyset$ exactly when λ has no nodes of residue α. This is equivalent to $h_{11}(\lambda) < p$. For $\hat{\lambda}$ to be a hook partition of p we need λ to have exactly one node of residue α. This is equivalent to the condition $h_{21}(\lambda) < p$. So $\hat{\lambda}$ is a hook exactly when $h_{11}(\lambda) \geq p$ and $h_{21}(\lambda) < p$. We know $h(\hat{\lambda}){=}h(\lambda)$ if $\hat{\lambda}$ is nonempty, so the criterion for $\hat{\lambda}$ and $\hat{\mu}$ to be adjacent is $\mid h(\lambda) - h(\mu) \mid = 1$. Thus we have proven Theorem 4.1.1.

We remark that if d is large relative to p, the conditions on $\lambda \vdash d$ stated in Theorem 4.1.1 are incompatible with λ being completely splittable. We make this precise in the following corollary:

Corollary 4.4.6. *Let D^λ and D^μ be completely splittable $K\Sigma_d-$ modules, where $d > p + p^2/4$. Then:*

$$\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) = 0.$$

Proof. We suppose $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{D}^\lambda, \mathrm{D}^\mu) \neq 0$ and prove $d \leq p + p^2/4$. Theorem 4.1.1 implies that $\hat{\lambda}$ and $\hat{\mu}$ are adjacent hook partitions of p, so assume $\hat{\lambda} = (p - r, 1^r)$ where $r > 0$. Since $h(\lambda) = h(\hat{\lambda})$, the lowest addable node of λ is the same as the lowest addable node of $\hat{\lambda}$, and has residue $p - r - 1$. From our description (after the definition of minimal core) of how to obtain the minimal core, we see that λ has no nodes of residue $p - r - 1$ except for the node $(1, p - r)$. In particular the node $(2, p - r + 1)$ is not in λ. This restriction plus the requirement that $\chi(\lambda) \leq p$ forces all the nodes of λ not in $\hat{\lambda}$ to be contained in the region marked with $\times' s$ in Figure 4.3, where we have illustrated the case $p = 11$ and $r = 4$.

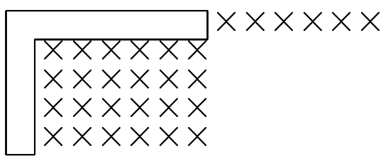


Figure 4.4: Possible nodes for λ with 11-core $(7, 1^4)$

It is easy to see from Figure 4.4 that the most nodes λ can have is $p + (r+1)(p - (r+1))$. This expression achieves its maximum value of $p + p^2/4$ when $r = p/2$. Thus $d \leq p + p^2/4$ as desired. $\square$

4.5 Application to $GL_n(K)$

We describe here how to prove Corollary 4.1.2. For a comprehensive discussion of background for this section see [7] and Section 1.6.

Let $M(n,d)$ denote the polynomial representations of $GL_n(K)$ of homogeneous degree d, for $d \leq n$. The simple modules in $M(n,d)$ are indexed by partitions $\lambda \in \Lambda^+(d)$, and denoted $L(\lambda)$. Recall that the Schur functor $\mathcal{F}$ is a covariant, exact functor from $M(n,d)$ to $K\Sigma_d-$ mod such that:

$$\mathcal{F}(L(\lambda)) = \begin{cases} D_\lambda & \text{if } \lambda \in \Lambda^p(n,d) \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Because (4.11) is in terms of D_λ and Theorem 4.1.1 is in terms of D^λ, we recall from (1.4) how the two parametrizations of simple $K\Sigma_d-$ modules are related:

$$\begin{array}{ccccc} D_\lambda & \cong & D^{\lambda'} \otimes \mathrm{sgn} & \cong & D^{m(\lambda')} \\ D^\lambda & \cong & D_{\lambda'} \otimes \mathrm{sgn} & \cong & D_{(m(\lambda))'} \end{array}. \tag{4.12}$$

$\mathcal{F}$ has a right adjoint functor $\mathcal{G}$, and this adjointness sometimes extends to Ext^1, as in the following:

Theorem 4.5.1. *[3]*

$$\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\mu) \cong \mathrm{Ext}^1_{M(n,d)}(L(\lambda), \mathcal{G}(D_\mu))$$

This theorem is especially useful for completely splittable modules, as the functor $\mathcal{G}$ behaves well on them:

Theorem 4.5.2. *[13] Let D_λ be nontrivial and completely splittable. Then*

$$\mathcal{G}(D_\lambda) = L(\lambda).$$

Theorems 4.1.1, 4.5.1, and 4.5.2 and Equation 4.12 immediately imply Corollary 4.1.2. Although Theorem 4.5.2 applies only to nontrivial modules, Corollary 4.1.2 still holds when $D^{\lambda} \cong D^{\mu} \cong K$, by the well known fact that $\mathrm{Ext}^1_{\mathrm{M(n,d)}}(\mathrm{L}(\lambda), \mathrm{L}(\lambda)) = 0$.

CHAPTER 5
THE HOOK CATEGORY

As we discussed briefly in Section 3.3, the module M which we defined in (3.4) has also appeared in [3] as a projective $K\Sigma_p-$ module. Joe Chuang observed to us that this module also has a similar structure to the projective modules for the Schur algebra $S(p,p)$, and suggested considering the category of $K\Sigma_{mp}-$ modules whose composition factors are all of the form D^λ for λ a hook. We prove here that this category is indeed equivalent to the module category for the principal block of $S(p,p)$, and raise several interesting open questions.

5.1 General considerations

We are interested in the category of $K\Sigma_{mp}-$ modules whose composition factors are all from the set $\{D^\lambda \mid \lambda \text{ is a hook}\}$. This is known to be the module category of an algebra $K\Sigma_{mp}/J$ for a suitable ideal J. We have not been able to find a description of this fact in the literature, so we present the details here.

Let G be a finite group and suppose the simple $KG-$ modules are partitioned into two disjoint subsets $\{S_i\}$ and $\{T_i\}$. Define J to be the minimal left submodule of KG such that KG/J has composition factors only containing the simple modules $\{S_i\}$. Notice that if U and V are submodules of KG having this property, it is easy to see that $U \cap V$ has the property as well, so J is a well-defined module. The property of having composition factors only containing the simple modules $\{S_i\}$ is clearly preserved under $KG-$ module maps. But the $KG-$ module endomorphisms of KG are given by right multiplications, so J is actually a two-sided ideal. We have:

Lemma 5.1.1. *The category $KG/J-$ mod is equivalent to the full subcategory of $KG-$ mod consisting only of modules with composition factors from $\{S_i\}$.*

Proof. There is a natural functor:

$$F : KG/J\text{–mod} \Rightarrow KG\text{–mod}.$$

To prove this functor gives the desired equivalence, it is enough to show that the $KG-$ modules with only $\{S_i\}$ composition factors are precisely those modules annihilated by J.

Suppose $U \in KG-$ mod and $JU = 0$. Let U have a basis $\{u_1, u_2, \ldots, u_t\}$ and let:

$$h : KG \oplus \cdots \oplus KG \Rightarrow U \tag{5.1}$$

be the surjection given by $h(g_1, \ldots, g_t) = \Sigma g_i u_i$. Clearly $h(J \oplus \cdots \oplus J) = 0$, so U is a homomorphic image of a direct sum of KG/J, and thus has all of its composition factors from $\{S_i\}$.

Conversely suppose U has all of its composition factors from $\{S_i\}$. We must show $JU = 0$. The minimality of J immediately implies:

Lemma 5.1.2. *$J/\text{rad}J$ has no composition factors from the set $\{S_i\}$.*

Now suppose $JU \neq 0$. We have:

$$0 \subseteq \text{rad}JU \subset JU \subseteq U \tag{5.2}$$

where $JU/\text{rad}JU \neq 0$ by Nakayama's lemma. The surjection h in (5.1) produces a surjection $\bigoplus J \to JU$ which yields a surjection:

$$J/\text{rad}J \oplus \cdots \oplus J/\text{rad}J \to JU/\text{rad}JU. \tag{5.3}$$

Lemma 5.1.2 together with (5.2) and (5.3) imply U has some composition factor from $\{T_i\}$, contradicting our assumption. Hence $JU = 0$. $\square$

5.2 The principal block of $S(p,p)$

The structure of the principal block of $S(p,p)$ is well known, and we describe it briefly here. The simple modules correspond to partitions of p with empty p-core, i.e. hooks. They are $L(1^p), L(2,1^{p-2}), \ldots, L(p)$. The structure of the Weyl modules $\Delta(\lambda)$ is also easy to determine (see [3] for example):

$$\Delta(1^p) = L(1^p),\ \ \Delta(2,1^{p-2}) = \begin{array}{c} L(2,1^{p-2}) \\ L(1^p) \end{array}, \cdots, \Delta(p) = \begin{array}{c} L(p) \\ L(p-1,1) \end{array}.$$

From these it is not difficult to determine the indecomposable projectives, which we label $P(\lambda)$:

$$P(1^p) = \begin{array}{c} L(1^p) \\ L(2,1^{p-2}) \\ L(1^p) \end{array}$$

$$P(a,1^{p-a}) = \begin{array}{c} L(a,1^{p-a}) \\ L(a-1,1^{p-a+1}) \ \ L(a+1,1^{p-a-1}) \\ L(a,1^{p-a}) \end{array} \qquad 2 \le a \le p-1 \tag{5.4}$$

$$P(p) = \Delta(p) = \begin{array}{c} L(p) \\ L(p-1,1). \end{array}$$

5.3 Projective modules in the hook category

Now let $m > 1$ and let J be the ideal in $K\Sigma_{mp}$ as in Lemma 5.1.1, where the modules $\{S_i\}$ are just $\{D^{(mp)}, D^{(mp-1,1)}, \ldots, D^{(mp-p+1,1^{p-1})}\}$. We would like to prove the algebras $K\Sigma_{mp}/J$ and $S(p,p)$ are Morita equivalent. We begin by determining the projective indecomposable $K\Sigma_{mp}/J-$ modules. First a lemma:

Lemma 5.3.1. *Let $d = mp$ for $m > 1$. Then:*

1. $$\mathrm{Res}_{\Sigma_d}^{\Sigma_{d+1}} D^{(d,1)} \cong \begin{matrix} D^{(d)} \\ D^{(d-1,1)} \\ D^{(d)} \end{matrix}$$

2. $$\mathrm{Res}_{\Sigma_d}^{\Sigma_{d+1}} D^{(d-a,1^{a+1})} \cong \begin{matrix} & D^{(d-a,1^a)} & \\ D^{(d-a+1,1^{a-1})} & & D^{(d-a-1,1^{a+1})} \\ & D^{(d-a,1^a)} & \end{matrix} \qquad 1 \leq a \leq p-2$$

3. $$S^{(d-p+1,1^{p-1})} \cong \begin{matrix} D^{(d-p+1,1^{p-1})} \\ D^{(d-p+2,1^{p-2})} \end{matrix}$$

Proof. Part three follows from Theorem 3.1.2. Parts one and two are consequences of Kleshchev's branching theorems and follow from the same argument used to determine the structure of M in Equation 3.4. □

We now show the modules in Lemma 5.3.1 are the projective indecomposable $K\Sigma_d/J-$ modules:

Theorem 5.3.2. *The modules in Lemma 5.3.1 are the projective indecomposable $K\Sigma_d/J-$ modules.*

Proof. All the modules have simple socles, so are indecomposable. Their heads are the desired simple $K\Sigma_d/J-$ modules, so we need only show they are projective. To do this it suffices to prove $\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{P}, \mathrm{D}^\lambda) = 0$ for each hook λ and each P from Lemma 5.3.1.

We begin with part three of Lemma 5.3.1, namely $P = S^{(d-p+1,1^{p-1})}$. Let $\mu = (d-p+1, 1^{p-1})$ and let $\lambda \neq \mu$ be a hook partition of d. So $P = S^\mu$ and we must show $\mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{P}, \mathrm{D}^\lambda) = 0$. The classical branching theorem gives:

$$0 \longrightarrow S^{(d-p,2,1^{p-2})} \longrightarrow \mathrm{Ind}^0 S^{(d-p,1^{p-1})} \longrightarrow S^\mu \longrightarrow 0. \tag{5.5}$$

Applying $\mathrm{Hom}_{K\Sigma_d}(-, D^\lambda)$ to (5.5) and using Eckmann-Shapiro gives:

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{K\Sigma_d}}(\mathrm{S}^\mu, \mathrm{D}^\lambda) \longrightarrow \mathrm{Ext}^1_{\mathrm{K\Sigma_{d-1}}}(\mathrm{S}^{(\mathrm{d-p},1^{\mathrm{p}-1})}, \mathrm{Res}_0 \mathrm{D}^\lambda). \tag{5.6}$$

But μ is the only p-regular hook with a normal node of residue 0, so $\mathrm{Res}_0 D^{\lambda} = 0$ by Theorem 2.2.1. Thus $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{S}^{\mu}, \mathrm{D}^{\lambda}) = 0$ as desired. The case $\lambda = \mu$ was handled in (3.9), where we proved $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{S}^{\mu}, \mathrm{D}^{\mu}) = 0$. Thus $P = S^{\mu}$ is projective.

Next we handle the generic case (part one and two of Lemma 5.3.1). Set

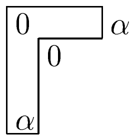
$$P = \mathrm{Res}^{\Sigma_{d+1}}_{\Sigma_d} D^{(d-a,1^{a+1})} \qquad 0 \le a \le p-2$$

and

$$\lambda = (d-b, 1^b) \qquad 0 \le b \le p-1.$$

We must prove $\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{P}, \mathrm{D}^{\lambda}) = 0$. We divide the proof into three cases, based on the value of b.

Case 1: $0 < b < p-1$ We have:

$\lambda =$ 

with p-residues labeled and $\alpha \neq 0, 1$. Then λ has one normal and two good addable nodes so Theorem 2.2.1(ii) implies:

$$\mathrm{Ind}^{\Sigma_{d+1}}_{\Sigma_d} D^{\lambda} = D^{(d-b,1^{b+1})} \bigoplus D^{(d-b,2,1^{b-1})}. \tag{5.7}$$

Then:

$$\begin{aligned}
\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{P}, \mathrm{D}^{\lambda}) &= \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d+1}}}(\mathrm{D}^{(\mathrm{d-a},1^{\mathrm{a+1}})}, \mathrm{Ind}\mathrm{D}^{\lambda}) \quad \text{by Eckmann-Shapiro} \\
&= \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d+1}}}(\mathrm{D}^{(\mathrm{d-a},1^{\mathrm{a+1}})}, \mathrm{D}^{(\mathrm{d-b},1^{\mathrm{b+1}})}) \oplus \\
&\qquad \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d+1}}}(\mathrm{D}^{(\mathrm{d-a},1^{\mathrm{a+1}})}, \mathrm{D}^{(\mathrm{d-b},2,1^{\mathrm{b-1}})}) \text{ by (5.7)} \\
&= 0 \oplus \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d+1}}}(\mathrm{D}^{(\mathrm{d-a},1^{\mathrm{a+1}})}, \mathrm{D}^{(\mathrm{d-b},2,1^{\mathrm{b-1}})}) \text{ by Theorem 4.1.1} \\
&= 0
\end{aligned}$$

because $D^{(d-a,1^{a+1})}$ and $D^{(d-b,2,1^{b-1})}$ are in different p-blocks by Nakayama's conjecture. To see this note that $a \leq p-2$ implies $(d-a,1^{a+1})$ has d/p nodes of residue 0 but $(d-b,2,1^{b-1})$ has $d/p+1$ nodes of residue 0.

Case 2: $b=0$

In this case $\lambda = (d)$ and $\mathrm{Ind}_{\Sigma_d}^{\Sigma_{d+1}} D^{\lambda} = D^{(d+1)} \oplus D^{(d,1)}$ so

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{P},\mathrm{D}^{(\mathrm{d})}) &= \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}+1}}(\mathrm{D}^{(\mathrm{d-a},1^{\mathrm{a}+1})},\mathrm{D}^{(\mathrm{d}+1)} \oplus \mathrm{D}^{(\mathrm{d},1)}) \\ &= 0 \text{ by Theorem 4.1.1.}\end{aligned}$$

Case 3: $b=p-1$

In this case $\lambda = (d-p+1,1^{p-1})$ has only one good addable node, at the $(2,2)$ position, and it has residue 0. Thus:

$$\mathrm{Ind}D^{\lambda} = \mathrm{Ind}^0 D^{\lambda}$$

is in the same block as the partition $\tau = (d-p+1,2,1^{p-2})$. But τ has three nodes of residue 0 and $(d-a,1^{a+1})$ does not so:

$$\begin{aligned}0 &= \mathrm{Ext}^1_{\mathrm{K}\Sigma_{\mathrm{d}+1}}(\mathrm{D}^{(\mathrm{d-a},1^{\mathrm{a}+1})},\mathrm{Ind}\mathrm{D}^{\lambda}) \\ &= \mathrm{Ext}^1_{\mathrm{K}\Sigma_\mathrm{d}}(\mathrm{P},\mathrm{D}^{\lambda}) \text{ by Eckmann-Shapiro.}\end{aligned}$$

This completes the proof of Theorem 5.3.2. □

5.4 The Morita equivalence

We now know the projective $K\Sigma_d/J-$ modules (Theorem 5.3.2) and the projective $B_0(S(p,p))-$ modules (5.4). Rather than work with $K\Sigma_d/J$ and $B_0(S(p,p))$, we will work with their *basic algebras*, which are Morita equivalent to the original algebras but have all simple modules of dimension one. For more information on basic algebras see [1, 2.2]. The basic algebra for an algebra A is $\mathrm{End}_A(\bigoplus P_i)$ where the direct sum contains one copy of each indecomposable projective. It is well known that:

Lemma 5.4.1. *Two algebras are Morita equivalent if and only if their basic algebras are isomorphic.*

We can use this to prove:

Theorem 5.4.2. *The algebras $B_0(S(p,p))$ and $K\Sigma_d/J$ are Morita equivalent.*

Proof. By Lemma 5.4.1, it suffices to show the following two algebras are isomorphic:

$$\mathrm{End}_{S(p,p)} \left(\begin{matrix} L(1^p) \\ L(2,1^{p-2}) \\ L(1^p) \end{matrix} \bigoplus \cdots \bigoplus \begin{matrix} & L(a,1^{p-a}) & \\ L(a+1,1^{p-a-1}) & & L(a-1,1^{p-a+1}) \\ & L(a,1^{p-a}) & \end{matrix} \cdots \bigoplus \right.$$

$$\left. \cdots \begin{matrix} & L(p-1,1) & \\ L(p) & & L(p-2,1^2) \\ & L(p-1,1) & \end{matrix} \bigoplus \begin{matrix} L(p) \\ L(p-1,1) \end{matrix} \right)$$

and

$$\mathrm{End}_{K\Sigma_d/J} \left(\begin{matrix} D^{(d)} \\ D^{(d-1,1)} \\ D^{(d)} \end{matrix} \bigoplus \cdots \bigoplus \begin{matrix} & D^{(d-a,1^a)} & \\ D^{(d-a-1,1^{a+1})} & & D^{(d-a+1,1^{a-1})} \\ & D^{(d-a,1^a)} & \end{matrix} \cdots \bigoplus \right.$$

$$\left. \cdots \begin{matrix} & D^{(d-p+2,1^{p-2})} & \\ D^{(d-p+1,1^{p-1})} & & D^{(d-p+3,1^{p-3})} \\ & D^{(d-p+2,1^{p-2})} & \end{matrix} \bigoplus \begin{matrix} D^{(d-p+1,1^{p-1})} \\ D^{(d-p+2,1^{p-2})} \end{matrix} \right)$$

We will label projective modules in the above direct sums by $\{P_i \mid i = 1, \ldots p\}$. Choose and fix a basis of either of these endomorphism algebras given by:

$\{e_i \mid i = 1, \dots p\}$	are the projections onto the ith component P_i.
$\{x_{ii} \mid i = 1, \dots p-1\}$	are maps from P_i onto $\mathrm{soc}(P_i)$.
$\{x_{ij} \mid \lvert i-j \rvert = 1\}$	are nontrivial maps from P_i to P_j.

Notice that all products of the above basis elements are 0 except those of the form $x_{ij}x_{ji}$ for $i \neq j$, and for the idempotents, where $e_i^2 = e_i$.

Schur's lemma implies

$$x_{ij}x_{ji} = \alpha_{ij}x_{ii}.$$

It is now a simple matter of replacing basis elements by scalar multiples in order to eliminate all the constants, proving that the algebra is uniquely determined up to isomorphism, and thus that the two endomorphism algebras are isomorphic. We sketch this here, and remark that a proof of a very similar result is given in detail in [15, 5.6.2]. First simplify the relations to eliminate half the constants $\{\alpha_{ij}\}$ and obtain :

$$\begin{aligned} x_{21}x_{12} &= \alpha_2 x_{23}x_{32} \\ x_{32}x_{23} &= \alpha_3 x_{34}x_{43} \\ x_{43}x_{34} &= \alpha_4 x_{45}x_{54} \\ &\vdots \\ x_{p-1,p-2}x_{p-2,p-1} &= \alpha_{p-1}x_{p-1,p}x_{p.p-1} \end{aligned}$$

Now replace $x_{p-2,p-1}$ by $x_{p-2,p-1}/\alpha_{p-1}$ to eliminate the constant α_{p-1}. Next scale $x_{p-3,p-2}$ to eliminate α_{p-2} and continue in this way until all the constants are eliminated. This proves the algebra is uniquely determined with no parameters, and thus the two algebras are isomorphic.

This algebra has arisen before in [5] and is discussed in [15]. It is one of a series of quasi-hereditary algebras of finite type introduced by Erdmann and denoted A_w. This algebra is the case $w = p$.

5.5 Open problems arising from the Morita equivalence

There are several interesting open problems which arise from the Morita equivalence given in the previous section. Notice that a functor realizing the equivalence of the two module categories was never given explicitly. Instead we proved the basic algebras of the two algebras were isomorphic. It is then natural to ask:

Open Problem 5.5.1. *Is there a nice functor realizing the Morita equivalence of Theorem 5.4.2?*

Problem 5.5.1 seems to be difficult. The "obvious" answer is to use the Schur functor to go between Schur algebras and the symmetric group algebra. However the Schur functor maps $S(p,p)-$ mod to $K\Sigma_p-$ mod, and the equivalence does not even hold here! We required $d = mp$ for $m > 1$ because $K\Sigma_p$ has only $p-1$ simple modules in its principal block, while $S(p,p)$ and $K\Sigma_{mp}$ for $m > 1$ have p simple modules in their principal block. So the "hook category" for $K\Sigma_p$ is not equivalent to that of $K\Sigma_{mp}$ for $m > 1$.

Theorem 5.4.2 also implies that $K\Sigma_{mp}/J$ and $K\Sigma_{mp+p}/J'$ are Morita equivalent, so one could search for a functor between these two categories. Again the "obvious" choice, restriction, fails in every variation we have attempted.

Once could also look for more examples of Erdmann's algebras A_w as quotients of $K\Sigma_d$ [4].

Open Problem 5.5.2. *Is there a quotient of $K\Sigma_{mp}$ which is Morita equivalent to A_w for $w \neq p$?*

Chuang has suggested [2] trying to generalize this equivalence in a way suggested by using the abacus representation. For our set of simples $\{S_i\}$ we used $\{D^\lambda \mid \lambda = (mp-r, 1^r)\}$. These partitions all have empty p-core, and when represented on the abacus they have a single bead moved down m places. The p distinct p-regular hooks correspond to the p different runners. Generalize this to the following question:

Open Problem 5.5.3. *Fix a p-core $\kappa \vdash r$ and a weight w. Consider the p partitions arising by moving a single bead on the abacus down w places. Form the ideal J as in Section 5.1. Is $K\Sigma_{r+wp}/J$ Morita equivalent to $B_0(S(p,p))$?*

We have been unable to find a counterexample to Problem 5.5.3 when w is sufficiently large so that the p partitions are all p-regular.

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