

Lectures on the ultraviolet problem for QED in $d=3$

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Program: We review some recent work on quantum electrodynamics on a three dimensional Euclidean spacetime, work which culminates in a proof of ultraviolet stability in a finite volume. The model is formulated on a fine lattice and bounds are obtained uniformly in the lattice spacing. The method is a renormalization group technique due to Balaban.

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1 Introduction

1.1 the model

We are interested in quantum electrodynamics in a three dimension Euclidean spacetime with finite volume. The model is presented as functional integrals which are ill defined due to short distance (ultraviolet) singularities. The goal is to show that the model can be renormalized non-perturbatively to tame the singularities and give a precise definition.

The model is first regularized by formulating it on a fine lattice, and the first goal is to get bounds uniformly in the lattice spacing. We particularly concentrate on the partition function, but the same methods should give bounds on correlation functions. The continuum limit as the lattice spacing goes to zero should also be feasible. The infinite volume limit and reconstruction of an operator theory on a three dimensional Minkowski spacetime are problems for the future.

Our tool is the renormalization group which is a systematic way of integrating out the short distance modes and tracking the result. The method is implemented with a block averaging technique due to Balaban. Balaban created the approach in his studies of (massive) scalar electrodynamics in $d=3$ [1] - [4], and Yang-Mills gauge theory in $d=3,4$. [5] - [15]. It was further developed in joint work with collaborators Imbrie, Jaffe [18] - [19] and O'Carroll, Shor [20] -[21].

The model is defined by the action

$$S(\mathcal{A}, \bar{\psi}, \psi) = \frac{1}{2} \|d\mathcal{A}\|^2 + \langle \bar{\psi}, (\mathfrak{D}_e(\mathcal{A}) + \bar{m})\psi \rangle + m^N \langle \bar{\psi}, \psi \rangle + \varepsilon^N \quad (1)$$

This is defined on a lattice with finite volume, and for definiteness we take unit volume. The lattice Λ_N has spacing $\varepsilon = L^{-N}$ for some fixed large positive odd integer L . We either take a toroidal lattice

$$\Lambda_N = (L^{-N}\mathbb{Z}/\mathbb{Z})^3 \quad (2)$$

or a cube

$$\Lambda_N = \left(L^{-N}\mathbb{Z} \cap [-1/2, 1/2] \right)^3 \quad (3)$$

We are interested in what happens as $N \rightarrow \infty$.

The gauge field \mathcal{A} is defined on bonds b in the lattice consisting of ordered nearest neighbor pairs $b = (x, x')$ with the convention that $\mathcal{A}(x, x') = -\mathcal{A}(x', x)$ The associated field strength $d\mathcal{A}$ defined on plaquettes (squares) in the lattice

$$d\mathcal{A}(p) = \varepsilon^{-1} \sum_{b \in \partial p} \mathcal{A}(b) \quad (4)$$

where ∂p is the boundary of p oriented counterclockwise. The gauge term in the action is

$$\frac{1}{2} \|d\mathcal{A}\|^2 = \frac{1}{2} \sum_p \varepsilon^3 |d\mathcal{A}(p)|^2 \quad (5)$$

The fermi fields $\bar{\psi}_\alpha(x), \psi_\alpha(x)$ are indexed by points x in the lattice and spinor indices α . They are totally anti-commuting and are the generators of a Grassman algebra. The pairing $\langle \bar{\psi}, \psi \rangle$ is defined by.

$$\langle \bar{\psi}, \psi \rangle = \sum_\alpha \sum_x \epsilon^3 \bar{\psi}_\alpha(x) \psi_\alpha(x) \quad (6)$$

A covariant forward derivative with gauge field \mathcal{A} and charge e in the direction e_μ is defined on spinors f by

$$(\partial_{\mathcal{A},\mu} f)(x) = \left(e^{ie\epsilon\mathcal{A}(x, x+\epsilon e_\mu)} f(x + \epsilon e_\mu) - f(x) \right) \epsilon^{-1} \quad (7)$$

This is a forward derivative. The transpose $\partial_{-\mathcal{A},\mu}^T$ is a backward derivative and we consider the symmetric derivative

$$\nabla_{\mathcal{A},\mu} = \frac{1}{2} \left(\partial_{\mathcal{A},\mu} - \partial_{-\mathcal{A},\mu}^T \right) \quad (8)$$

and the covariant Laplacian

$$\Delta_{\mathcal{A}} = -(\partial_{-\mathcal{A}})^T \partial_{\mathcal{A}} \quad (9)$$

Now let $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ for $\mu = 0, 1, 2$ be a representation of the three dimensional Clifford algebra. The Dirac operator on spinors is

$$\mathfrak{D}_\epsilon(\mathcal{A}) = \gamma \cdot \nabla_{\mathcal{A}} - \frac{1}{2} \epsilon \Delta_{\mathcal{A}} = \sum_{\mu=0}^2 \gamma_\mu \nabla_{\mathcal{A},\mu} - \frac{1}{2} \epsilon \Delta_{\mathcal{A}} \quad (10)$$

The extra term $\frac{1}{2} \epsilon \Delta_{\mathcal{A}}$ was added by Wilson to prevent doubling of fermion species in the continuum limit. The operator can also be written

$$\begin{aligned} & (\mathfrak{D}_\epsilon(\mathcal{A})f)(x) \\ &= -\epsilon^{-1} \sum_\mu \left[\left(\frac{1-\gamma_\mu}{2} \right) e^{ie\epsilon\mathcal{A}(x, x+\epsilon e_\mu)} f(x + \epsilon e_\mu) + \left(\frac{1+\gamma_\mu}{2} \right) e^{ie\epsilon\mathcal{A}(x, x-\epsilon e_\mu)} f(x - \epsilon e_\mu) - f(x) \right] \end{aligned} \quad (11)$$

The vacuum energy density ϵ^N and the mass m^N are counterterms and will be chosen to depend on N to cancel singularities in the model. With these definitions the $N \rightarrow \infty$ limit formally gives the standard continuum theory.

The partition function is defined by

$$\mathcal{Z}_N(e) = \int \exp(-\mathcal{S}(\mathcal{A}, \bar{\psi}, \psi)) D\psi D\mathcal{A} \quad (12)$$

where

$$D\psi = \prod_{x,\alpha} d(\bar{\psi}_\alpha(x)) d(\psi_\alpha(x)) \quad D\mathcal{A} = \prod_b d(\mathcal{A}(b)) \quad (13)$$

The fermion integral is the standard Grassman integral: projection onto the highest element. The integral over the gauge field is not yet well-defined since the integrand is gauge invariant and so there is no convergence in pure gauge directions. One has to specify some gauge fixing which we will discuss presently. Once that is settled the goal is to show that for e sufficiently small the partition function is bounded above and below uniformly in the lattice spacing N .

1.2 the scaled model

Let us now specialize to the toroidal lattice. We introduce the notation

$$\mathbb{T}_M^{-N} = (L^{-N}\mathbb{Z}/L^M\mathbb{Z})^3 \quad (14)$$

for the lattice with spacing L^{-N} and width L^M . So we begin with the lattice $\Lambda_N = \mathbb{T}_0^{-N}$ with spacing L^{-N} and unit volume. But it for some purposes it is easier to work with a unit lattice so we scale up to the unit lattice \mathbb{T}_N^0 with width L^N and volume L^{3N} . Let $\Psi(x), \bar{\Psi}(x)$ be elements of a Grassmann algebra indexed by $x \in \mathbb{T}_N^0$ and let $A(b)$ be a real valued functions on bonds b in \mathbb{T}_N^0 . The scaled action is defined by

$$S_0(A, \bar{\Psi}, \Psi) = S(A_{L^{-N}}, \bar{\Psi}_{L^{-N}}, \Psi_{L^{-N}}) \quad (15)$$

where

$$\begin{aligned} A_{L^{-N}}(b) &= L^{\frac{1}{2}N} A(L^N b) \\ \Psi_{L^{-N}}(x) &= L^N \Psi(L^N x) \\ \bar{\Psi}_{L^{-N}}(x) &= L^N \bar{\Psi}(L^N x) \end{aligned} \quad (16)$$

are fields on the original lattice \mathbb{T}_0^{-N} . The scaling factors here are chosen to preserve the free kinetic terms $\|dA\|^2$ and $\langle \bar{\psi}, \mathfrak{D}(0)\psi \rangle$. and we have

$$S_0(A, \bar{\Psi}, \Psi) = \frac{1}{2} \|dA\|^2 + \langle \bar{\Psi}, (\mathfrak{D}_{e_0^N}(A) + \bar{m}_0^N)\Psi \rangle + m_0^N \langle \bar{\Psi}, \Psi \rangle + \varepsilon_0^N \text{Vol}(\mathbb{T}_N^0) \quad (17)$$

Now lattice sums are unweighted and derivatives are unit lattice derivatives. The new parameters are the tiny values

$$e_0^N = L^{-\frac{1}{2}N} e \quad \bar{m}_0^N = L^{-N} \bar{m} \quad m_0^N = L^{-N} m^N \quad \varepsilon_0^N = L^{-3N} \varepsilon^N \quad (18)$$

For example $m^N \langle \bar{\Psi}, \Psi \rangle$ becomes

$$\begin{aligned} m^N \langle \bar{\Psi}_{L^{-N}}, \Psi_{L^{-N}} \rangle &= m^N \sum_{x \in \mathbb{T}_0^{-N}} L^{-3N} L^N \bar{\Psi}(L^N x) L^N \Psi(L^N x) \\ &= L^{-N} m^N \sum_{x' \in \mathbb{T}_N^0} \bar{\Psi}(x') \Psi(x') \\ &= m_0^N \langle \bar{\Psi}, \Psi \rangle \end{aligned} \quad (19)$$

Note also that parallel translation in the derivative $\exp(i e L^{-N} A(x, x + L^{-N} e_\mu))$ becomes $\exp(i e L^{-N/2} A(L^N x, L^N x + e_\mu))$ or in unit lattice points $\exp(i e L^{-\frac{1}{2}N} A(x', x' + e_\mu))$. This accounts for the $L^{-\frac{1}{2}N}$ in $e_0^N = L^{-\frac{1}{2}N} e$. In the following we omit the superscript N in the parameters and just write $e_0, \bar{m}_0, m_0, \varepsilon_0$, the zero indicating that this is our starting point.

The $N \rightarrow \infty$ limit, formerly an ultraviolet (short distance) problem, has now been recast as an infrared (long distance) problem with scaled parameters.

1.3 RG transformations

The partition function now has the form, (up to scaling factors, and with Ψ_0 standing for $\bar{\Psi}_0, \Psi_0$)

$$\begin{aligned} Z_N(e) &= \int \rho_0(A_0, \Psi_0) D A_0 D \Psi_0 \\ \rho_0(A_0, \Psi_0) &= \exp\left(-S_0(A_0, \Psi_0)\right) \end{aligned} \quad (20)$$

The fields all live on the torus \mathbb{T}_N^0 . This still needs gauge fixing but we proceed formally with specifying how it is accomplished.

The difficulty with the problem is that we have an unbounded number of variables. The renormalization group method consists of integrating out a few variables at a time, and keeping careful track of the effective actions at each stage. In Balaban's formulation one defines new fields $A_1 = \mathcal{Q}A_0$ and $\Psi_1 = Q(A_0)\Psi_0$ defined on the L -lattice \mathbb{T}_N^1 where the operators $\mathcal{Q}, Q(A_0)$ average over blocks of width L (precise definition later). Then one integrates over all fields A_0, Ψ_0 with roughly fixed values of A_1, Ψ_1 by

$$\begin{aligned} \tilde{\rho}_1(A_1, \Psi_1) &= \mathbf{N}_0 \int D A_0 D \Psi_0 \delta(A_1 - \mathcal{Q}A_0) \\ &\exp\left(-b/L < \bar{\Psi}_1 - Q(-A_0)\bar{\Psi}_0, \Psi_1 - Q(A_0)\Psi_0 >\right) \rho_0(A_0, \Psi_0) \end{aligned} \quad (21)$$

If we had taken a delta function $\delta(\Psi_1 - Q(A_0)\Psi_0)$ then the integral would be exactly over fixed values of A_1, Ψ_1 , but this does not make sense for Grassmann variables and so we have taken a Gaussian type factor to approximate a delta function. The normalization factor \mathbf{N}_0 is chosen so $\mathbf{N}_0 \int \exp(-b/L < \bar{\Psi}_1, \Psi_1 >) D \Psi_1 = 1$. Then we have

$$\int D A_1 D \Psi_1 \tilde{\rho}_1(A_1, \Psi_1) = \int D A_0 D \Psi_0 \rho_0(A_0, \Psi_0) = Z_N(e) \quad (22)$$

and so a new representation of the partition function. Next we scale back to the unit lattice and define for A_1, Ψ_1 on \mathbb{T}_{N-1}^0

$$\rho_1(A_1, \Psi_1) = \tilde{\rho}_1(A_{1,L}, \Psi_{1,L}) \quad (23)$$

where $A_{1,L}(b) = L^{-\frac{1}{2}} A_1(b/L)$ and $\Psi_{1,L}(x) = L^{-1} \Psi_1(x/L)$ are fields on \mathbb{T}_N^1 .

We repeat the transformation many times generating a sequence of densities $\rho_0, \rho_1, \rho_2, \dots$. After k steps $\rho_k(A_k, \Psi_k)$ is a function on fields on \mathbb{T}_{N-k}^0 . We pass from ρ_k to ρ_{k+1} defining for A_{k+1}, Ψ_{k+1} on \mathbb{T}_{N-k}^1

$$\begin{aligned} \tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) = N_k \int DA_k D\Psi_k \delta(A_{k+1} - Q A_k) \\ \exp\left(-b/L < \bar{\Psi}_{k+1} - Q(-A_k)\bar{\Psi}_k, \Psi_{k+1} - Q(A_k)\Psi_k >\right) \rho_k(A_k, \Psi_k) \end{aligned} \quad (24)$$

Then for A_{k+1}, Ψ_{k+1} on \mathbb{T}_{N-k-1}^0 define

$$\rho_{k+1}(A_{k+1}, \Psi_{k+1}) = \tilde{\rho}_{k+1}(A_{k+1,L}, \Psi_{k+1,L}) \quad (25)$$

We stop the iteration when $k = K \approx N$. Then we have a density $\rho_K(A_K, \Psi_K)$ on \mathbb{T}_{N-K}^0 and

$$Z_N(e) = \int \rho_K(A_K, \Psi_K) DA_K D\Psi_K \quad (26)$$

If we have control over the density ρ_K this is now a tractable integral over a small number of variables and should yield good estimates.

So the issue is to keep control over the sequence $\rho_0, \rho_1, \rho_2, \dots$. To accomplish this it is desirable to write it as the exponential of an effective action $\rho_k(A_k, \Psi_k) = \exp(-S_k(A_k, \Psi_k))$ as much as possible. It turns out this is only possible when the gauge field is bounded in in certain sense. To accomodate this fact we will have to continuously split the field space into large and small field sectors, and show that the large field sectors make a negligible contribution. With effective actions defined for small fields the problem is to control the sequence S_0, S_1, S_2, \dots . After k steps the the leading terms S_k should look something like the original action with new parameters $e_k, \bar{m}_k, m_k, \varepsilon_k$. These start at tiny values and grow. The issue is to chose counter terms so they do not grow too quickly. The other contributions to S_k are smaller but complicated multi-fields terms and their growth must be limited as well. To keep track of this it is important that they have all the symmetries of the original action, that they have a local structure, and that they have some residual gauge invariance in spite of the gauge fixing. These ideas will be explored in detail in subsequent lectures.

1.4 gauge fixing

We make some preliminary remarks on gauge fixing. We are interested in integrals of the form

$$\int f(A) \exp\left(-\frac{1}{2}\|dA\|^2\right) DA \quad DA = \prod_{b \in \Lambda} A(b) \quad (27)$$

For definiteness suppose that the space time Λ is a cube in \mathbb{Z}^3 . and that $f(A)$ is gauge invariant, $f(A^\lambda) = f(A - d\lambda) = f(A)$. The gauge invariance of $\|dA\|^2$ means we have

a large null space and such integrals will not converge. The integrand is really defined on the orbit space of gauge transformations $A \rightarrow A - d\lambda$. To define the integral we need to pick coordinates for the orbit space, that is pick a representative of each orbit. We now make a definite choice called an *axial gauge*.

Let T be a tree on the lattice with rectilinear branches joining all points. The case $d=2$ is illustrated in figure 1

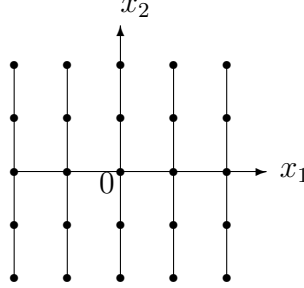


Figure 1: A tree in $d=2$

A gauge field A is said to be axial if it vanishes on the tree.

Proposition 1. *Every A on Λ is gauge equivalent to a unique axial $A' = A - d\lambda$.*

Proof. For any lattice point $x \in \Lambda$, let $\Gamma_{0,x}$ be the path in the tree joining the origin to the point x and let

$$(\tau A)(x) = A(\Gamma_{0,x}) = \sum_{b \in \Gamma_{0,x}} A(b) \quad (28)$$

Then for a bond $(x, x + e_\mu)$ on the tree

$$A(x, x + e_\mu) = (\tau A)(x + e_\mu) - (\tau A)(x) \quad (29)$$

So $(\tau A)(x) = 0$ for all x is equivalent to A axial.

Now given A define $\lambda(x) = (\tau A)(x)$ and $A' = A - d\lambda$. Then for $(x, x + e_\mu)$ on the tree

$$(d\lambda)(x, x + e_\mu) = \lambda(x + e_\mu) - \lambda(x) = A(x, x + e_\mu) \quad (30)$$

and so

$$A'(x, x + e_\mu) = A(x, x + e_\mu) - d\lambda(x, x + e_\mu) = 0 \quad (31)$$

This establishes equivalence to the axial field A' .

For uniqueness suppose that $A \sim A_1$ axial and that $A \sim A_2$ axial. Then $A_1 \sim A_2$ or $A_1 = A_2 - d\lambda$ and $A_1 = A_2 = 0$ on the tree. Hence $d\lambda = 0$ on the tree so λ is constant on the tree and hence everywhere. Therefore $A_1 = A_2$. This completes the proof.

Now we interpret the integral (27) as integration over axial fields only. We write this as

$$\int f(A) \delta(\tau(A)) \exp\left(-\frac{1}{2} \|dA\|^2\right) DA \quad (32)$$

where

$$\delta(\tau(A)) = \prod_{x \in \Lambda} \delta((\tau A)(x)) \quad (33)$$

With the gauge fixing the integral will converge. This is so because dA has no null space. To see this use that principle that if $A = 0$ on three sides of a plaquette p and $dA(p) = 0$, then $A = 0$ on the fourth side as well. Using this principle and moving outward from the origin one can argue that $dA = 0$ and $\tau A = 0$ imply $A = 0$.

However the expression (32) is not a good starting point for our analysis. It does not have good ultraviolet properties. We actually use something else which we now develop.

2 Gauge fields

2.1 block averaging

For A on a unit lattice define an averaged field $\mathcal{Q}A$ bonds in an L lattice by

$$(\mathcal{Q}A)(y, y + Le_\mu) = \sum_{x \in B(y)} L^{-4} A(\Gamma_{x, x+Le_\mu}) \quad (34)$$

Here $B(y)$ is the L -block of L^3 lattice points centered on y . The $\Gamma_{x, x+Le_\mu}$ is the straight line between the indicated points and $A(\Gamma) = \sum_{b \in \Gamma} A(b)$. This is designed to be gauge covariant, that is

$$\mathcal{Q}(A - d\lambda) = \mathcal{Q}(A) - d(\mathcal{Q}\lambda) \quad (35)$$

where $\mathcal{Q}\lambda$ is the average of the the scalar λ

$$(\mathcal{Q}\lambda)(y) = \sum_{x \in B(y)} \lambda(x) \quad (36)$$

Similarly gauge fields on any rectangular lattice can be averaged to gauge fields on a coarser lattice.

Given a density on the unit lattice $\rho_0(A_0)$ on our starting unit lattice \mathbb{T}_N^0 we define a new density $\tilde{\rho}_1(A)$ of the L -lattice \mathbb{T}_N^1 defining for A_1 on \mathbb{T}_N^1

$$\tilde{\rho}_1(A_1) = \int \delta(A_1 - \mathcal{Q}A_0) \rho_0(A_0) DA_0 \quad (37)$$

(If $\rho_0(A_0)$ is gauge invariant we need some gauge fixing here, but we postpone dealing with this for now.) Then $\int \tilde{\rho}_1(A_1) DA_1 = \int \rho_0(A_0) DA_0$. Next we scale back down to the unit lattice \mathbb{T}_{N-1}^0 defining for A_1 on \mathbb{T}_{N-1}^0

$$\rho_1(A_1) = \tilde{\rho}_1(A_{1,L}) \quad A_{1,L}(b) = L^{-\frac{1}{2}} A(L^{-1}b) \quad (38)$$

Then up to scaling factors from DA_0 ¹

$$\int \rho_1(A_1) DA_1 = \int \rho_0(A_0) DA_0 \quad (39)$$

We repeat this process. Given $\rho_k(A_k)$ on \mathbb{T}_{N-k}^0 we define $\tilde{\rho}_{k+1}(A_{k+1})$ on \mathbb{T}_{N-k}^1 by

$$\tilde{\rho}_{k+1}(A_{k+1}) = \int \delta(A_{k+1} - \mathcal{Q}A_k) \rho_k(A_k) DA_k \quad (40)$$

and then $\rho_{k+1}(A_{k+1})$ on \mathbb{T}_{N-k-1}^0 by

$$\rho_{k+1}(A_{k+1}) = \tilde{\rho}_{k+1}(A_{k+1,L}) \quad (41)$$

¹We are not keeping tracking of these scaling factors which take care of themselves. Those arising from the original scaling up are compensated step by step as we scale back down.

Again the integral is preserved

$$\int \rho_{k+1}(A_{k+1})DA_{k+1} = \int \rho_k(A_k)DA_k \quad (42)$$

The result of the iteration has an explicit representation. The delta functions compose by

$$\int \delta(A_2 - \mathcal{Q}A_1)\delta(A_1 - \mathcal{Q}A_0)DA_1 = \delta(A_2 - \mathcal{Q}^2A_0) \quad (43)$$

so after k steps we generate operators $\mathcal{Q}_k = \mathcal{Q}^k$. Taking advantage of the scale invariance of \mathcal{Q} we find

$$\rho_k(A_k) = \int \delta(A_k - \mathcal{Q}_k\mathcal{A})\rho_{0,L^{-k}}(\mathcal{A})DA \quad (44)$$

where $\rho_{0,L^{-k}}(\mathcal{A}) = \rho_0(\mathcal{A}_{L^k})$. Here \mathcal{A} is defined on bonds in \mathbb{T}_{N-k}^{-k} and $\mathcal{Q}_k\mathcal{A}$ is defined on bonds in \mathbb{T}_{N-k}^0 by

$$(\mathcal{Q}_k\mathcal{A})(y, y + e_\mu) = \sum_{x \in B_k(y)} L^{-4k} \mathcal{A}(\Gamma_{x, x+e_\mu}) \quad (45)$$

Here $B_k(y)$ is the L^k block of L^{3k} sites in \mathbb{T}_{N-k}^{-k} . It has unit volume. The $\mathcal{A}(\Gamma_{x, x+e_\mu})$ is an unweighted sum over bonds in $\Gamma_{x, x+e_\mu}$.

2.2 axial gauge

The discussion of block averaging has not dealt with the necessity of gauge fixing. A key idea of Balaban's method is that one does not fix the gauge at the start, but piece by piece as one makes the RG transformations. We carry this out with a modified axial gauge.

In the first step we construct a tree in each L -block as in figure 1. A partial axial gauge is imposed by setting $A_0(b) = 0$ for each bond b in any such tree. Bonds joining different blocks are not restricted. This is implemented with the delta function

$$\delta(\tau A_0) = \prod_{y \in \mathbb{T}_N^1} \prod_{x \in B(y), x \neq y} \delta\left((\tau A_0)(y, x)\right) \quad (46)$$

where

$$(\tau A_0)(y, x) = A_0(\Gamma_{y, x}) \quad (47)$$

with $\Gamma_{y, x}$ the path from the center y to the point x along the tree. This also generalizes to any rectangular lattice

Now we define the first RG transformation by

$$\tilde{\rho}_1(A_1) = \int \delta(A_1 - \mathcal{Q}A_0)\delta(\tau A_0)\rho_0(A_0) DA_0 \quad (48)$$

followed by scaling back to the unit lattice. We are particularly interested in initial densities of the form

$$\rho_0(A_0) = F_0(A_0)e^{-\frac{1}{2}\|dA_0\|^2} \quad (49)$$

where say F_0 is bounded. For such functions the above integral converges as the following proposition shows (in the case $A_1 = 0$)

Proposition 2. *On the subspace $\mathcal{Q}A_0 = 0$, $\tau A_0 = 0$ if $dA_0 = 0$ then $A_0 = 0$ and hence $\|dA_0\|^2 \geq C\|A_0\|^2$ for some constant C .*

Proof. $dA_0 = 0$ and $\tau A_0 = 0$ imply $A_0 = 0$ on bonds in each block as explained earlier. This also implies the A_0 is constant for bonds in any face joining two blocks, But now QA_0 only depends on such bonds and it is evaluated as a multiple of the same constant. Then $QA_0 = 0$ says the constant is zero so $A_0 = 0$.

The inequality follows since in a finite dimensional vector space a positive definite quadratic form is bounded below. This completes the proof. (This simple proof does not give control over the constant, put a further analysis shows that the constant only depends on L and is $\mathcal{O}(L^{-4})$ [6].

We again repeat the operation and generate a sequence of densities $\rho_k(A_k)$ on \mathbb{T}_{N-k}^0 by

$$\begin{aligned} \tilde{\rho}_{k+1}(A_{k+1}) &= \int \delta(A_{k+1} - \mathcal{Q}A_k)\delta(\tau A_k)\rho_k(A_k) DA_k \\ \rho_{k+1}(A_{k+1}) &= \tilde{\rho}_{k+1}(A_{k+1,L}) \end{aligned} \quad (50)$$

The result of the iteration again has an explicit representation. Taking advantage of the scale invariance of τA it is found to be.

$$\rho_k(A_k) = \int \delta(A_k - \mathcal{Q}_k \mathcal{A})\delta(\tau_k \mathcal{A})\rho_{0,L-k}(\mathcal{A})D\mathcal{A} \quad (51)$$

where now \mathcal{A} is defined on bonds in \mathbb{T}_{N-k}^{-k} . The gauge fixing function now has a hierarchical structure:

$$\delta(\tau_k \mathcal{A}) = \prod_{j=0}^{k-1} \delta(\tau \mathcal{Q}_j \mathcal{A}) \quad (52)$$

In the case $\rho_0(A_0) = F_0(A_0)e^{-\frac{1}{2}\|dA_0\|^2}$ the last expression becomes

$$\rho_k(A_k) = \int \delta(A_k - \mathcal{Q}_k \mathcal{A}) \delta(\tau_k \mathcal{A})F_{0,L-k}(\mathcal{A})e^{-\frac{1}{2}\|d\mathcal{A}\|^2} D\mathcal{A} \quad (53)$$

Here we used the scale invariance $\|d\mathcal{A}_{L^k}\|^2 = \|d\mathcal{A}\|^2$. We analyze this further. One can show that the quadratic form $\|d\mathcal{A}\|^2$ on \mathbb{T}_{N-k}^{-k} , as restricted by the constraints of the delta functions, has a unique minimum. It is a linear function of A_k and is denoted

$\mathcal{A}_k^\times = \mathcal{H}_k^\times A_k$. Expanding around the minimizer by $\mathcal{A} = \mathcal{A}_k^\times + \mathcal{Z}$ the quadratic form splits (there is no term linear in \mathcal{Z})

$$\frac{1}{2}\|d\mathcal{A}\|^2 = \frac{1}{2}\|d\mathcal{A}_k^\times\|^2 + \frac{1}{2}\|\mathcal{Z}\|^2 \quad (54)$$

and one finds

$$\rho_k(A_k) = Z_k F_k(\mathcal{A}_k^\times) e^{-\frac{1}{2}\|d\mathcal{A}_k^\times\|^2} \quad (55)$$

where

$$\begin{aligned} F_k(\mathcal{A}) &= Z_k^{-1} \int \delta(\mathcal{Q}_k \mathcal{Z}) \delta(\tau_k \mathcal{Z}) F_{0,L-k}(\mathcal{A} + \mathcal{Z}) \exp\left(-\frac{1}{2}\|d\mathcal{Z}\|^2\right) D\mathcal{Z} \\ Z_k &= \int \delta(\mathcal{Q}_k \mathcal{Z}) \delta(\tau_k \mathcal{Z}) \exp\left(-\frac{1}{2}\|d\mathcal{Z}\|^2\right) D\mathcal{Z} \end{aligned} \quad (56)$$

In arriving at this expression we used that $\mathcal{Q}_k \mathcal{A}_k^\times = A_k$ and that \mathcal{A}_k^\times is axial.

Note that if the initial function $F_0(A_0)$ is gauge invariant then we also have gauge invariance for the function $F_k(\mathcal{A})$, this in spite of the gauge fixing we were obliged to insert. If we average over the orientation of the trees when they are first introduced, then we also have invariance under all lattice symmetries.

The previous discussion is useful for identifying the type of structures that arise after multiple RG transformations. But to really track the flow we have to understand the individual steps better. We consider how one passes from the representation for ρ_k to the representation for ρ_{k+1} .

Suppose we are starting with the expression (55) for $\rho_k(A_k)$. In the next step we have

$$\tilde{\rho}_{k+1}(A_{k+1}) = Z_k \int \delta(A_{k+1} - \mathcal{Q}A_k) \delta(\tau A_k) F_k(\mathcal{A}_k^\times) \exp\left(-\frac{1}{2}\|d\mathcal{A}_k^\times\|^2\right) D\mathcal{A}_k \quad (57)$$

Let

$$A_k^{\min} = H_k^\times A_{k+1} = \text{minimizer for } \|d\mathcal{A}_k^\times\|^2 \text{ on the subspace } \mathcal{Q}A_k = A_{k+1}, \tau A_k = 0 \quad (58)$$

We expand around the minimizer with $A_k = A_k^{\min} + Z_k$. Then $\mathcal{A}_k^\times = \tilde{\mathcal{A}}_{k+1}^\times + \mathcal{Z}_k$ where

$$\begin{aligned} \tilde{\mathcal{A}}_{k+1}^\times &= \mathcal{H}_k^\times H_k^\times A_{k+1} \\ \mathcal{Z}_k &= \mathcal{H}_k^\times Z_k \end{aligned} \quad (59)$$

The quadratic form splits

$$\frac{1}{2}\|d\mathcal{A}_k^\times\|^2 = \frac{1}{2}\|d\tilde{\mathcal{A}}_{k+1}^\times\|^2 + \frac{1}{2}\|d\mathcal{Z}_k\|^2 \quad (60)$$

We get

$$\tilde{\rho}_{k+1}(A_k) = \mathbf{Z}_k \delta \mathbf{Z}_k \tilde{F}_{k+1}(\tilde{\mathcal{A}}_{k+1}^\times) \exp\left(-\frac{1}{2} \|d\tilde{\mathcal{A}}_{k+1}^\times\|^2\right) \quad (61)$$

where

$$\begin{aligned} \tilde{F}_{k+1}(\mathcal{A}) &= \delta \mathbf{Z}_k^{-1} \int \delta(\mathcal{Q}Z_k) \delta(\tau Z_k) F_k(\mathcal{A} + \mathbf{Z}_k) \exp\left(-\frac{1}{2} \|d\mathbf{Z}_k\|^2\right) DZ_k \\ \delta \mathbf{Z}_k &= \int \delta(\mathcal{Q}Z_k) \delta(\tau Z_k) \exp\left(-\frac{1}{2} \|d\mathbf{Z}_k\|^2\right) DZ_k \end{aligned} \quad (62)$$

This is a fluctuation integral. It is a Gaussian integral on a unit lattice and can be further analyzed depending on the particular function F_k .

Finally we scale by $\rho_{k+1}(A_{k+1}) = \tilde{\rho}_{k+1}(A_{k+1,L})$ and get the expected

$$\rho_{k+1}(A_{k+1}) = \mathbf{Z}_{k+1} F_{k+1}(\mathcal{A}_{k+1}^\times) e^{-\frac{1}{2} \|d\mathcal{A}_{k+1}^\times\|^2} \quad (63)$$

provided we make the identifications

$$\begin{aligned} \mathbf{Z}_{k+1} &= \mathbf{Z}_k \delta \mathbf{Z}_k \\ \mathcal{A}_{k+1,L}^\times &= \mathcal{H}_k^\times H_k^\times A_{k+1,L} \\ F_{k+1}(\mathcal{A}) &= \tilde{F}_{k+1}(\mathcal{A}_L) \end{aligned} \quad (64)$$

2.3 parametrization of fluctuation integrals

Our fluctuation integrals have the form with $\Delta_k = \mathcal{H}_k^{\times,T} \delta d\mathcal{H}_k^\times$ and $\delta = d^T$

$$\int f(Z) \delta(\mathcal{Q}Z) \delta(\tau Z) e^{-\frac{1}{2} \langle Z, \Delta_k Z \rangle} dZ \quad [f = 1] \quad (65)$$

It is an integral over the subspace $\mathcal{Q}Z = 0, \tau Z = 0$. This subspace can be parametrized as follows. The $\delta(\tau Z)$ sets $Z = 0$ on the bonds in each tree. For the remaining bonds we take $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2)$ where

- \tilde{Z}_1 is defined on bonds in each block not on a tree .
- \tilde{Z}_2 is defined on bonds joining adjacent block, except the central bond.

There is a bijective map $Z = C\tilde{Z}$ to the subspace $\mathcal{Q}Z = 0, \tau Z = 0$ defined by choosing the value on the central bond so that $\mathcal{Q}Z = 0$. The integral is then evaluated as the Gaussian integral

$$\int f(C\tilde{Z}) e^{-\frac{1}{2} \langle C\tilde{Z}, \Delta_k C\tilde{Z} \rangle} d\tilde{Z} \quad [f = 1] = \int f(C\tilde{Z}) d\mu_{C_k}(\tilde{Z}) \quad (66)$$

where

$$C_k = (C^T \Delta_k C)^{-1} \quad (67)$$

2.4 changing gauges

We are going to change from the axial gauge to covariant gauges which have better ultraviolet properties. First we introduce gauge fixing functions for both the axial and covariant gauges

For the axial gauge we define for \mathcal{A}, λ on \mathbb{T}_{N-k}^{-k}

$$\mathcal{I}_k(\mathcal{A}) = \int D\lambda \delta(Q_k \lambda) \delta(\tau_k \mathcal{A}^\lambda) = \int D\lambda \delta(Q_k \lambda) \prod_{j=0}^{k-1} \delta(\tau \mathcal{Q}_j \mathcal{A}^\lambda) \quad (68)$$

Here $Q_k = Q^k$. For $y \in \mathbb{T}_{N-k}^0$ it is given by

$$(Q_k f)(y) = L^{-3k} \sum_{x \in B_k(y)} f(x) \quad (69)$$

where $B_k(y)$ is the unit block centered on y with L^k sites on a side. The adjoint is

$$(Q_k^T f)(x) = f(y) \text{ for } x \in B_k(y) \quad (70)$$

Proposition 3. $\mathcal{I}_k \equiv \mathcal{I}_k(\mathcal{A})$ is constant in \mathcal{A} .

Remark. We could have used \mathcal{I}_k to obtain the axial gauge formally from an expression with no gauge fixing.

Proof. [25] In $\delta(\tau \mathcal{Q}_j(\mathcal{A} - d\lambda))$ we use $\mathcal{Q}_j d\lambda = d\mathcal{Q}_j \lambda$ and for $x \in B(y)$ the identity $\tau d\mathcal{Q}_j \lambda(y, x) = -L^{N-j}(Q_j \lambda(x) - Q_j \lambda(y))$ to obtain

$$\mathcal{I}(A) = \int D\lambda \delta(Q_k \lambda) \prod_{j=0}^{k-1} \prod_{y_j} \prod_{x_j \in B(y_j), x_j \neq y_j} \delta\left((\tau \mathcal{Q}_j \mathcal{A})(y_j, x_j) - L^{N-j}(Q_j \lambda(x_j) - Q_j \lambda(y_j))\right) \quad (71)$$

Now change variables from λ to $Q_k \lambda$ and $\{Q_j \lambda(x_j) - Q_j \lambda(y_j)\}_{0 \leq j \leq k-1}$. The Jacobian is independent of \mathcal{A} and we get the result.

The covariant gauges depend on a parameter $\alpha > 0$ and have the gauge fixing function

$$\begin{aligned} \mathfrak{G}_k(\mathcal{A}) &= \int D\lambda \delta(Q_k \lambda) \exp\left(-\frac{1}{2\alpha} \|\delta \mathcal{A}^\lambda\|^2\right) \\ &= \int D\lambda \delta(Q_k \lambda) \exp\left(-\frac{1}{2\alpha} \|\delta \mathcal{A} - \Delta \lambda\|^2\right) \\ &= \int D\lambda \delta(Q_k \lambda) \exp\left(-\frac{1}{2\alpha} \|\delta \mathcal{A} - \Delta'_k \lambda\|^2\right) \end{aligned} \quad (72)$$

Here $\delta = d^T$ for d on scalars. The scalar $\delta\mathcal{A}$ is the divergence. Also we have identified the positive Laplacian $\Delta\lambda = \delta d\lambda$ on scalars, and have defined for any $a > 0$ the operator

$$\Delta'_k = \Delta + aQ_k^T Q_k \quad (73)$$

Note that the quadratic form

$$\langle f, \Delta'_k f \rangle = \|df\|^2 + a\|Q_k f\|^2 \quad (74)$$

is positive definite. Indeed if it vanishes then $df = 0$ implies f is a constant in which case $Q_k f = f$ is that constant which must be zero. Hence in our finite dimensional space $\langle f, \Delta'_k f \rangle \geq c\|f\|^2$ for some $c > 0$ and hence $\|\Delta'_k f\| \geq c\|f\|$. Thus the inverse operator

$$G_k = (\Delta'_k)^{-1} = (\Delta + aQ_k^T Q_k)^{-1} \quad (75)$$

exists. It is called the Greens function or propagator. Properties of this operator are developed later.

Proposition 4. $\mathfrak{G}_k(\mathcal{A}) = \mathfrak{G}_k(0) \exp\left(-\frac{1}{2\alpha}\|P_k \delta\mathcal{A}\|^2\right)$ where P_k is the orthogonal projection

$$P_k = G_k Q_k^T (Q_k G_k^2 Q_k^T)^{-1} Q_k G_k \quad (76)$$

The projection is independent of a , and in fact $R_k = I - P_k$ is the projection onto $\Delta \ker Q_k$.

Proof. [5] Take $\alpha = 1$ for simplicity. To compute the integral we look for a minimum of the exponential on the subspace $Q_k \lambda = 0$. For the constraint we introduce the Lagrange multiplier $\langle \omega, Q_k \lambda \rangle$ and seek to minimize

$$\begin{aligned} g_k(\lambda, \omega) &= \frac{1}{2} \|\delta\mathcal{A} - \Delta'_k \lambda\|^2 + \langle \omega, Q_k \lambda \rangle \\ &= \frac{1}{2} \|\delta\mathcal{A}\|^2 - \langle \Delta'_k \delta\mathcal{A}, \lambda \rangle + \frac{1}{2} \|\Delta'_k \lambda\|^2 + \langle Q_k^T \omega, \lambda \rangle \end{aligned} \quad (77)$$

The minimum comes when

$$\begin{aligned} D_\lambda g_k &= -\Delta'_k \delta\mathcal{A} + \Delta'_k{}^2 \lambda + Q_k^T \omega = 0 \\ D_\omega g_k &= Q_k \lambda = 0. \end{aligned} \quad (78)$$

Then the first equation is solved by

$$\lambda_0 = G_k \delta\mathcal{A} - G_k^2 Q_k^T \omega \quad (79)$$

Then the second equation says

$$Q_k \lambda_0 = Q_k G_k \delta\mathcal{A} - Q_k G_k^2 Q_k^T \omega = 0 \quad (80)$$

The symmetric non-negative operator $Q_k G_k^2 Q_k^T$ is invertible. Indeed if $\langle f, Q_k G_k^2 Q_k^T f \rangle = 0$, then $\|G_k Q_k^T f\|^2 = 0$, then $G_k Q_k^T f = 0$, then $Q_k^T f = 0$, then $Q_k Q_k^T f = 0$, and since $Q_k Q_k^T = I$ this implies $f = 0$. Hence we can solve the last equation by

$$\omega = (Q_k G_k^2 Q_k^T)^{-1} Q_k G_k \delta \mathcal{A} \quad (81)$$

Thus

$$\lambda_0 = G_k \delta \mathcal{A} - G_k^2 Q_k^T (Q_k G_k^{-2} Q_k^T)^{-1} Q_k G_k \delta \mathcal{A} = G_k (I - P_k) \delta \mathcal{A} \quad (82)$$

and so

$$\Delta'_k \lambda_0 = (I - P_k) \delta \mathcal{A} \quad (83)$$

At the minimum we have

$$\|\delta \mathcal{A} - \Delta'_k \lambda_0\|^2 = \|P_k \delta \mathcal{A}\|^2 \quad (84)$$

Expanding the exponential in $\mathfrak{G}_k(\mathcal{A})$ around λ_0 by $\lambda = \lambda_0 + \lambda'$ yields

$$\mathfrak{G}_k(\mathcal{A}) = \int D\lambda \delta(Q_k \lambda) \exp\left(-\frac{1}{2}\|P_k \delta \mathcal{A}\|^2 - \frac{1}{2}\|\Delta'_k(\lambda - \lambda_0)\|^2\right) \quad (85)$$

Since $Q_k \lambda_0 = 0$ a change of variables $\lambda \rightarrow \lambda + \lambda_0$ shows that we can remove the λ_0 in the second term and find the announced

$$\begin{aligned} \mathfrak{G}_k(\mathcal{A}) &= \exp\left(-\frac{1}{2}\|P_k \delta \mathcal{A}\|^2\right) \int D\lambda \delta(Q_k \lambda) \exp\left(-\frac{1}{2}\|\Delta'_k \lambda\|^2\right) \\ &= \exp\left(-\frac{1}{2}\|P_k \delta \mathcal{A}\|^2\right) \mathfrak{G}_k(0) \end{aligned} \quad (86)$$

We characterize R_k as the projection onto $\Delta \ker Q_k$. If $f \in \Delta \ker Q_k$ then $f = \Delta \omega$ and $Q_k \omega = 0$. Then we have $f = \Delta'_k \omega$ and $Q_k \omega = 0$ and hence $P_k f = 0$ and $R_k f = f$. On the other hand if $R_k f = f$ then $P_k f = 0$ and applying $Q_k \Delta'_k$ gives $Q_k G_k f = 0$. Let $\omega = G_k f$. Then $Q_k \omega = 0$ and $f = \Delta'_k \omega$, so $f \in \Delta'_k \ker Q_k = \Delta \ker Q_k$.

Proposition 5. *For gauge fields \mathcal{A} on \mathbb{T}_{N-k}^{-k} let $f(\mathcal{A})$ be bounded and gauge invariant under transformations $\mathcal{A} \rightarrow \mathcal{A} - d\lambda$ with $Q_k \lambda = 0$. Then*

$$\begin{aligned} &\int \delta(A_k - Q_k \mathcal{A}) f(\mathcal{A}) \delta(\tau_k \mathcal{A}) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2\right) D\mathcal{A} \\ &= \frac{\mathcal{I}_k}{\mathfrak{G}_k(0)} \int \delta(A_k - Q_k \mathcal{A}) f(\mathcal{A}) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2 - \frac{1}{2\alpha}\|R_k \delta \mathcal{A}\|^2\right) D\mathcal{A} \end{aligned} \quad (87)$$

Remark. Equation (53) was of this form with $f(\mathcal{A}) = F_{0,L-k}(\mathcal{A})$

Proof. [5] The following is a Fadeev-Popov procedure. We insert

$$\mathfrak{G}_k(\mathcal{A}) \mathfrak{G}_k(\mathcal{A})^{-1} = \int D\lambda \delta(Q_k \lambda) \exp\left(-\frac{1}{2\alpha}\|\delta \mathcal{A}^\lambda\|^2\right) \mathfrak{G}_k(\mathcal{A})^{-1} \quad (88)$$

into the axial integral and after changing the order of integration we find

$$\int D\lambda \delta(Q_k\lambda) \left[\int D\mathcal{A} \delta(A_k - \mathcal{Q}_k\mathcal{A}) f(\mathcal{A}) \delta(\tau_k\mathcal{A}) \exp\left(-\frac{1}{2\alpha}\|d\mathcal{A}\|^2 - \frac{1}{2\alpha}\|\delta\mathcal{A}^\lambda\|^2\right) \mathfrak{G}_k(\mathcal{A})^{-1} \right] \quad (89)$$

In the bracketed expression we make the gauge transformation $\mathcal{A} \rightarrow \mathcal{A}^{-\lambda}$ and with $Q_k\lambda = 0$. The gauge function $\mathfrak{G}_k(\mathcal{A})$ is invariant since either by a change of variables in the integral expression, or by observing that $P_k\delta\mathcal{A}$ is invariant since $P_k\delta d\lambda = P_k\Delta\lambda = (1-R_k)\Delta\lambda = 0$. Furthermore $\delta(A_k - \mathcal{Q}_k\mathcal{A})$ is invariant since $\mathcal{Q}_k(\mathcal{A} + d\lambda) = \mathcal{Q}_k\mathcal{A} + d\mathcal{Q}_k\lambda = \mathcal{Q}_k\mathcal{A}$. So the expression becomes

$$\int D\lambda \delta(Q_k\lambda) \left[\int D\mathcal{A} \delta(A_k - \mathcal{Q}_k\mathcal{A}) f(\mathcal{A}) \delta(\tau_k\mathcal{A}^{-\lambda}) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2 - \frac{1}{2\alpha}\|\delta\mathcal{A}\|^2\right) \mathfrak{G}_k(\mathcal{A})^{-1} \right] \quad (90)$$

We change the order of integration again and identify $\int D\lambda \delta(Q_k\lambda) \delta(\tau_k\mathcal{A}^{-\lambda}) = \mathcal{I}_k(\mathcal{A}) = \mathcal{I}_k$. We are left with

$$\mathcal{I}_k \int D\mathcal{A} \delta(A_k - \mathcal{Q}_k\mathcal{A}) f(\mathcal{A}) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2 - \frac{1}{2\alpha}\|\delta\mathcal{A}\|^2\right) \mathfrak{G}_k(\mathcal{A})^{-1} \quad (91)$$

Now put the calculated value

$$\mathfrak{G}_k(\mathcal{A})^{-1} = \mathfrak{G}_k^{-1}(0) \exp\left(\frac{1}{2\alpha}\|P_k\delta\mathcal{A}\|^2\right) = \mathfrak{G}_k(0)^{-1} \exp\left(\frac{1}{2\alpha} \langle \delta\mathcal{A}, P_k\delta\mathcal{A} \rangle\right) \quad (92)$$

and identify $\exp\left(-\frac{1}{2\alpha} \langle \delta\mathcal{A}, R_k\delta\mathcal{A} \rangle\right) = \exp\left(-\frac{1}{2\alpha}\|R_k\delta\mathcal{A}\|^2\right)$ to complete the proof.

Does the integral on the right side of (87) makes sense? Take $\alpha = 1$ for simplicity. The issue is whether the quadratic form $\|d\mathcal{A}\|^2 + \|R_k\delta\mathcal{A}\|^2 = \langle \mathcal{A}, (\delta d + dR_k\delta)\mathcal{A} \rangle$ is positive definite. Since we are constrained by $\mathcal{Q}_k\mathcal{A} = A_k$ we can add a term $a\|\mathcal{Q}_k\mathcal{A}\|^2 = \langle \mathcal{A}, a\mathcal{Q}_k^T\mathcal{Q}_k\mathcal{A} \rangle$ and ask the same question. Then we question is answered by the following

Proposition 6. $\delta d + dR_k\delta + a\mathcal{Q}_k^T\mathcal{Q}_k$ is a positive definite operator

Proof. [5] Since $R_k = I - P_k$ and $\Delta = \delta d + d\delta$ the operator can also be written as $\Delta - dP_k\delta + \mathcal{Q}_k^T\mathcal{Q}_k$. We need to show that

$$\langle \mathcal{A}, (\Delta - dP_k\delta)\mathcal{A} \rangle + a\|\mathcal{Q}_k\mathcal{A}\|^2 = 0 \implies \mathcal{A} = 0 \quad (93)$$

This is a sum of two symmetric non-negative operators so it suffices to show that

$$(\Delta - dP_k\delta)\mathcal{A} = 0 \quad \text{and} \quad \mathcal{Q}_k\mathcal{A} = 0 \implies \mathcal{A} = 0 \quad (94)$$

The space $\ell^2(\mathbb{T}_{N-k}^-)$ space is a direct sum of constants and functions orthogonal to constants. The operators Δ, Q_k, Q_k^T preserve these subspaces. The function $\delta\mathcal{A}$ is orthogonal to constants so we are concerned with P_k on this subspace. Then we can take $a = 0$ in the expression for P_k . Then $\Delta'_k = \Delta$ and we write it as

$$P_k = \Delta^{-1} Q_k^T (Q_k \Delta^{-2} Q_k^T)^{-1} Q_k \Delta^{-1} \quad (95)$$

The first equation is then

$$\Delta\mathcal{A} - d\Delta^{-1} Q_k^T (Q_k \Delta^{-2} Q_k^T)^{-1} Q_k \Delta^{-1} \delta\mathcal{A} = 0 \quad (96)$$

This has the form $\Delta\mathcal{A} = f$ with f orthogonal to constants. Solutions have the form $\mathcal{A} = (\Delta)^{-1} f + \mathcal{A}_0$ for constant \mathcal{A}_0 and so

$$\mathcal{A} = d\Delta^{-2} Q_k^T (Q_k \Delta^{-2} Q_k^T)^{-1} Q_k \Delta^{-1} \delta\mathcal{A} + \mathcal{A}_0 \quad (97)$$

Since $Q_k d = dQ_k$ and $Q_k \mathcal{A}_0 = \mathcal{A}_0$ the second equation $Q_k \mathcal{A} = 0$ says

$$dQ_k \Delta^{-1} \delta\mathcal{A} + \mathcal{A}_0 = 0 \quad (98)$$

Apply δ to this equation and get $\Delta Q_k \Delta^{-1} \delta\mathcal{A} = 0$ and hence $Q_k \Delta^{-1} \delta\mathcal{A} = 0$. Then $\mathcal{A}_0 = 0$ by (98) and $\mathcal{A} = 0$ by (97).

2.5 covariant gauges

We study the covariant gauges in more detail with. Instead of the axial gauge expression (53), the k -step RG transformation would now be given by

$$\rho_k(A_k) = \int \delta(A_k - Q_k \mathcal{A}) F_{0,L-k}(\mathcal{A}) \exp\left(-\frac{1}{2} \|d\mathcal{A}\|^2 - \frac{1}{2\alpha} \|R_k \delta\mathcal{A}\|^2\right) D\mathcal{A} \quad (99)$$

We analyze this further taking $\alpha = 1$ for simplicity. The quadratic form in the exponential is $\langle \mathcal{A}, (\delta d + dR_k \delta) \mathcal{A} \rangle$ and we look for the minimizer. subject to $A_k = Q_k \mathcal{A}$. Again this is the same as the minimizer of $\langle \mathcal{A}, (\delta d + dR_k \delta + Q_k^T Q_k) \mathcal{A} \rangle$ subject to $A_k = Q_k \mathcal{A}$. To find it we introduce Lagrange multipliers and ask for the minimizer of

$$\langle \mathcal{A}, (\delta d + dR_k \delta + Q_k^T Q_k) \mathcal{A} \rangle + \langle \omega, (A_k - Q_k \mathcal{A}) \rangle \quad (100)$$

Taking derivatives in \mathcal{A} and ω we find

$$\begin{aligned} (\delta d + dR_k \delta + Q_k^T Q_k) \mathcal{A} - Q_k^T \omega &= 0 \\ A_k - Q_k \mathcal{A} &= 0 \end{aligned} \quad (101)$$

The solution of the first equation is $\mathcal{A}_k = \mathcal{G}_k Q_k^T \omega$ where.

$$\mathcal{G}_k = (\delta d + dR_k \delta + Q_k^T Q_k)^{-1} \quad (102)$$

Then the second equation says $A_k = \mathcal{Q}_k \mathcal{G}_k \mathcal{Q}_k^T \omega$ or $\omega = (\mathcal{Q}_k \mathcal{G}_k \mathcal{Q}_k^T)^{-1} A_k$. (The inverse exists as before.) Therefore the minimizer is

$$\mathcal{A}_k = \mathcal{H}_k A_k \equiv \mathcal{G}_k \mathcal{Q}_k^T (\mathcal{Q}_k \mathcal{G}_k \mathcal{Q}_k^T)^{-1} A_k \quad (103)$$

As for the axial gauge we expand around the minimizer by $\mathcal{A} = \mathcal{A}_k + \mathcal{Z}$. The quadratic form splits and one finds

$$\rho_k(A_k) = Z_k F_k(\mathcal{A}_k) \exp\left(-\frac{1}{2}\|d\mathcal{A}_k\|^2 - \frac{1}{2}\|R_k \delta\mathcal{A}\|^2\right) \quad (104)$$

where

$$\begin{aligned} F_k(\mathcal{A}) &= Z_k^{-1} \int \delta(\mathcal{Q}_k \mathcal{Z}) F_{0,L-k}(\mathcal{A} + \mathcal{Z}) \exp\left(-\frac{1}{2}\|d\mathcal{Z}\|^2 - \frac{1}{2}\|R_k \delta\mathcal{Z}\|^2\right) D\mathcal{Z} \\ Z_k &= \int \delta(\mathcal{Q}_k \mathcal{Z}) \exp\left(-\frac{1}{2}\|d\mathcal{Z}\|^2 - \frac{1}{2}\|R_k \delta\mathcal{Z}\|^2\right) D\mathcal{Z} \end{aligned} \quad (105)$$

A special case is $F_{0,L-k}(\mathcal{A}) = \mathcal{A}(b)$ for some fixed bond $b \in \mathbb{T}_{N-k}^{-k}$. Then in (105) $F_{0,L-k}(\mathcal{A} + \mathcal{Z}) = \mathcal{A}(b) + \mathcal{Z}(b)$ and the $\mathcal{Z}(b)$ term contributes nothing since we are integrating an odd function. Thus we have in this case

$$\rho_k(A_k) = Z_k \mathcal{A}_k(b) e^{-\frac{1}{2}\|d\mathcal{A}_k\|^2} \quad (106)$$

On the other hand if $F_{0,L-k}(\mathcal{A}) = 1$ we get $\rho_k(A_k) = Z_k e^{-\frac{1}{2}\|d\mathcal{A}_k\|^2}$. Thus \mathcal{A}_k is the ratio of these two. Returning to the representation (99) we have for the covariant minimizer

$$\mathcal{A}_k(b) = \mathcal{H}_k A_k(b) = \frac{\int \delta(A_k - \mathcal{Q}_k \mathcal{A}) \mathcal{A}(b) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2 - \frac{1}{2}\|R_k \delta\mathcal{A}\|^2\right) D\mathcal{A}}{\int \delta(A_k - \mathcal{Q}_k \mathcal{A}) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2 - \frac{1}{2}\|R_k \delta\mathcal{A}\|^2\right) D\mathcal{A}} \quad (107)$$

Similarly for the axial minimizer

$$\mathcal{A}_k^\times(b) = \mathcal{H}_k^\times A_k(b) = \frac{\int \delta(A_k - \mathcal{Q}_k \mathcal{A}) \delta(\tau_k \mathcal{A}) \mathcal{A}(b) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2\right) D\mathcal{A}}{\int \delta(A_k - \mathcal{Q}_k \mathcal{A}) \delta(\tau_k \mathcal{A}) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2\right) D\mathcal{A}} \quad (108)$$

The covariant minimizers $\mathcal{H}_k A_k$ will play a big role in our development. These representations enable us to show it is equivalent to the axial gauge minimizer $\mathcal{H}_k^\times A_k$.

Proposition 7. *There exists a scalar function $\lambda^* = \lambda^*(A_k)$ such that*

$$\mathcal{H}_k^\times A_k = \mathcal{H}_k A_k + d\lambda^* \quad (109)$$

Proof. [18] To compare the two we proceed as in Propostion 5. Insert $\mathfrak{G}_k(\mathcal{A})\mathfrak{G}_k(\mathcal{A})^{-1}$ in the axial expression (108), change the order of integration and find

$$\begin{aligned} \mathcal{H}_k^\times A_k(b) = & Z_k^\times(A_k)^{-1} \int D\lambda \delta(Q_k \lambda) \\ & \left[\int D\mathcal{A} \delta(A_k - \mathcal{Q}_k \mathcal{A}) \mathcal{A}(b) \delta(\tau_k \mathcal{A}) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2 - \frac{1}{2}\|\delta\mathcal{A}^\lambda\|^2\right) \mathfrak{G}_k(\mathcal{A})^{-1} \right] \end{aligned} \quad (110)$$

where $Z_k^\times(A_k)$ be the denominator in (108). Now let $\mathcal{A} \rightarrow \mathcal{A}^{-\lambda}$. As before it removes the λ from $\|\delta\mathcal{A}^\lambda\|^2$ and changes $\delta(\tau_k \mathcal{A})$ to $\delta(\tau_k \mathcal{A}^{-\lambda})$. New is the $\mathcal{A}(b)$ which becomes $\mathcal{A}(b) + d\lambda(b)$.

In the $\mathcal{A}(b)$ term we identify \mathcal{I}_k as before and put in the expression for $\mathfrak{G}_k(\mathcal{A})^{-1}$. We then use the representation (107) to identify $\mathcal{H}_k A_k(b)$. If $Z_k(A_k)$ is the denominator in (107) we find that the $\mathcal{A}(b)$ term is

$$Z_k^\times(A_k)^{-1} \mathcal{I}_k \mathfrak{G}_k(0)^{-1} Z_k(A_k) \mathcal{H}_k A_k(b) = \mathcal{H}_k A_k(b) \quad (111)$$

Here we use the identity $Z_k^\times(A_k) = \mathcal{I}_k \mathfrak{G}_k(0)^{-1} Z_k(A_k)$ which is (87) with $f(\mathcal{A}) = 1$,

For the $d\lambda(b)$ term suppose $b = (x, x + L^{-k} e_\mu)$. Then this term is revealed to be $d\lambda^*(b) = (\lambda^*(x + L^{-k} e_\mu) - \lambda^*(x))/L^{-k}$ where

$$\begin{aligned} \lambda^*(x) = & Z_k^\times(A_k)^{-1} \int D\lambda \delta(Q_k \lambda) \\ & \left[\int D\mathcal{A} \delta(A_k - \mathcal{Q}_k \mathcal{A}) \lambda(x) \delta(\tau_k \mathcal{A}^{-\lambda}) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2 - \frac{1}{2}\|\delta\mathcal{A}\|^2\right) \mathfrak{G}_k(\mathcal{A})^{-1} \right] \end{aligned} \quad (112)$$

2.6 Landau gauge

In (99) if we take the limit $\alpha \rightarrow \infty$ we get an integral over the subspace $R_k \delta\mathcal{A} = 0$. This is the Landau gauge. It can be defined directly by the k -step RG transformation

$$\rho_k(A_k) = \int \delta(A_k - \mathcal{Q}_k \mathcal{A}) \delta(R_k \delta\mathcal{A}) F_{0,L^{-k}}(\mathcal{A}) \exp\left(-\frac{1}{2}\|d\mathcal{A}\|^2\right) D\mathcal{A} \quad (113)$$

One can show it is equivalent to the other gauges by a Fadeev-Popov argument.

The minimizer in the Landau gauge can be computed as before now with two Lagrange multipliers. One finds that it is the same as the covariant gauges, which are therefore independent of α . Without going through the calculation we can understand why they are the same.

Proposition 8. *The minimizers \mathcal{H}_k are the same in the Landau and covariant gauges*

Proof. [17] We work in the subspace of gauge fields \mathcal{A} satisfying $\mathcal{Q}\mathcal{A} = A_k$. We want to argue that.

$$\begin{aligned}
\mathcal{H}_k A_k &= \text{the unique minimizer of } \|d\mathcal{A}\|^2 + \frac{1}{2\alpha}\|R_k\delta\mathcal{A}\|^2 + a\|\mathcal{Q}_k\mathcal{A}\|^2 \\
&= \text{the unique minimizer of } \|d\mathcal{A}\|^2 + a\|\mathcal{Q}_k\mathcal{A}\|^2 \text{ with } R_k\delta\mathcal{A} = 0 \\
&= \mathcal{H}_k^{\text{Landau}} A_k
\end{aligned} \tag{114}$$

It suffices to show that the minimizer in the first line satisfies $R_k\delta\mathcal{A} = 0$ since then we can restrict the definition of that minimizer to that subspace. The proof is by contradiction. Suppose the minimizer in the first line has $R_k\delta\mathcal{A} \neq 0$. There is a restricted gauge transformation $\mathcal{Q}_k\lambda = 0$ such that $R_k\delta\mathcal{A}^\lambda = 0$. Indeed this equation says

$$R_k\delta\mathcal{A} - R_k\Delta\lambda = R_k\delta\mathcal{A} - \Delta\lambda = 0 \tag{115}$$

and since $R_k\delta\mathcal{A} = \Delta\omega$ for some $\mathcal{Q}_k\omega = 0$, we can just take $\lambda = \omega$. But the restricted gauge transformation leaves invariant the condition $\mathcal{Q}\mathcal{A} = A_k$ and the functions $\|d\mathcal{A}\|^2$ and $\|\mathcal{Q}_k\mathcal{A}\|^2$. Thus we have lower the overall value by replacing the positive $(2\alpha)^{-1}\|R_k\delta\mathcal{A}\|^2$ by zero. This contradicts the fact that we had a minimum.

3 Fermi fields

3.1 block averaging

The block averaging operator for fermions on the unit lattice \mathbb{T}_N^0 with a background field A_0 and coupling constant e_0 is

$$(Q(A)\Psi)(y) = L^{-3} \sum_{x \in B(y)} e^{ie_0 A(\Gamma_{y,x})} \Psi(x) \quad y \in \mathbb{T}_N^1 \quad (116)$$

As before $B(y)$ is the L -block centered on the point y in L -lattice \mathbb{T}_N^1 and $\Gamma_{y,x}$ is the path in the tree from the center y to $x \in B(y)$. This is constructed to be gauge covariant. If $\Psi^\lambda = e^{ie_0 \lambda} \Psi$ and $A^\lambda = A - d\lambda$ then

$$Q(A^\lambda)\Psi^\lambda = (Q(A)\Psi)^{\lambda^{(1)}} \quad (117)$$

where $\lambda^{(1)}$ is λ restricted to the lattice \mathbb{T}_N^1 . For the conjugate field $\bar{\Psi}^\lambda = e^{-ie_0 \lambda} \bar{\Psi}$ and it is $Q(-A)\bar{\Psi}$ which is covariant. The transpose operator maps functions Ψ on \mathbb{T}_N^1 to functions on \mathbb{T}_N^0 . It is computed with sums on \mathbb{T}_N^1 weighted by L^3 and is given by

$$(Q^T(A)\Psi)(x) = \Psi(y) e^{ie_0 A(\Gamma_{y,x})} \quad x \in B(y) \quad (118)$$

Then we have $Q(A)Q^T(-A) = I$ and $Q^T(-A)Q(A)$ is a projection operator. $Q(A)$ is defined similarly on any finer lattice.

Suppose we start with a density $\rho_0(A, \Psi_0)$ on \mathbb{T}_N^0 with fermion field Ψ_0 and background gauge field A on \mathbb{T}_N^0 . Delta functions are not an option for fermions, and instead we implement the block averaging with an exponential. We define a new density on \mathbb{T}_N^1 by

$$\tilde{\rho}_1(A, \Psi_1) = N_1 \int \exp\left(-\frac{b}{L} |[\Psi_1 - Q(A)\Psi_0]|^2\right) \rho_0(A, \Psi_0) D\Psi_0 \quad (119)$$

Here $b > 0$ is an arbitrary constant and we have introduced the notation

$$|[\Psi_1 - Q(A)\Psi_0]|^2 = \left\langle \bar{\Psi}_1 - Q(-A)\bar{\Psi}_0, \Psi_1 - Q(A)\Psi_0 \right\rangle \quad (120)$$

where $\langle \bar{\Psi}_1, \Psi_1 \rangle = \sum_x L^3 \bar{\Psi}_1(x) \Psi_1(x)$, etc., Nothing is actually being squared here, the exponent two is just a reminder that this is a quadratic form. The normalization factor $N_1^{-1} = \int e^{-bL^{-1} \langle \bar{\Psi}_1, \Psi_1 \rangle} D\Psi_1$ is chosen so that

$$\int \tilde{\rho}_1(A, \Psi_1) D\Psi_1 = \int \rho_0(A, \Psi_0) D\Psi_0 \quad (121)$$

Next one scales back to the unit lattice defining for Ψ_1 on \mathbb{T}_{N-1}^0 and \mathcal{A} on \mathbb{T}_{N-1}^{-1}

$$\rho_1(\mathcal{A}, \Psi_1) = \text{const } \tilde{\rho}_0(\mathcal{A}_L, \Psi_{1,L}) \quad (122)$$

where

$$\mathcal{A}_L(b) = L^{-1/2}\mathcal{A}(L^{-1}b) \quad \Psi_{1,L}(x) = L^{-1}\Psi_1(L^{-1}x) \quad (123)$$

We repeat the operation k times and we get a density $\rho_k(\mathcal{A}, \Psi_k)$ defined for \mathcal{A} on \mathbb{T}_{N-k}^{-k} and Ψ_k on \mathbb{T}_{N-k}^0 . After some work we find it is given by

$$\rho_k(\mathcal{A}, \Psi_k) = \mathcal{N}_k \int \exp\left(-b_k|[\Psi_k - Q_k(\mathcal{A})\psi]|^2\right) \rho_0(\mathcal{A}_{L^k}, \psi_{L^k}) D\psi \quad (124)$$

for some constants b_k, \mathcal{N}_k . The integral here is over ψ on \mathbb{T}_{N-k}^{-k} . The averaging operator is the composition

$$Q_k(\mathcal{A}) = Q(\mathcal{A}) \circ \dots \circ Q(\mathcal{A}) \quad (k \text{ times}) \quad (125)$$

This is not a simple as it looks since the factors are acting on different scales. We can give a more explicit expression. Suppose $x \in \mathbb{T}_{N-k}^{-k}$ and $y \in \mathbb{T}_{N-k}^0$ satisfy $x \in B_k(y)$, which is the same as $|x-y| < \frac{1}{2}$. There is an associated sequence $x = y_0, y_1, y_2, \dots, y_k = y$ such that $y_j \in \mathbb{T}_{N-k}^{-k+j}$ and $x \in B_j(y_j)$. Define

$$\mathcal{A}(\Gamma_{y,x}^k) = \sum_{j=0}^{k-1} \mathcal{A}(\Gamma_{y_{j+1}, y_j}) \quad (126)$$

Here each $\mathcal{A}(\Gamma_{y_{j+1}, y_j})$ is an unweighted sum over bonds in \mathbb{T}_{N-k}^{-k} , but the paths Γ_{y_{j+1}, y_j} live on trees in different scales. Then with $\eta = L^{-k}$

$$(Q_k(\mathcal{A})\psi)(y) = \sum_{x \in B_k(y)} \eta^3 e^{ie_k \eta \mathcal{A}(\Gamma_{y,x}^k)} \psi(x) \quad (127)$$

Here e_0 has scaled to

$$e_k = L^{k/2} e_0 = L^{-(N-k)/2} e \quad (128)$$

This running coupling constant which will make frequent appearances in the following. For future reference the background mass scales to

$$\bar{m}_k = L^k \bar{m}_0 = L^{-(N-k)} m \quad (129)$$

3.2 Dirac type effective actions

Now consider an initial density which is a perturbation of the free fermion action:

$$\rho_0(\mathcal{A}, \Psi_0) = F_0(\Psi_0) \exp\left(-\left\langle \bar{\Psi}, (\mathfrak{D}_{e_0}(\mathcal{A}) + \bar{m}_0)\Psi \right\rangle\right) \quad (130)$$

Insert this in (124). The Dirac operator with e_0, m_0 scales to the Dirac operator with e_k, m_k . Then with $F_{0,L-k}(\psi) = F_0(\psi_{L^k})$ we have

$$\rho_k(\mathcal{A}, \Psi_k) = \mathcal{N}_k \int F_{0,L-k}(\psi) \exp\left(-b_k|[\Psi_k - Q_k(\mathcal{A})\psi]|^2 - \left\langle \bar{\psi}, (\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k)\psi \right\rangle\right) D\psi \quad (131)$$

We next diagonalize the quadratic form in the exponential. To accomplish this we temporarily regard $\bar{\psi}, \psi$ as functions and look for critical points of the quadratic form in these variables. Setting the derivative in $\bar{\psi}$ equal to zero says.

$$b_k Q_k^T(-\mathcal{A}) \left(\Psi_k - Q_k(\mathcal{A})\psi \right) - (\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k)\psi = 0 \quad (132)$$

This has the solution $\psi = \psi_k(\mathcal{A})$ where

$$\psi_k(\mathcal{A}) \equiv \mathcal{H}_k(\mathcal{A})\Psi_k \equiv b_k S_k(\mathcal{A}) Q_k^T(-\mathcal{A})\Psi_k \quad (133)$$

where $S_k(\mathcal{A})$ is the Green's function (assuming it exists)

$$S_k(\mathcal{A}) = \left(\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k + b_k Q_k^T(-\mathcal{A}) Q_k(\mathcal{A}) \right)^{-1} \quad (134)$$

There is a similar expression for the critical point $\bar{\psi} = \bar{\psi}_k(\mathcal{A})$.

Now we change variables from $\bar{\psi}, \psi$ to new Grassmann elements $\bar{\mathcal{Z}}, \mathcal{Z}$ by expanding around the critical point

$$\psi = \psi_k(\mathcal{A}) + \mathcal{Z} \quad \bar{\psi}_k(\mathcal{A}) = \bar{\psi}_k(\mathcal{A}) + \bar{\mathcal{Z}} \quad (135)$$

The quadratic form splits and becomes

$$\mathfrak{S}_k(\mathcal{A}, \Psi_k, \psi_k(\mathcal{A})) + \langle \mathcal{Z}, S_k(\mathcal{A})^{-1} \mathcal{Z} \rangle \quad (136)$$

where

$$\mathfrak{S}_k(\mathcal{A}, \Psi_k, \psi_k(\mathcal{A})) \equiv b_k |[\Psi_k - Q_k(\mathcal{A})\psi_k(\mathcal{A})]|^2 - \left\langle \bar{\psi}_k(\mathcal{A}), \left(\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k \right) \psi_k(\mathcal{A}) \right\rangle \quad (137)$$

We find that

$$\rho_k(\mathcal{A}, \Psi_k) = \mathcal{N}_k Z_k(\mathcal{A}) F_k(\psi_k(\mathcal{A})) \exp \left(- \mathfrak{S}_k(\mathcal{A}, \Psi_k, \psi_k(\mathcal{A})) \right) \quad (138)$$

where

$$F_k(\psi_k(\mathcal{A})) = \int F_{0,L-k}(\psi_k(\mathcal{A}) + \mathcal{Z}) d\mu_{S_k}(\psi) \quad (139)$$

$$Z_k(\mathcal{A}) = \det(S_k(\mathcal{A}))^{-1}$$

The fluctuation integral is now a Gaussian integral with covariance $S_k(\mathcal{A})$.

Let us examine the assumption that the inverse $S_k(\mathcal{A})$ exists. The operator in question can be written with $P_k(\mathcal{A}) = b_k Q_k^T(-\mathcal{A}) Q_k(\mathcal{A})$

$$\begin{aligned} \mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k + P_k(\mathcal{A}) &= \left(\mathfrak{D}_{e_k}(0) + \bar{m}_k + P_k(0) \right) \\ &+ \left(\mathfrak{D}_{e_k}(\mathcal{A}) - \mathfrak{D}_{e_k}(0) \right) + \left(P_k(\mathcal{A}) - P_k(0) \right) \end{aligned} \quad (140)$$

The $\mathcal{A} = 0$ term is invertible and has nice properties. Indeed the inverse has a kernel with strong exponential decay. (Not from the \bar{m}_k which is too small to be useful, but from the $P_k(0)$ which provides an effective mass. More on this later) The second term has the form with $\eta = L^{-k}$

$$\left((\mathfrak{D}_{e_k}(\mathcal{A}) - \mathfrak{D}_{e_k}(0))f \right)(x) = - \sum_{\mu} \left(\frac{1 - \gamma_{\mu}}{2} \right) \left(\frac{(e^{ie_k \eta \mathcal{A}(x, x + \eta e_{\mu})} - 1)}{\eta} \right) f(x + \eta e_{\mu}) + \dots \quad (141)$$

If both e_k and \mathcal{A} are small then the operator is small. The same is true for $P_k(\mathcal{A}) - P_k(0)$. Thus under these assumptions we have a small perturbation of an invertible operator and so the sum is invertible. However the assumption that \mathcal{A} is globally small cannot be realized. At best we can get bounds on $d\mathcal{A}$. This is a key issue which we will explore further.

Nevertheless we continue with the next step. We first note that since $\psi_k(\mathcal{A}) = \mathcal{H}_k(\mathcal{A})\Psi_k$, the leading term $\mathfrak{S}(\mathcal{A}, \Psi_k, \psi_k(\mathcal{A}))$ is a quadratic form in Ψ_k and can be calculated as

$$\mathfrak{S}_k(\mathcal{A}, \Psi_k, \psi_k(\mathcal{A})) = \left\langle \bar{\Psi}_k, D_k(\mathcal{A})\Psi_k \right\rangle \quad (142)$$

where

$$D_k(\mathcal{A}) = b_k - b_k^2 Q_k(\mathcal{A}) S_k(\mathcal{A}) Q_k^T(-\mathcal{A}) \quad (143)$$

If we start with the expression (138) for ρ_k and apply another renormalization transformation we again get ρ_{k+1} . We have first

$$\begin{aligned} \tilde{\rho}_{k+1}(\mathcal{A}, \Psi_{k+1}) &= \mathcal{N}_k N_k Z_k(\mathcal{A}) \\ &\int F_k(\psi_k(\mathcal{A})) \exp \left(-\frac{b}{L} |[\Psi_{k+1} - Q(\mathcal{A})\Psi_k]|^2 - \left\langle \bar{\Psi}_k, D_k(\mathcal{A})\Psi_k \right\rangle \right) D\Psi_k \end{aligned} \quad (144)$$

Here Ψ_{k+1}, Ψ_k are fields on $\mathbb{T}_{N-k}^1, \mathbb{T}_{N-k}^0$ respectively. The critical point for the quadratic form in the exponential is

$$\begin{aligned} \Psi_k^{crit} &= H_k(\mathcal{A})\Psi_{k+1} \equiv bL^{-1} \Gamma_k(\mathcal{A}) Q^T(-\mathcal{A})\Psi_{k+1} \\ \Gamma_k(\mathcal{A}) &= \left(D_k(\mathcal{A}) + \frac{b}{L} Q^T(-\mathcal{A}) Q(\mathcal{A}) \right)^{-1} \end{aligned} \quad (145)$$

and similarly for $\bar{\Psi}_k$. Again we diagonalize the quadratic form by expanding around the critical point by $\Psi_k = H_k(\mathcal{A})\Psi_{k+1} + W_k$ and $\bar{\Psi}_k = H_k(\mathcal{A})\bar{\Psi}_{k+1} + \bar{W}_k$. Under this change of variables $\psi_k(\mathcal{A}) = \mathcal{H}_k(\mathcal{A})\Psi_k$ becomes $\tilde{\psi}_{k+1}(\mathcal{A}) + \mathcal{H}_k(\mathcal{A})W_k$ where

$$\tilde{\psi}_{k+1}(\mathcal{A}) = \mathcal{H}_k(\mathcal{A})H_k(\mathcal{A})\Psi_{k+1} \quad (146)$$

The quadratic form splits and after a calculation can be written in the form

$$\tilde{\mathfrak{S}}_{k+1}(\mathcal{A}, \Psi_{k+1}, \tilde{\psi}_{k+1}(\mathcal{A})) + \left\langle \bar{W}_k, \Gamma_k(\mathcal{A})^{-1} W_k \right\rangle \quad (147)$$

The integral over W_k can be identified as a Gaussian integral with covariance $\Gamma_k(\mathcal{A})$ and we find

$$\tilde{\rho}_{k+1}(\mathcal{A}, \Psi_{k+1}) = \mathcal{N}_k N_k Z_k(\mathcal{A}) \delta Z_k(\mathcal{A}) \tilde{F}_{k+1}(\tilde{\psi}_{k+1}(\mathcal{A})) \exp\left(-\tilde{\mathfrak{S}}_{k+1}(\mathcal{A}, \Psi_{k+1}, \tilde{\psi}_{k+1}(\mathcal{A}))\right) \quad (148)$$

where

$$\begin{aligned} \tilde{F}_{k+1}(\psi) &= \int F_k(\psi + \mathcal{H}_k(\mathcal{A})W_k) d\mu_{\Gamma_k(\mathcal{A})}(W_k) \\ \delta Z_k(\mathcal{A}) &= \det(\Gamma_k(\mathcal{A}))^{-1} \end{aligned} \quad (149)$$

Now we scale by $\rho_{k+1}(\mathcal{A}, \Psi_k) = \tilde{\rho}_{k+1}(\mathcal{A}_L, \Psi_{k+1,L})$. One finds that $\tilde{\psi}_{k+1}(\mathcal{A})$ scales to

$$\mathcal{H}_k(\mathcal{A}_L)H_k(\mathcal{A}_L)\Psi_{k+1,L} = [\mathcal{H}_{k+1}(\mathcal{A})\Psi_{k+1}]_L = [\psi_{k+1}(\mathcal{A})]_L \quad (150)$$

that $\tilde{\mathfrak{S}}_{k+1}(\mathcal{A}, \Psi_{k+1}, \tilde{\psi}_{k+1}(\mathcal{A}))$ scales to $\mathfrak{S}_{k+1}(\mathcal{A}, \Psi_{k+1}, \psi_{k+1}(\mathcal{A}))$ and that (up to scaling factors)

$$\mathcal{N}_k N_k Z_k(\mathcal{A}_L) \delta Z_k(\mathcal{A}_L) = \mathcal{N}_{k+1} Z_{k+1}(\mathcal{A}) \quad (151)$$

Then we get the expected

$$\rho_{k+1}(\mathcal{A}, \Psi_{k+1}) = \mathcal{N}_{k+1} Z_{k+1}(\mathcal{A}) F_{k+1}(\psi_{k+1}(\mathcal{A})) \exp\left(-\mathfrak{S}_{k+1}(\mathcal{A}, \Psi_{k+1}, \psi_{k+1}(\mathcal{A}))\right) \quad (152)$$

where now

$$F_{k+1}(\psi) = \int F_k(\psi_L + \mathcal{H}_k(\mathcal{A})W_k) d\mu_{\Gamma_k(\mathcal{A})}(W_k) \quad (153)$$

This is the basic fluctuation integral. Analyzing such integrals is one of our main tasks, and this depends heavily on properties of $\mathcal{H}_k(\mathcal{A})$ and $\Gamma_k(\mathcal{A})$

4 Random walk expansions

We need detailed control over the gauge field propagators \mathcal{G}_k and Dirac propagators $\mathcal{S}_k(\mathcal{A})$. Results on these operators will give control over other important operators $\mathcal{H}_k, \Delta_k, C_k$ for gauge fields and $\mathcal{H}_k(\mathcal{A}), D_k(\mathcal{A}), \Gamma_k(\mathcal{A})$ for fermions. To accomplish this we employ random walk expansions. We start with a simpler case.

4.1 A simple case

In the Dirac propagator $\mathcal{S}_k(\mathcal{A})$ and critical operator $\mathcal{H}_k(\mathcal{A})$ eliminate the background field \mathcal{A} , replace the Dirac operator by the Laplacian, and drop the mass term. So we are interested on the operators

$$\begin{aligned} G_k &= (\Delta + a_k Q_k^T Q_k)^{-1} \\ \mathcal{H}_k &= a_k G_k Q_k^T \end{aligned} \tag{154}$$

which act on functions on $\mathbb{T}_{N-k}^{-k}, \mathbb{T}_{N-k}^0$ respectively. With constants $a_k > 0$ bounded above and below, these are in fact the Greens functions and minimizer for a massless scalar field theory. We want to develop estimates and random walk expansions for these operators. We start with a restricted operator.

Partition the lattice into M -blocks (cubes) where $M = L^m$ is much larger than L . Then enlarge each M -block to a $3M$ -block by including all of its nearest neighbors. Then we have a covering of the lattice with $3M$ -blocks \square .

Lemma 1. *Let \square be an $3M$ -block, let $[\Delta + a_k Q_k^T Q_k]_{\square}$ be the indicated operator on $\ell^2(\square)$ with Neumann boundary conditions and define*

$$G_k(\square) = [\Delta + a_k Q_k^T Q_k]_{\square}^{-1} \tag{155}$$

Then there is a constant $\mathcal{O}(1)$ such that

$$\|G_k(\square)f\|_2 \leq \mathcal{O}(1)\|f\|_2 \quad \|\partial_{\mu}G_k(\square)f\|_2 \leq \mathcal{O}(1)\|f\|_2 \tag{156}$$

The same bounds hold for $G_k = [\Delta + a_k Q_k^T Q_k]^{-1}$ on the torus \mathbb{T}_{N-k}^{-k} or infinite lattice $L^{-k}\mathbb{Z}^3$.

Remark. Neumann boundary conditions for Δ_{\square} means that on bonds entirely in \square contribute. As a quadratic form with $\eta = L^{-k}$

$$\langle f, \Delta_{\square} f \rangle = \sum_{(x,y) \in \square} \eta^3 \left(|f(x) - f(y)| / \eta \right)^2 \tag{157}$$

Proof. [4] First consider a unit block $\Delta \subset \square$ and consider the operator $[\Delta + a_k Q_k^T Q_k]_{\Delta}$ on $\ell^2(\Delta)$ with Neumann conditions. The Hilbert space splits into constants and the

orthogonal complement which we write $\ell^2(\Delta) = [1] \oplus [1]^\perp$. Under the decomposition we have $\Delta = 0 \oplus (\Delta)$ and $\Delta \geq \pi^2$ on $[1]^\perp$. On the other hand $Q_k^T Q_k = I \oplus 0$; it is the projection onto $[1]$. So since a_k is bounded below

$$[\Delta + a_k Q_k^T Q_k]_\Delta \geq \inf_k \{\pi^2, a_k\} \geq c_0 \quad (158)$$

Now for the M -cube we can drop the contribution of bonds joining different unit blocks and estimate

$$\begin{aligned} \left\langle f, [\Delta + a_k Q_k^T Q_k]_{\square} f \right\rangle &\geq \sum_{\Delta \subset \square} \left\langle f, [\Delta + a_k Q_k^T Q_k]_{\Delta} f \right\rangle \\ &\geq c_0 \sum_{\Delta \subset \square} \|f\|_{\Delta}^2 = c_0 \|f\|_{\square}^2 \end{aligned} \quad (159)$$

This implies $\|[\Delta + a_k Q_k^T Q_k]_{\square} f\| \geq c_0 \|f\|$ and hence $\|G_k(\square) f\| \leq c_0^{-1} \|f\|$.

For the second bound we first note

$$\begin{aligned} \|\partial_\mu G_k^{\frac{1}{2}}(\square) f\|^2 &\leq \left\langle f, G_k^{\frac{1}{2}}(\square) [\Delta]_{\square} G_k^{\frac{1}{2}}(\square) f \right\rangle \\ &\leq \left\langle f, G_k^{\frac{1}{2}}(\square) \left[\Delta + a_k Q_k^T Q_k \right]_{\square} G_k^{\frac{1}{2}}(\square) f \right\rangle = \|f\|^2 \end{aligned} \quad (160)$$

It follows that

$$\|\partial_\mu G_k(\square) f\| \leq \|G_k^{\frac{1}{2}}(\square) f\| \leq c_0^{-\frac{1}{2}} \|f\| \quad (161)$$

This completes the proof.

We now quote a sharpened version of these bounds. The exponential decay in the following can be interpreted as saying that the $a_k Q_k^T Q_k$ in $G_k(\square) = [\Delta + a_k Q_k^T Q_k]_{\square}^{-1}$ provides an effective mass.

Lemma 2. *There are constants C, γ depending only on L such that for unit cubes Δ, Δ'*

$$|1_{\Delta} G_k(\square) 1_{\Delta'} f|, |1_{\Delta} \partial G_k(\square) 1_{\Delta'} f| \leq C e^{-\gamma d(\Delta, \Delta')} \|f\|_{\infty} \quad (162)$$

Remark. We could have stated the result in terms of kernels $G_k(\square, x, y)$, but then we would have to keep track of integrable short distance singularities in some detail. Stating the result as above efficiently deals with these. Also if we have two operators which satisfy a bound of the form (162), then the composition also satisfies a bound of the form (162). Note also that these imply the global estimates

$$|G_k(\square) f|, |\partial G_k(\square) f| \leq C \|f\|_{\infty} \quad (163)$$

Proof. [4] We sketch the proof. First consider the kernel $G_k(x, y)$ of the operator $G_k = (\Delta + a_k Q_k^T Q_k)^{-1}$ on the infinite lattice $L^{-k} \mathbb{Z}^3$. This operator G_k has a kernel

which can be written as Fourier transform

$$\begin{aligned} G_k(x, y) &= (2\pi)^{-3} \int_{|p_\mu| \leq L^k \pi} e^{ip(x-y)} \tilde{G}_k(p) \\ \tilde{G}_k(p) &= \sum_x L^{-3k} e^{-ipx} G_k(x) \end{aligned} \tag{164}$$

Corresponding to the fact that G_k is a bounded operator one finds that $\tilde{G}_k(p)$ is a bounded function, and it is in fact $\mathcal{O}(|p|^{-2})$ as $p \rightarrow \infty$. Closer inspection of the explicit formula for $\tilde{G}_k(p)$ reveals that it is an analytic function of any component p_μ in a strip of width γ around the real axis. Then a contour deformation $p_\mu \rightarrow p_\mu \pm i\gamma$ (the sign depending on the sign of $x_\mu - y_\mu$) yields a decay factor $e^{-\gamma|x_\mu - y_\mu|}$. These ideas lead to the bounds (162) for the infinite lattice.

The result for $G_k(\square, x, y)$ with Neumann boundary conditions follows by the method of reflections. More is true. The method of reflections also gives the result for $G_k(\square, x, y)$ with Dirichlet or periodic boundary conditions, and also for $G_k(x, y)$ on the full torus \mathbb{T}_{N-k}^{-k} .

Armed with $G_k(\square)$ we can now develop the random walk expansion. We have a covering of the lattice \mathbb{T}_{N-k}^{-k} with $3M$ blocks \square . Let h_\square be a continuum partition of unity subordinate to this covering. More precisely we require $\sum_\square h_\square^2 = 1$ and $\text{supp } h_\square \subset \square$. We can choose h_\square so that

$$|\partial h_\square| \leq \mathcal{O}(1)M^{-1} \tag{165}$$

For example take a covering \square' by blocks of width 3 and an associated partition of unity $h'_{\square'}$. Then define $h_\square(x) = h'_{M^{-1}\square}(M^{-1}x)$.

We define a parametrix for $\Delta + a_k Q_k^T Q_k$ by

$$G_k^* = \sum_\square h_\square G_k(\square) h_\square \tag{166}$$

On $\text{supp } h_\square$ the boundary conditions are irrelevant and so $(\Delta + a_k Q_k^T Q_k)G(\square)f = f$. Therefore

$$(\Delta + a_k Q_k^T Q_k)G_k^* = I - \sum_\square K_\square G_k(\square) h_\square \equiv I - K \tag{167}$$

where

$$K_\square = -\left[(\Delta + a_k Q_k^T Q_k), h_\square\right] \tag{168}$$

Then

$$G_k = G_k^* (I - K)^{-1} = G_k^* \sum_{n=0}^{\infty} K^n \tag{169}$$

if the series converges. This can be written as the random walk expansion

$$G_k = \sum_\omega G_{k,\omega} \tag{170}$$

where a path ω is a sequence of blocks $\omega = (\square_0, \square_1, \dots, \square_n)$ such that $\square_i \cap \square_{i+1} \neq \emptyset$ and

$$G_{k,\omega} = (h_{\square_0} G_k(\square_0) h_{\square_0})(K_{\square_1} G_k(\square_1) h_{\square_1}) \cdots (K_{\square_n} G_k(\square_n) h_{\square_n}) \quad (171)$$

Lemma 3. *If M is sufficiently large the random walk expansion for G_k converges and again satisfies*

$$|1_{\Delta} G_k 1_{\Delta'} f|, |1_{\Delta} \partial G_k 1_{\Delta'} f| \leq C e^{-\gamma d(\Delta, \Delta')} \|f\|_{\infty} \quad (172)$$

Proof. $[\Delta, h_{\square}]$ is a first order differential operator that has at least one derivative of h_{\square} and so the coefficients are $\mathcal{O}(M^{-1})$. The operator $Q_k^T Q_k$ is localized in unit squares and thereby one can show $[Q_k^T Q_k, h_{\square}]$ is also $\mathcal{O}(M^{-1})$. These considerations lead to the bound

$$|K_{\square} f| \leq \mathcal{O}(1) M^{-1} (\|f\|_{\infty} + \|\partial f\|_{\infty}) \quad (173)$$

and hence by (163)

$$|K_{\square} G_k(\square) f| \leq \mathcal{O}(1) M^{-1} \|f\|_{\infty} \quad (174)$$

These imply that if $|\omega| = n$ then

$$|G_{k,\omega} f| \leq C (CM^{-1})^n \|f\|_{\infty} \quad (175)$$

This is sufficient to establish the convergence of the expansion for M sufficiently large. Each block has 3^3 neighbors so the number of paths with a fixed length n is bounded by $(3^3)^n$. We have

$$\begin{aligned} |G_k f| &\leq \sum_{n=0}^{\infty} \sum_{\omega: |\omega|=n} |G_{k,\omega} f| \\ &\leq \sum_{n=0}^{\infty} \sum_{\omega: |\omega|=n} C (CM^{-1})^n \|f\|_{\infty} \\ &\leq \sum_{n=0}^{\infty} (3^3)^n C (CM^{-1})^n \|f\|_{\infty} \\ &\leq C \|f\|_{\infty} \end{aligned} \quad (176)$$

Similarly $|\partial G_k f| \leq C \|f\|_{\infty}$

For the local version we use instead of (174) the sharper estimate

$$|1_{\Delta} K_{\square} G_k(\square) 1_{\Delta'} f| \leq CM^{-1} e^{-\gamma d(\Delta, \Delta')} \|f\|_{\infty} \quad (177)$$

The decay factors combine to give an overall decay factor and the result follows.

4.2 Dirac propagators

Let us define

$$P_k(\mathcal{A}) = Q_k^T(-\mathcal{A})Q_k(\mathcal{A}) \quad (178)$$

so that the Dirac Greens function and critical operator are

$$\begin{aligned} S_k(\mathcal{A}) &= (\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k + b_k P_k(\mathcal{A}))^{-1} \\ \mathcal{H}_k(\mathcal{A}) &= b_k S_k(\mathcal{A}) Q_k^T(-\mathcal{A}) \end{aligned} \quad (179)$$

We need random walk expansions for these operators, and it suffices to consider $S_k(\mathcal{A})$.

First we need some restrictions on the background gauge field \mathcal{A} . We will only have good control over $d\mathcal{A}$ and we need to transfer it to bounds and \mathcal{A} and its derivatives. This can only be done locally. Here is an example

Lemma 4. *Let A be a gauge field on a block of arbitrary size in a unit lattice centered on y . Then there is a gauge transformation to $A' = A - d\lambda$ such that for any bond in the block*

$$|A'(b)| \leq d(b, y) \|dA\|_\infty \quad (180)$$

Proof. We can assume that A is axial gauge relative to a tree rooted on x . If b is on the tree the result is trivial. Otherwise $b = (x, x + e_\mu)$ with $x, x + e_\mu$ on different branches. Note that

$$\mathcal{A}(\Gamma_{y,x}) + \mathcal{A}(x, x + e_\mu) - \mathcal{A}(\Gamma_{y,x+e_\mu}) \quad (181)$$

is a sum around a closed curve. Thus by the lattice version of Stokes theorem it is equal to $d\mathcal{A}(\Sigma_{y,x})$ for some surface $\Sigma_{y,x}$ of width one and length $d(y, x)$. But in the axial gauge $\mathcal{A}(\Gamma_{y,x}) = \mathcal{A}(\Gamma_{y,x+e_\mu}) = 0$. Then

$$A(x, x + e_\mu) = dA(\Sigma_{y,x}) = \sum_{p \in \Sigma_{y,x}} dA(p) \quad (182)$$

which gives the result.

We want to generalize this to some kind of global statement about gauge fields on finer lattices. We particularly want to consider the minimizers $\mathcal{A}_k = \mathcal{H}_k A_k$.

First a definition For each $3M$ -block \square let $\tilde{\square}$ be an enlargement, say to a $5M$ -block.

Definition \mathcal{R}_k is the space of all real fields \mathcal{A} on \mathbb{T}_{N-k}^{-k} such that in every block $\tilde{\square}$ the \mathcal{A} is gauge equivalent to a field \mathcal{A}' such that

$$|\mathcal{A}'|, |\partial\mathcal{A}'| < e_k^{-\frac{3}{4}} \quad (183)$$

Complex \mathcal{R}_k is the space of all \mathcal{A} of the form $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ with $\mathcal{A}_0 \in \mathcal{R}_k$ and \mathcal{A}_1 complex and satisfying the bounds (183).

Note that $|e_k \mathcal{A}'| |e_k \partial\mathcal{A}'| \leq e_k^{\frac{1}{4}}$ are small under this restriction.

Lemma 5. [26]

1. $\mathcal{A}_k = \mathcal{H}_k \mathcal{A}_k$ has the property that in each $\tilde{\square}$ it is gauge equivalent to some \mathcal{A}' satisfying

$$|\mathcal{A}'|, |\partial \mathcal{A}'| \leq CM \|d\mathcal{A}_k\|_\infty \quad (184)$$

2. If $\|d\mathcal{A}_k\|_\infty \leq (CM)^{-1} e_k^{-\frac{3}{4}}$ then $\mathcal{A}_k \in \mathcal{R}_k$

The first point is like lemma 4, but requires some work. Then the second point is immediate. The condition on $\|d\mathcal{A}_k\|_\infty$ can be arranged as we will see. Next we note that the condition $\mathcal{A} \in \mathcal{R}_k$ is sufficient for the existence of $S_k(\mathcal{A})$

Lemma 6. [21] Let $\mathcal{A} \in \mathcal{R}_k$. Then for each $5M$ -block $\tilde{\square}$ there exists an operator $S_k(\square, \mathcal{A})$ on functions f on $\tilde{\square}$ such that

$$\left(\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k + b_k P_k(\mathcal{A}) \right) S_k(\square, \mathcal{A}) f = f \text{ on } \square \quad (185)$$

The operator $S_k(\square, \mathcal{A})$ only depends on \mathcal{A} in $\tilde{\square}$ and satisfies

$$|1_\Delta S_k(\mathcal{A}) 1_{\Delta'} f| \leq C e^{-\gamma d(\Delta, \Delta')} \|f\|_\infty \quad (186)$$

The idea of the proof is to first prove it for $S_k(\square, 0)$. Then prove it for $S_k(\square, \mathcal{A}')$ with the small transformed field $\mathcal{A}' = \mathcal{A} - d\lambda$ by expanding in $e_k \mathcal{A}'$ as sketched previously. Finally translate this result to \mathcal{A} by using the gauge covariance

$$S_k(\square, \mathcal{A}) = e^{ie_k \lambda} S_k(\square, \mathcal{A}') e^{-ie_k \lambda} \quad (187)$$

The creation of $S_k(\square, 0)$ needs more work. The construction given by Balaban, O'Carroll, and Shor [21] uses a multi-scale random walk expressions which are of a type we consider later and which are rather complicated. Possibly this construction could be improved.

Using the operators $S_k(\square, \mathcal{A})$ we can develop a random walk expansion. Again take a partition of unity h_\square with $\text{supp } h_\square \subset \square$ and $|\partial h_\square| \leq \mathcal{O}(1)M^{-1}$. We define a parametrix by

$$S_k^*(\mathcal{A}) = \sum_{\square} h_\square S_k(\square, \mathcal{A}) h_\square \quad (188)$$

Then

$$\left(\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k + b_k P_k(\mathcal{A}) \right) S_k^*(\mathcal{A}) = I - \sum_{\square} K_\square(\mathcal{A}) S_k(\square) h_\square \equiv I - K(\mathcal{A}) \quad (189)$$

where

$$K_\square(\mathcal{A}) = - \left[\left(\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k + b_k P_k(\mathcal{A}) \right), h_\square \right] = \mathcal{O}(M^{-1}) \quad (190)$$

Now as before $S_k(\mathcal{A}) = S_k^*(\mathcal{A})(I - K)^{-1} = S_k^*(\mathcal{A}) \sum_{n=0}^{\infty} K^n(\mathcal{A})$ This can be written as the random walk expansion

$$S_k(\mathcal{A}) = \sum_{\omega} S_{k,\omega}(\mathcal{A}) \quad (191)$$

where a path ω is a sequence of blocks $\omega = (\square_0, \square_1, \dots, \square_n)$ such that $\square_i \cap \square_{i+1} \neq \emptyset$ and

$$S_{k,\omega}(\mathcal{A}) = h_{\square_0} S_k(\mathcal{A}, \square_0) h_{\square_0} \left(K_{\square_1}(\mathcal{A}) S_k(\mathcal{A}, \square_1) h_{\square_1} \right) \cdots \left(K_{\square_n}(\mathcal{A}) S_k(\mathcal{A}, \square_n) h_{\square_n} \right) \quad (192)$$

This only depends on \mathcal{A} in ω .

Lemma 7. [21] *Let $\mathcal{A} \in \mathcal{R}_k$ and let M be sufficiently large. Then $S_k(\mathcal{A})$ exists, the random walk expansion for $S_k(\mathcal{A})$ converges, and satisfies*

$$|1_{\Delta} S_k(\mathcal{A}) 1_{\Delta'} f| \leq C e^{-\gamma d(\Delta, \Delta')} \|f\|_{\infty} \quad (193)$$

The same holds for $\mathcal{H}_k(\mathcal{A})$.

4.3 gauge propagators

We will also need random walk expansions for the gauge Greens function and minimizer

$$\begin{aligned} \mathcal{G}_k &= (\delta d + d R_k \delta + a \mathcal{Q}_k^T \mathcal{Q}_k)^{-1} \\ \mathcal{H}_k &= \mathcal{G}_k \mathcal{Q}_k^T (\mathcal{Q}_k \mathcal{G}_k \mathcal{Q}_k^T)^{-1} \end{aligned} \quad (194)$$

As in the scalar case the operator $\mathcal{Q}_k^T \mathcal{Q}_k$ provides an effective mass (although it is not now a projection operator).

Now there are new difficulties connected with the non-local projection operator R_k in \mathcal{G}_K Dealing with this non-locality requires some substantial workarounds. Also in the minimizer the operator $(\mathcal{Q}_k \mathcal{G}_k \mathcal{Q}_k^T)^{-1}$ is not the inverse of a local operator and requires special treatment. Nevertheless one can develop a random walk expansion

$$\mathcal{G}_k = \sum_{\omega} \mathcal{G}_{k,\omega} \quad (195)$$

However the elementary local building blocks have to be generalized from $3M$ blocks \square to something more general.

Lemma 8. [8] *If M is sufficiently large the random walk expansion for \mathcal{G}_k converges and again satisfies*

$$|1_{\Delta} \mathcal{G}_k 1_{\Delta'} f|, |1_{\Delta} \partial \mathcal{G}_k 1_{\Delta'} f| \leq C e^{-\gamma d(\Delta, \Delta')} \|f\|_{\infty} \quad (196)$$

The same holds for the minimizer \mathcal{H}_k .

5 Norms and polymer functions

5.1 definitions

We collect some definitions for use later

Consider a Grassmann algebra generated by element $\Psi(x)$ on a finite unit lattice. (For example the fundamental fields Ψ_k on \mathbb{T}_{N-k}^0). The general element of the algebra has the form.

$$E(\Psi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1, \dots, x_n} E_n(x_1, \dots, x_n) \Psi(x_1) \cdots \Psi(x_n) \quad (197)$$

Here there is an implicit sum over spin indices and over the choice of Ψ or $\bar{\Psi}$. Then for $h > 0$ a norm is defined by

$$\|E\|_h = \sum_{n=0}^{\infty} \frac{h^n}{n!} \sum_{x_1, \dots, x_n} |E_n(x_1, \dots, x_n)| \quad (198)$$

This satisfies $\|EF\|_h \leq \|E\|_h \|F\|_h$.

More generally suppose we have Grassman elements ψ on a finite lattice with spacing L^{-k} (For example the smeared fields $\psi_k(\mathcal{A})$ on \mathbb{T}_{N-k}^{-k}). Then a general element has the form

$$E(\psi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1, \dots, x_n} L^{-3kn} E_n(x_1, \dots, x_n) \psi(x_1) \cdots \psi(x_n) \quad (199)$$

This has a norm

$$\|E\|_h = \sum_{n=0}^{\infty} \frac{h^n}{n!} \sum_{x_1, \dots, x_n} L^{-3kn} |E_n(x_1, \dots, x_n)| \quad (200)$$

and again $\|EF\|_h \leq \|E\|_h \|F\|_h$.

A *polymer* X is a connected union of M -blocks with the convention that two blocks are connected if they have a face in common. A *polymer function* $E(X) = E(X, \psi)$ is an element of the Grassman algebra which only depends on ψ in X . Equivalently the kernels $E_n(X, x_1, \dots, x_n)$ have support in $X \times \cdots \times X$. We will be considering elements $E(\psi)$ which have polymer expansions $E(\psi) = \sum_X E(X, \psi)$.

More generally we also allow dependence on a gauge field \mathcal{A} and we consider elements of the form

$$E(\mathcal{A}, \psi) = \sum_X E(X, \mathcal{A}, \psi) \quad (201)$$

now with the requirement that $E(X, \mathcal{A}, \psi)$ only depend on \mathcal{A}, ψ in X . This has kernels $E_n(X, \mathcal{A}, x_1, \dots, x_n)$ and for a set \mathcal{R} of gauge fields we define

$$\|E_n(X)\|_{\mathcal{R}} = \sup_{\mathcal{A} \in \mathcal{R}} \sum_{x_1, \dots, x_n} L^{-3kn} |E_n(X, \mathcal{A}, x_1, \dots, x_n)| \quad (202)$$

Then we define

$$\|E(X)\|_{\mathcal{R},h} = \sum_{n=0}^{\infty} \frac{h^n}{n!} \|E_n(X)\|_{\mathcal{R}} \quad (203)$$

We expect that $E(X)$ will have tree decay on scale M of the form $E(X) \sim e^{-\kappa d_M(X)}$ for some constant $\kappa = \mathcal{O}(1)$ and

$$Md_M(X) = \text{length of the shortest continuum tree joining the blocks in } X \quad (204)$$

Correspondingly we introduce the norm on the family $\{E(X, \mathcal{A}, \psi)\}$ by

$$\|E\|_{\mathcal{R},h,\kappa} = \sup_X \|E(X)\|_{\mathcal{R},h} e^{\kappa d_M(X)} \quad (205)$$

We note that there is a constant κ_0 depending only on the dimension such that

$$\sum_{X \supset \square} e^{-\kappa_0 d_M(X)} \leq \mathcal{O}(1) \quad (206)$$

We assume that κ is a substantial multiple of κ_0 and hence $e^{-\kappa d_M(X)}$ is strongly summable.

Now suppose we specifically want to allow fields $\mathcal{A} = \mathcal{A}_k$ and $\psi = \psi_k(\mathcal{A})$ defined on \mathbb{T}_{N-k}^{-k} after k iterations of the RG group. We take $\mathcal{R} = \mathcal{R}_k$ as defined earlier with the running coupling constants. It is also convenient to let allow the parameter h to depend on the running coupling constant e_k by taking $h_k = e_k^{-\frac{1}{4}}$. so we define for a family $\{E(X, \mathcal{A}, \psi)\}$

$$\|E\|_k = \|E\|_{\mathcal{R}_k, h_k, \kappa} \quad (207)$$

The choice $h = h_k$ means that if E is bounded then the kernels E_n are $\mathcal{O}(h_k^{-n}) = \mathcal{O}(e_k^{\frac{1}{4}n})$. They are automatically small, and more fields means even smaller. This is convenient. But hiding some of the smallness in the definition of the norm means we have less in the size of the norm. Estimates with this norm with tend to have fractional powers of e_k . (This choice $h = h_k$ is probably optional. Very likely we could take a fixed h and estimates would have integral powers of e_k)

5.2 scaling

It is important to know how polymer functions scale. So we have polymer functions $E(X, \mathcal{A}, \psi)$ on \mathbb{T}_{N-k}^{-k} with norms $\|E\|_k$ as defined above. Before scaling we reblock. Let \bar{X} defined to be the smallest union of LM -blocks containing X and define for LM polymers X'

$$E'(X', \mathcal{A}, \psi) = \sum_{X: \bar{X}=X'} E(X, \mathcal{A}, \psi) \quad (208)$$

Then we scale down defining $\mathcal{L}E(Y)$ on $\mathbb{T}_{N-k-1}^{-k-1}$ for M -polymers Y

$$(\mathcal{L}E)(Y, \mathcal{A}, \psi) = E'(LY, \mathcal{A}_L, \psi_L) = \sum_{X: \bar{X}=LY} E(X, \mathcal{A}_L, \psi_L) \quad (209)$$

Lemma 9.

$$\|\mathcal{L}E\|_{k+1} \leq \mathcal{O}(1)L^3\|E\|_k \quad (210)$$

or more precisely

$$\|\mathcal{L}E\|_{\mathcal{R}_{k+1}, h_{k+1}, \kappa} \leq \mathcal{O}(1)L^3\|E\|_{L^{-\frac{7}{8}}\mathcal{R}_k, L^{-\frac{9}{8}}h_k, \kappa} \leq \mathcal{O}(1)L^3\|E\|_{\mathcal{R}_k, h_k, \kappa} \quad (211)$$

Proof. We have

$$\begin{aligned} \|E(X, (\cdot)_L, (\cdot)_L)\|_{\mathcal{R}_{k+1}, h_{k+1}} &\leq \|E(X)\|_{L^{-\frac{1}{2}}\tilde{\mathcal{R}}_{k+1}, L^{-1}h_{k+1}} \\ &\leq \|E(X)\|_{L^{-\frac{7}{8}}\mathcal{R}_k, L^{-\frac{9}{8}}h_k} \end{aligned} \quad (212)$$

Here $\tilde{\mathcal{R}}_{k+1}$ is \mathcal{R}_k but with bounds $e_k^{-\frac{3}{4}}$ rather than $e_k^{-\frac{3}{4}}$. Since $e_{k+1} = L^{\frac{1}{2}}e_k$ we have $e_{k+1}^{-\frac{3}{4}} = L^{-\frac{3}{8}}e_k^{-\frac{3}{4}}$ which accounts for the improvement $L^{-\frac{1}{2}} \rightarrow L^{-\frac{7}{8}}$. Similarly $h_{k+1} = L^{-\frac{1}{8}}h_k$ accounts for the improvement $L^{-1} \rightarrow L^{-\frac{9}{8}}$.

$$\begin{aligned} \|(\mathcal{L}E)(Y)\|_{\mathcal{R}_{k+1}, h_{k+1}} &\leq \sum_{X:\bar{X}=LY} \|E(X)\|_{L^{-\frac{7}{8}}\mathcal{R}_k, L^{-\frac{9}{8}}h_k} \\ &\leq \|E\|_{L^{-\frac{7}{8}}\mathcal{R}_k, L^{-\frac{9}{8}}h_k, \kappa_0+2L^{-1}\kappa} \sum_{X:\bar{X}=LY} e^{-(\kappa_0+2L^{-1}\kappa)d_M(X)} \end{aligned} \quad (213)$$

But a tree on X is also a tree on \bar{X} and so $Md_M(X) \geq LMd_{LM}(\bar{X})$. Thus if $\bar{X} = LY$

$$d_M(X) \geq Ld_{LM}(LY) = Ld_M(Y) \quad (214)$$

Then we can extract a factor $e^{-2\kappa d_M(Y)}$ from the sum and we are left with

$$\sum_{X:X \subset LY} e^{-\kappa_0 d_M(X)} \leq \sum_{\square \subset LY} \sum_{X \supset \square} e^{-\kappa_0 d_M(X)} \leq \mathcal{O}(1)|LY|_M \leq \mathcal{O}(1)L^3|Y|_M \quad (215)$$

Now use $|Y|_M \leq \mathcal{O}(1)(1 + d_M(Y)) \leq e^{\kappa d_M(Y)}$ and get

$$\|(\mathcal{L}E)(Y)\|_{\mathcal{R}_{k+1}, h_{k+1}} \leq \mathcal{O}(1)L^3\|E\|_{L^{-\frac{7}{8}}\mathcal{R}_k, L^{-\frac{9}{8}}h_k, \kappa_0+2L^{-1}\kappa} e^{-\kappa d_M(Y)} \quad (216)$$

which gives

$$\|\mathcal{L}E\|_{k+1} \leq \|\mathcal{L}E\|_{\mathcal{R}_{k+1}, h_{k+1}, \kappa} \leq \mathcal{O}(1)L^3\|E\|_{L^{-\frac{7}{8}}\mathcal{R}_k, L^{-\frac{9}{8}}h_k, \kappa_0+2L^{-1}\kappa} \quad (217)$$

Since $L^{-\frac{7}{8}}\mathcal{R}_k \subset \mathcal{R}_k$, $L^{-\frac{9}{8}}h_k < h_k$, and $\kappa_0 + 2L^{-1}\kappa < \kappa$ the right side is less than $\|E\|_{\mathcal{R}_k, h_k, \kappa} = \|E\|_k$ which gives the result.

Thus under scaling the overall size of the polymer functions can increase by as much as L^3 which represents dangerous growth. But if the polymer functions $E(X)$ have relevant parts removed the factors $L^{-\frac{7}{8}}$ for gauge fields and $L^{-\frac{9}{8}}$ for fermions in (217) can compensate, even for marginal terms.

In addition if $d_M(Y) > 0$ we can modify the above proof (replace $\kappa_0 + 2L^{-1}\kappa$ by $\kappa_0 + 1 + 2L^{-1}\kappa$) and gain a factor $e^{-d_M(X)} \leq e^{-Ld_M(Y)}$ which dominates the L^3 . Thus we do not have to remove relevant parts from $E(X)$ for large sets X such that $Y = L^{-1}\bar{X}$ satisfies $d_M(Y) > 0$.

6 RG with bounded fields

6.1 the first step

We now discuss the first step in some detail. It has the form

$$\tilde{\rho}_1(A_1, \Psi_1) = N_1 \int \delta(A_1 - \mathcal{Q}A_0) \delta(\tau A_0) \exp\left(-\frac{b}{L} [|\Psi_1 - Q(A_0)\Psi_0|^2 - \mathcal{S}_0(A_0, \Psi_0)]\right) D\Psi_0 D A_0 \quad (218)$$

where

$$\mathcal{S}_0(A_0, \Psi_0) = \frac{1}{2} \|dA_0\|^2 + \left\langle \bar{\Psi}_0, (\mathfrak{D}_{e_0}(A_0) + \bar{m}_0)\Psi_0 \right\rangle + m_0 \left\langle \bar{\Psi}_0, \Psi_0 \right\rangle + \varepsilon_0 \quad (219)$$

In $\rho_0(A_0)$ we have a factor $e^{-\frac{1}{2}\|dA_0\|^2}$ which is going to suppress large values of dA_0 . To translate this fact into hard bounds on dA_0 (which we have seen that we need) we again divide the torus into blocks of size $M = L^m$. Then we introduce the characteristic functions for each M -block \square by

$$\chi_0(\square, A_0) = \chi\left(\sup_{p \in \square} |dA_0(p)| \leq p(e_0)\right) \quad (220)$$

Here

$$p(e_0) = (\log e_0^{-1})^p \quad (221)$$

for some positive integer p . Since e_0 is tiny $p(e_0)$ is large, but not too large in the sense that $e_0 p(e_0)$ is still very small. Then with $\zeta_0(\square) = 1 - \chi_0(\square)$ we write

$$1 = \prod_{\square} \zeta_0(\square) + \chi_0(\square) = \sum_{\Omega} \prod_{\square \subset \Omega^c} \zeta_0(\square) \prod_{\square \subset \Omega} \chi_0(\square) \equiv \sum_{\Omega} \zeta_0(\Omega^c) \chi_0(\Omega) \quad (222)$$

where Ω is an arbitrary union of M -cubes. For the term Ω the characteristic function $\chi_0(\Omega)$ enforces that $|dA_0(p)| \leq p(e_0)$ everywhere in Ω . The characteristic function $\zeta_0(\Omega^c)$ enforces that in each M -cube \square in Ω^c there is at least one plaquette p such that $|dA_0(p)| > p(e_0)$.

We insert the identity under the integral sign in (218) and then the sum taken outside the integral. The integrand approximately factorizes and we have

$$\begin{aligned} \tilde{\rho}_1(A_1, \Psi_1) &= \sum_{\Omega} N_1 \int \delta_{\Omega_1^c}(A_1 - \mathcal{Q}A_0) \delta_{\Omega_1^c}(\tau A_0) \zeta_0(\Omega^c) \\ &\quad \exp\left(-\frac{b}{L} [|\Psi_1 - Q(A_0)\Psi_0|_{\Omega_1^c}^2 - \mathcal{S}_{0, \Omega_1^c}(A_0, \Psi_0)]\right) D\Psi_{0, \Omega_1^c} D A_{0, \Omega_1^c} \\ &\quad \left[\int \delta_{\Omega_1}(A_1 - \mathcal{Q}A_0) \delta_{\Omega_1}(\tau A_0) \chi_0(\Omega) \right. \\ &\quad \left. \exp\left(-\frac{b}{L} [|\Psi_1 - Q(A_0)\Psi_0|_{\Omega_1}^2 - \mathcal{S}_{0, \Omega_1}(A_0, \Psi_0)]\right) D\Psi_{0, \Omega_1} D A_{0, \Omega_1} \right] \end{aligned} \quad (223)$$

Of course this is not a perfect split. There are bonds and plaquettes that intersect both Ω and Ω^c . The convention is that Ω includes such bonds and plaquettes but Ω^c does not. The the integral over the Ω fields depends on fields in Ω^c near the boundary.

Most of our attention will be focused on the small field region Ω , but first consider the large field region Ω^c . A block \square in Ω^c has at least one plaquette with $|dA_0(p)| > p(e_0)$ and so

$$\zeta_0(\square)e^{-\frac{1}{4}\|dA_0\|_{\square}^2} \leq e^{-\frac{1}{4}p(e_0)^2}. \quad (224)$$

Note that for any positive integer n_0 and $p \geq 2$

$$e_0^{-n_0}e^{-\frac{1}{4}p(e_0)^2} = \exp\left(n_0(\log e_0^{-1}) - \frac{1}{4}(\log e_0^{-1})^p\right) \leq c_{n_0} \quad (225)$$

Thus $e^{-\frac{1}{4}p(e_0)^2} \leq c_{n_0}e_0^{n_0}$ for any n_0 ; it is a very small number. The bound (224) gives that

$$\zeta_0(\Omega^c)e^{-\frac{1}{4}\|dA_0\|_{\Omega^c}^2} \leq e^{-\frac{1}{4}p(e_0)^2|\Omega^c|_M} \quad (226)$$

where $|\Omega^c|_M$ is the number of M blocks in Ω^c . This is enough for the convergence of the sum over Ω for we have

$$\sum_{\Omega} e^{-\frac{1}{4}p(e_0)^2|\Omega^c|_M} = \prod_{\square} (1 + e^{-\frac{1}{4}p(e_0)^2}) \leq \prod_{\square} \exp(c_{n_0}e_0^{n_0}) \leq \exp\left(c_{n_0}e_0^{n_0}|\mathbb{T}_N^0|_M\right) \quad (227)$$

Note that $e_0^{n_0}|\mathbb{T}_N^0|_M$ is tiny uniformly in N since $e_0^{n_0} = L^{-n_0N/2}e^{n_0}$ beats $|\mathbb{T}_N^0|_M = L^{3N}M^{-3}$.

In subsequent steps we make similar splits in successively smaller regions $\Omega = \Omega_1 \supset \Omega_2 \supset \dots$ with the region Ω_k defined so that $|dA_k| \leq p(e_k)$. This gives tiny factors $e^{-\frac{1}{4}p(e_k)^2}$ in each block in Ω_k^c . As above the sum over Ω_k is bounded by $\exp(c_{n_0}e_k^{n_0}|\mathbb{T}_{N-k}^0|_M)$ which is tiny. A bound on on the contribution from all steps $1 \leq k \leq N$ is the exponential of

$$\sum_{k=1}^N c_{n_0}e_k^{n_0}|\mathbb{T}_{N-k}^0|_M \leq c_{n_0}M^{-3} \sum_{k=1}^N \left(L^{-(N-k)/2}e\right)^{n_0} L^{3(N-k)} \quad (228)$$

which is bounded uniformly in N for $n_0 \geq 7$.

A final remark. For technical reasons it is useful to replace M -blocks in these expansions by larger $M[r(e_k)]$ blocks where $r(e_k) = (-\log e_k)^r$ for some integer $r < p$. The above discussion still holds with this modification.

6.2 the general step

The first step is iterated many times. As noted at each step we need a fresh splitting of the integrated gauge field into large and small field regions. This can get rather complicated and for a first pass we consider the case where at each stage the small field region is the entire torus. This is the dominant contribution. Our discussion can also be taken over almost word for word to the analysis of the small field region when it is not the whole torus.

With this understanding suppose that after k transformations the (partial) density ρ_k has the form for $|d\mathcal{A}_k| \leq p(e_k)$ on the lattice \mathbb{T}_{N-k}^0

$$\begin{aligned} \rho_k(A_k, \Psi_k) = & \mathcal{N}_k Z_k \bar{Z}_k(0) \exp \left(-\frac{1}{2} \|d\mathcal{A}_k\|^2 - \mathfrak{S}_k(A_k, \Psi_k, \psi_k(A_k)) \right. \\ & \left. - \varepsilon_k \text{Vol}(\mathbb{T}_{N-k}) - m_k \langle \bar{\psi}_k(A_k), \psi_k(A_k) \rangle + E_k(A_k, \psi_k(A_k)) \right) \end{aligned} \quad (229)$$

with $E_k = \sum_X E_k(X)$.

We introduce characteristic functions in a different fashion than that suggested in the discussion of the first step and define

$$\begin{aligned} \tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) = \\ N_k \int \chi_k \hat{\chi}_k \delta(A_{k+1} - \mathcal{Q}A_k) \delta(\tau A_k) \exp \left(-\frac{b}{L} [|\Psi_{k+1} - Q(A_k)\Psi_k|^2] \right) \rho_k(A_k, \Psi_k) D\Psi_k D A_k \end{aligned} \quad (230)$$

Here characteristic function χ_k enforces globally

$$|d\tilde{\mathcal{A}}_{k+1}| \leq L^{-\frac{3}{2}} p(e_{k+1}) \quad p(e_k) = (\log e_k^{-1})^p \quad (231)$$

We have also inserted another characteristic function $\hat{\chi}_k$ which enforces

$$|A_k - A_k^{\min}| \leq p_0(e_k) \quad p_0(e_k) = \log(e_k^{-1})^{p_0} \quad (232)$$

with $p_0 < p$. Here $A_k^{\min} = H_k^x A_{k+1}$ is the minimizer of the quadratic form introduced previously.

When we translate to the minimum the factor $\exp(-\frac{1}{2}\|d\mathcal{A}_k\|^2)$ becomes the product $\exp(-\frac{1}{2}\|d\tilde{\mathcal{A}}_k\|^2) \exp(-\frac{1}{2}\|d\mathcal{Z}_k\|^2)$. The first factor facilitates the bound on $|d\tilde{\mathcal{A}}_{k+1}|$. The second factor, which is the same as $\exp(-\frac{1}{2} \langle Z_k, \Delta_k Z_k \rangle)$, together with the constraints, facilitates the bound on $Z_k = |A_k - A_k^{\min}|$.

Furthermore the bounds $|d\tilde{\mathcal{A}}_{k+1}| \leq L^{-\frac{3}{2}} p(e_{k+1})$ and $|Z_k| \leq p_0(e_k)$ give $|d\mathcal{Z}_k| \leq Cp_0(e_k)$ and hence

$$|d\mathcal{A}_k| \leq |d\tilde{\mathcal{A}}_{k+1}| + |d\mathcal{Z}_k| \leq L^{-\frac{3}{2}} p(e_{k+1}) + Cp_0(e_k) \leq p(e_k) \quad (233)$$

which is the bound discussed in the first step, and which we need for the expression for ρ_k to be well-defined.

The first main result is the following:

Theorem 1. *Let L be sufficiently large, let M be sufficiently large (depending on L), and let e_k be sufficiently small (depending on L, M). Suppose that $\rho_k(A_k, \Psi_k)$ has the representation (229) for $|d\mathcal{A}_k| \leq p(e_k)$. Suppose the polymer function $E_k(X, \mathcal{A}, \psi_k(\mathcal{A}))$ is defined and analytic in complex \mathcal{R}_k as defined in section 4.2, is gauge invariant, is charge conjugation invariant, and is invariant under all lattice symmetries. Suppose also that*

$$|m_k| \leq 1 \quad \|E_k\|_k \leq 1 \quad (234)$$

Then $\rho_{k+1}(A_{k+1}, \Psi_{k+1})$ defined by (230) has a representation of the same form for $|d\mathcal{A}_{k+1}| \leq p(e_{k+1})$, now with $e_{k+1} = L^{1/2}e_k$ and

$$\begin{aligned} \varepsilon_{k+1} &= L^3(\varepsilon_k + \varepsilon_k^*) \\ m_{k+1} &= L(m_k + m_k^*) \\ E_{k+1} &= \mathcal{L}(\mathcal{R}E_k + E_k^*) \end{aligned} \quad (235)$$

Here ε_k^, m_k^* are linear in E_k , and E_k^* is a function of e_k, m_k, E_k . They satisfy*

$$|\varepsilon_k^*| \leq \mathcal{O}(1) \quad |m_k^*| \leq \mathcal{O}(e_k^{1/2}) \quad \|E_k^*\|_{k+1} \leq \mathcal{O}(e_k^{1/4}) \quad (236)$$

When we iterate this result all these quantities are going to grow. The main issue is that they do not grow too quickly. Things are arranged so the polymer functions stay bounded. The point is that $\mathcal{R}E_k$ has the relevant parts removed so when we scale with the operator \mathcal{L} we get shrinkage. The E_k^* provides a fresh small term and $\|E_{k+1}\|_{k+1} < 1$. Control of the growth of ε_k^* and particularly m_k^* is not assured and this is where renormalization enters, a topic still to come.

Note that the assumption $|d\mathcal{A}_k| \leq p(e_k)$ easily implies $\mathcal{A}_k \in \mathcal{R}_k$ by lemma 5.

We now sketch the proof of the theorem.

Proof. [27]

Step I: extraction We remove the relevant parts of E_k defining $\mathcal{R}E_k$ by the identity

$$E_k(X, \mathcal{A}, \psi) = \alpha_0(X)\text{Vol}(X) + \langle \bar{\psi}, \alpha_2(X)\psi \rangle + (\mathcal{R}E_k)(X, \mathcal{A}, \psi) \quad (237)$$

where

$$\begin{aligned} \alpha_0(X) &= \frac{1}{\text{Vol}(X)} E_{k,0}(X, 0) \\ \alpha_2(X) &= \frac{1}{\text{Vol}(X)} \sum_{x,y \in X} L^{-6k} E_{k,2}(X, 0; x, y) \end{aligned} \quad (238)$$

Then we sum over X and get

$$E_k(\mathcal{A}, \psi) = -\varepsilon_k^* \text{Vol}(\mathbb{T}_{N-k}^0) - m_k^* \langle \bar{\psi}, \psi \rangle + (\mathcal{R}E_k)(\mathcal{A}, \psi) \quad (239)$$

where

$$\begin{aligned}\varepsilon_k^* &= - \sum_{X \supset \square} \alpha_0(X) \\ m_k^* &= - \sum_{X \supset \square} \alpha_2(X)\end{aligned}\tag{240}$$

Here invariance under lattice symmetries is important to insure that ε_k^*, m_k^* are independent of \square and are Dirac scalars. One can show

$$\begin{aligned}|\varepsilon_k^*| &\leq \mathcal{O}(1) \|E_k\|_k \\ |m_k^*| &\leq \mathcal{O}(1) e_k^{\frac{1}{2}} \|E_k\|_k \\ \|\mathcal{R}E_k\|_k &\leq \mathcal{O}(1) \|E_k\|_k\end{aligned}\tag{241}$$

here with $\|E_k\|_k \leq 1$.

Step II: gauge field translation. Now we expand around the minimum of $\|d\mathcal{A}\|^2$ as constrained by $A_{k+1} = \mathcal{Q}A_k$ and $\tau A_k = 0$. As explained earlier the minimum is $A_k^{\min} = H_k^\times A_{k+1}$ and we write $A_k = H_k^\times A_{k+1} + Z_k$. We take advantage of the fact that the minimizer \mathcal{H}_k^\times and the covariant minimizer \mathcal{H}_k are gauge equivalent to evaluate $\mathcal{A}_k = \mathcal{H}_k A_{k+1}$ as

$$\begin{aligned}\mathcal{A}_k &= \mathcal{H}_k H_k^\times A_{k+1} + \mathcal{H}_k Z_k \\ &\sim \mathcal{H}_k^\times H_k^\times A_{k+1} + \mathcal{H}_k Z_k \\ &= \tilde{\mathcal{A}}_{k+1}^\times + \mathcal{H}_k Z_k \\ &\sim \tilde{\mathcal{A}}_{k+1} + \mathcal{Z}_k\end{aligned}\tag{242}$$

where $\mathcal{Z}_k = \mathcal{H}_k Z_k$. In the gauge invariant effective action we make the replacement $\mathcal{A}_k = \tilde{\mathcal{A}}_{k+1} + \mathcal{Z}_k$ and integrate over Z_k instead of A_k . Again we have the split

$$\frac{1}{2} \|d\mathcal{A}_k\|^2 = \frac{1}{2} \|d\tilde{\mathcal{A}}_{k+1}\|^2 + \frac{1}{2} \|d\mathcal{Z}_k\|^2 = \frac{1}{2} \|d\tilde{\mathcal{A}}_{k+1}\|^2 + \frac{1}{2} \langle Z_k, \Delta_k Z_k \rangle\tag{243}$$

where $\Delta_k = \mathcal{H}_k^T \delta d\mathcal{H}_k$. We also have with $\mathcal{A} = \tilde{\mathcal{A}}_{k+1}$

$$\begin{aligned}\mathfrak{S}_k(\mathcal{A} + \mathcal{Z}_k, \Psi_k, \psi_k(\mathcal{A} + \mathcal{Z}_k)) &= \mathfrak{S}_k(\mathcal{A}, \Psi_k, \psi_k(\mathcal{A})) + [\dots] \\ \langle \bar{\psi}_k(\mathcal{A} + \mathcal{Z}), \psi_k(\mathcal{A} + \mathcal{Z}) \rangle &= \langle \bar{\psi}_k(\mathcal{A}), \psi_k(\mathcal{A}) \rangle + [\dots] \\ \mathcal{R}E_k(\mathcal{A} + \mathcal{Z}_k, \psi_k(\mathcal{A} + \mathcal{Z}_k)) &= \mathcal{R}E_k(\mathcal{A}, \psi_k(\mathcal{A})) + [\dots]\end{aligned}\tag{244}$$

The quantities $[\dots]$ are collected into a function $E'(\mathcal{A}, \psi_k(\mathcal{A}), \mathcal{Z}_k)$ which has a polymer expansion but not yet in the variables we want.

As noted in section 2.3 the integral over Z_k is Gaussian and parametrized by fields \tilde{Z}_k by

$$\int f(Z_k) \delta(\mathcal{Q}Z_k) \delta(\tau Z_k) \exp\left(-\frac{1}{2} \langle Z_k, \Delta_k Z_k \rangle\right) dZ_k = \int f(C\tilde{Z}_k) d\mu_{C_k}(\tilde{Z}_k) \quad (245)$$

where C maps to the subspace $\mathcal{Q}Z_k = 0, \tau Z_k = 0$ and $C_k = (C^T \Delta_k C)^{-1}$. We make a further change to an identity covariance by

$$\int f(C\tilde{Z}_k) d\mu_{C_k}(\tilde{Z}_k) = \int f(CC_k^{\frac{1}{2}} \tilde{Z}_k) d\mu_I(\tilde{Z}_k) \quad (246)$$

After all these changes we now have with $m'_k = m_k + m_k^*$ and $\varepsilon'_k = \varepsilon_k + \varepsilon_k^*$ and $\mathcal{Z}'_k = \mathcal{H}_k C C_k^{\frac{1}{2}} \tilde{Z}_k$

$$\begin{aligned} \tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) &= (Z_k \delta Z_k) (\mathcal{N}_k N_k Z_k(0)) \exp\left(-\frac{1}{2} \|d\mathcal{A}\|^2\right) \\ &\int D\Psi_k \exp\left(-\frac{b}{L} [|\Psi_{k+1} - Q(\mathcal{A}_{k+1})\Psi_k|^2 - \mathfrak{S}_k(\mathcal{A}, \Psi_k, \psi_k(\mathcal{A}))\right. \\ &\quad \left. - \varepsilon'_k \text{Vol}(\mathbb{T}_{N-k}) - m'_k \langle \bar{\psi}_k(\mathcal{A}), \psi_k(\mathcal{A}) \rangle + \mathcal{R}E_k(\mathcal{A}, \psi_k(\mathcal{A}))\right) \\ &\cdot \int d\mu_I(\tilde{Z}_k) \hat{\chi}_k(CC_k^{\frac{1}{2}} \tilde{Z}_k) \exp\left(E'_k(\mathcal{A}, \psi_k(\mathcal{A}), \mathcal{Z}'_k)\right) \Big|_{\mathcal{A}=\tilde{\mathcal{A}}_{k+1}} \end{aligned} \quad (247)$$

Step III: localization. A major issue going forward is keeping the interaction part of the effective action localized, i.e. expressed as a polymer expansion. This was compromised when we expanded around the minimizers and we have to make adjustments. The problem occurs in the expression E'_k . There are three contributions to E'_k in (244), but to illustrate the remedy we only consider the first. So we consider

$$E'_k(\mathcal{A}, \mathcal{Z}'_k, \Psi_k,) = \mathfrak{S}_k(\mathcal{A} + \mathcal{Z}'_k, \Psi_k, \psi_k(\mathcal{A} + \mathcal{Z}'_k)) - \mathfrak{S}_k(\mathcal{A}, \Psi_k, \psi_k(\mathcal{A})) \quad (248)$$

This can also be written as

$$\begin{aligned} E'_k(\mathcal{A}, \mathcal{Z}'_k, \Psi_k) &= \left\langle \bar{\Psi}_k, \left(D_k(\mathcal{A} + \mathcal{Z}'_k) - D_k(\mathcal{A})\right) \Psi_k \right\rangle \\ &= \left\langle \bar{\Psi}_k, \left(\hat{D}_k(\mathcal{A} + \mathcal{Z}'_k) - \hat{D}_k(\mathcal{A})\right) \Psi_k \right\rangle \end{aligned} \quad (249)$$

where

$$\hat{D}(\mathcal{A}) = -b_k^2 Q_k(\mathcal{A}) S_k(\mathcal{A}) Q_k^T(-\mathcal{A}) \quad (250)$$

omits the constant term in $D(\mathcal{A})$.

We first localize $\left\langle \bar{\Psi}_k, \hat{D}_k(\mathcal{A}) \Psi_k \right\rangle$ in \mathcal{A} . Here we use the random walk expansion $S_k(\mathcal{A}) = \sum_{\omega} S_{k,\omega}(\mathcal{A})$ for the Green's function. Every path $\omega = (\square_0, \square_1, \dots, \square_n)$ determines a polymer X_{ω} which is the union the $3M$ -blocks in ω . Then we can write

$$S_k(\mathcal{A}) = \sum_X \sum_{\omega: X_{\omega}=X} S_{k,\omega}(\mathcal{A}) \equiv \sum_X S_k(X, \mathcal{A}) \quad (251)$$

and we have the estimate from the expansion $\|S_k(X, \mathcal{A})f\| \leq Ce^{-\kappa d_M(X)}\|f\|_\infty$. This gives a local expansion

$$\left\langle \bar{\Psi}_k, \hat{D}_k(\mathcal{A})\Psi_k \right\rangle = \sum_X \left\langle \bar{\Psi}_k, \hat{D}_k(X, \mathcal{A})\Psi_k \right\rangle \quad (252)$$

The $\left\langle \bar{\Psi}_k, \hat{D}_k(X, \mathcal{A})\Psi_k \right\rangle$ only depends on \mathcal{A}, Ψ_k in X and satisfies

$$\left\| \left\langle \bar{\Psi}_k, \hat{D}_k(X, \mathcal{A})\Psi_k \right\rangle \right\|_{h_k} \leq Ch_k^2 e^{-\kappa d_M(X)} \quad (253)$$

This also localizes the difference. We have $E'_k = \sum_X E_k^0(X)$ where.

$$E_k^0(X, \mathcal{A}, \mathcal{Z}'_k, \Psi_k) = \left\langle \bar{\Psi}_k, \left(\hat{D}_k(X, \mathcal{A} + \mathcal{Z}'_k) - \hat{D}_k(X, \mathcal{A}) \right) \Psi_k \right\rangle \quad (254)$$

We also need to know that this is small. Consider the operator $D_k(X, \mathcal{A} + t\mathcal{Z}'_k)$ for t complex. From the bound $|CC_k^{\frac{1}{2}}\tilde{Z}_k| \leq p_0(e_k)$ from $\hat{\chi}_k$ and our bound on \mathcal{H}_k we have that $\mathcal{Z}'_k = \mathcal{H}_k CC_k^{\frac{1}{2}}\tilde{Z}_k = \mathcal{O}(p_0(e_k))$ is logarithmic in e_k . But for $\mathcal{A} \in \mathcal{R}_k$ the $D_k(X, \mathcal{A})$ can accept complex fields as large as $e_k^{-\frac{3}{4}}$. Then $\hat{D}_k(X, \mathcal{A} + t\mathcal{Z}'_k)$ is analytic in say complex $|t| \leq e_k^{-\frac{3}{4}+\epsilon}$ and we can write

$$E_k^0(X, \mathcal{A}, \mathcal{Z}'_k, \Psi_k) = \frac{1}{2\pi i} \int_{|t|=e_k^{-\frac{3}{4}+\epsilon}} \frac{dt}{t(t-1)} \left\langle \bar{\Psi}_k, \hat{D}_k(X, \mathcal{A} + t\mathcal{Z}'_k)\Psi_k \right\rangle \quad (255)$$

The $\left\langle \bar{\Psi}_k, \hat{D}_k(X, \mathcal{A} + t\mathcal{Z}'_k)\Psi_k \right\rangle$ satisfies the bound (253) and we get

$$\|E_k^0(X, \mathcal{A}, \mathcal{Z}'_k, \Psi_k)\|_{h_k} \leq Ch_k^2 e_k^{\frac{3}{4}-\epsilon} e^{-\kappa d_M(X)} = Ce_k^{\frac{1}{4}-\epsilon} e^{-\kappa d_M(X)} \quad (256)$$

This is not the end of the story. We have localized in $\mathcal{Z}'_k = \mathcal{H}_k CC_k^{\frac{1}{2}}\tilde{Z}_k \equiv \mathcal{H}'_k\tilde{Z}_k$. But for the fluctuation integral we want to localize in \tilde{Z}_k .

Here we again need our random walk expansions. Recall that the gauge propagator has the expansion $\mathcal{G}_k = \sum_\omega \mathcal{G}_{k,\omega}$ where $\omega = (\square_0, \square_1, \square_2, \dots)$. We separate off the paths consisting of a single block \square_0 and for the rest introduce a weakening parameter $0 \leq s_\square \leq 1$. Then we define

$$\mathcal{G}_k(s) = \sum_{\square_0} h_{\square_0} \mathcal{G}_k h_{\square_0} + \sum_{\omega: |\omega| \geq 1} s_\omega \mathcal{G}_{k,\omega} \quad (257)$$

where

$$s_\omega = \prod_{\square \subset X_\omega} s_\square \quad (258)$$

This again satisfies the bounds (186). But recall that each block \square in ω contributes a factor M^{-1} to the convergence. This is more than we need and one could in fact let s_\square be complex and satisfy say $|s_\square| \leq M^{\frac{1}{2}}$ and still have convergence.

With more work the same remarks hold for $\mathcal{H}'_k = \mathcal{H}_k CC_k^{\frac{1}{2}}$. We have a random walk expansion $\mathcal{H}'_k = \sum_\omega \mathcal{H}'_{k,\omega}$ and again we weaken it introducing $\mathcal{H}'_k(s) = \mathcal{H}_k(s) CC_k^{\frac{1}{2}}(s)$ as above. Then we have

$$E_k^0(X, \mathcal{A}, \mathcal{Z}'_k, \Psi_k) = \left[E_k^0(X, \mathcal{A}, \mathcal{H}'_k(s) \tilde{Z}_k, \Psi_k) \right]_{s_\square=1} \quad (259)$$

Now in each block \square in X^c we interpolate between $s_\square = 1$ and $s_\square = 0$ by

$$f(s_\square = 1) = f(s_\square = 0) + \int_0^1 \frac{\partial}{\partial s_\square} f(s_\square) \quad (260)$$

This yields

$$E_k^0(X, \mathcal{A}, \mathcal{Z}'_k, \Psi_k) = \sum_{Y \supset X} \int ds_{Y-X} \frac{\partial}{\partial s_{Y-X}} \left[E_k^0(X, \mathcal{A}, \mathcal{H}'_k(s) \tilde{Z}_k, \Psi_k) \right]_{s_{Y^c}=0, s_X=1} \quad (261)$$

where $\int ds_Z = \prod_{\square \subset Z} \int_0^1 ds_\square$ and $\partial/\partial s_X = \prod_{\square \subset X} \partial/\partial s_\square$. But with $s_{Y^c} = 0, s_X = 1$ the operator $\mathcal{H}'(s)$ has no coupling between the connected components of Y . If the connected components are $Y_0 \supset X$ and $\{Y_\beta\}$ then we have

$$\mathcal{H}'_k(s) = \mathcal{H}'_k^{Y_0}(s) \oplus \left(\oplus_\beta \mathcal{H}'_k^{Y_\beta}(s) \right) \quad (262)$$

Only the first term contributes since for the others we have $Y_\beta \cap X = \emptyset$, and the derivative is

$$\frac{\partial}{\partial s_{Y_0-X}} \prod_\beta \frac{\partial}{\partial s_{Y_\beta}} \left[E_k^0(X, \mathcal{A}, \mathcal{H}'_k^{Y_0}(s) \tilde{Z}_k, \Psi_k) \right]_{s_{Y^c}=0, s_X=1} \quad (263)$$

which vanishes unless there are no Y_β . Thus in the sum we can restrict to connected $Y \supset X$.

Now in $E' = \sum_X E_k^0(X)$ insert this sum and change the order of summation, Then we have $E'_k = \sum_Y E'_k(Y)$ with

$$E'_k(Y, \mathcal{A}, \tilde{Z}_k, \Psi_k) = \sum_{X \subset Y} \int ds_{Y-X} \frac{\partial}{\partial s_{Y-X}} \left[E_k^0(X, \mathcal{A}, \mathcal{H}'_k(s) \tilde{Z}_k, \Psi_k) \right]_{s_{Y^c}=0, s_X=1} \quad (264)$$

This accomplishes the localization as $E'_k(Y)$ only depends on \tilde{Z}_k in Y .

As noted $\mathcal{H}_k(s)$ is analytic in $|s_\square| \leq M^{\frac{1}{2}}$. So for $|s_\square| \leq 1$ each derivative gains a factor $M^{-\frac{1}{2}} \leq e^{-\kappa}$ by Cauchy inequalities. This leads to the estimate

$$\|E'_k(Y)\|_{h_k} \leq C e_k^{\frac{1}{4}-\epsilon} \sum_{X \subset Y} \exp \left(-\kappa |Y - X|_M - \kappa d_M(X) \right) \quad (265)$$

But $|Y - X|_M + d_M(X) \geq d_M(Y)$ so we can extract a factor $e^{-(\kappa - \kappa_0)d_M(Y)}$ and still have $e^{-\kappa_0 d_M(X)}$ left for the convergence of the sum over X . The sum over X is estimated as in (215) by

$$\sum_{X \subset Y} e^{-\kappa_0 d_M(X)} \leq \mathcal{O}(1)|Y|_M \leq \mathcal{O}(1)e^{d_M(Y)} \quad (266)$$

This gives the estimate

$$\|E'_k(Y)\|_{h_k} \leq C e_k^{\frac{1}{4} - \epsilon} e^{-(\kappa - \kappa_0 - 1)d_M(Y)} \quad (267)$$

The same bound holds for the other contributions to E'_k , but with $\frac{1}{2}h_k$ instead of h_k . We also note that there is a local linear operator $T_k(\mathcal{A})$ so that $\Psi_k = T_k(\mathcal{A})\psi_k(\mathcal{A})$ and so we can write our function as

$$E'_k(\mathcal{A}, \tilde{Z}_k, \psi_k(\mathcal{A})) = \sum_X E'_k(X, \mathcal{A}, \tilde{Z}_k, \psi_k(\mathcal{A})) \quad (268)$$

Remark. There is also a localization problem with the characteristic function $\hat{\chi}_k(CC_k^{\frac{1}{2}}\tilde{Z}_k)$. One might try to localize it by introducing weakened versions $\hat{\chi}_k(CC_k^{\frac{1}{2}}(s)\tilde{Z}_k)$ and expanding around $s_\square = 1$ as above. However estimates on the construction break down. One cannot use Cauchy inequalities since $\hat{\chi}$ is not analytic.

This problem is dealt with in different ways in [27] and [29]. The method of [29] seems more general and is probably preferable. It involves introducing a local version $C_k^{\frac{1}{2}, \text{loc}}$ of $C_k^{\frac{1}{2}}$ with a small error. Then the characteristic function $\hat{\chi}_k(CC_k^{\frac{1}{2}, \text{loc}}\tilde{Z}_k)$ stays local. This function behaves like $\hat{\chi}(\tilde{Z}_k)$ and we simplify things by replacing it by $\hat{\chi}(\tilde{Z}_k)$. This is not really a cheat since in the full expansion we can arrange that it is actually $\hat{\chi}(\tilde{Z}_k)$ that appears, at the cost of introducing another large field/small field split.

Step IV: fermion translation. Now in the expression (247) we expand around the critical point in fermions by $\Psi_k = H_k(\mathcal{A})\Psi_{k+1} + W_k$ and hence $\psi_k(\mathcal{A}) = \tilde{\psi}_{k+1}(\mathcal{A}) + \mathcal{H}_k(\mathcal{A})W_k$. As before the quadratic form splits

$$\tilde{\mathfrak{S}}_{k+1}(\mathcal{A}, \Psi_{k+1}, \tilde{\psi}_{k+1}(\mathcal{A})) + \left\langle \bar{W}_k, \left(D_k(\mathcal{A}) + \frac{b}{L}Q^T(-\mathcal{A})Q(\mathcal{A}) \right) W_k \right\rangle \quad (269)$$

and we identify the fluctuation integral as Gaussian integral with covariance

$$\Gamma_k(\mathcal{A}) = \left(D_k(\mathcal{A}) + \frac{b}{L}Q^T(-\mathcal{A})Q(\mathcal{A}) \right)^{-1} \quad (270)$$

and the formal measure is

$$\exp \left(- \left\langle \bar{W}_k, \Gamma_k(\mathcal{A})^{-1}W_k \right\rangle \right) DW_k = \delta Z_k(\mathcal{A}) d\mu_{\Gamma_k(\mathcal{A})}(W_k) \quad (271)$$

We also use the identity

$$\int f(\bar{W}_k, W_k) d\mu_{\Gamma_k(\mathcal{A})}(W_k) = \int f(\bar{W}_k, \Gamma_k(\mathcal{A})W_k) d\mu_I(W_k) \quad (272)$$

The translation now becomes

$$\begin{aligned} \psi_k(\mathcal{A}) &= \tilde{\psi}_{k+1}(\mathcal{A}) + \mathcal{H}_k(\mathcal{A})\Gamma_k(\mathcal{A})W_k \\ \bar{\psi}_k(\mathcal{A}) &= \overline{\tilde{\psi}_{k+1}(\mathcal{A})} + \mathcal{H}_k(\mathcal{A})\bar{W}_k \end{aligned} \quad (273)$$

and we summarize both as $\psi_k(\mathcal{A}) = \tilde{\psi}_{k+1}(\mathcal{A}) + \mathcal{H}'_k(\mathcal{A})W_k$ where $\mathcal{H}'_k(\mathcal{A})$ is either $\mathcal{H}_k(\mathcal{A})\Gamma_k(\mathcal{A})$ or $\mathcal{H}_k(\mathcal{A})$.

Under this translation we get $E'_k(\mathcal{A}, Z_k, \tilde{\psi}_{k+1}(\mathcal{A}) + \mathcal{H}'_k(\mathcal{A})W_k)$ and similarly for some other terms in the action. These are no longer well localized in W_k . We again localize by introducing weakened random walk expansions $\mathcal{H}'(s, \mathcal{A})$ for $\mathcal{H}'_k(\mathcal{A})$, and expand around $s_\square = 1$. The result is that E' becomes E'' with

$$E''_k(\mathcal{A}, Z_k, \tilde{\psi}_{k+1}(\mathcal{A}), W_k) = \sum_X E''_k(X, \mathcal{A}, \tilde{Z}_k, \tilde{\psi}_{k+1}(\mathcal{A}), W_k) \quad (274)$$

with $E''(X)$ localized in the indicated fields. It has a bound for a new smaller κ'

$$\|E''_k(Y, \mathcal{A}, \tilde{Z}_k)\|_{\frac{1}{4}h_k, \frac{1}{4}h_k} \leq C e_k^{\frac{1}{4}-\epsilon} e^{-\kappa' d_M(Y)} \quad (275)$$

After these changes we have

$$\begin{aligned} \tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) &= \left(Z_k \delta Z_k \right) \left(\mathcal{N}_k N_k Z_k(0) \delta Z_k(\mathcal{A}) \right) \exp \left(-\frac{1}{2} \|d\mathcal{A}\|^2 \right) \\ &\exp \left(-\tilde{\mathfrak{S}}_{k+1}(\mathcal{A}, \Psi_{k+1}, \tilde{\psi}_{k+1}(\mathcal{A})) - \varepsilon'_k \text{Vol}(\mathbb{T}_{N-k}) - m'_k \left\langle \overline{\tilde{\psi}_{k+1}(\mathcal{A})}, \tilde{\psi}_{k+1}(\mathcal{A}) \right\rangle \right) \\ &\exp \left(\mathcal{R}E_k(\mathcal{A}, \tilde{\psi}_{k+1}(\mathcal{A})) \Xi_k(\mathcal{A}, \tilde{\psi}_{k+1}(\mathcal{A})) \right) \Big|_{\mathcal{A}=\tilde{A}_{k+1}} \end{aligned} \quad (276)$$

where the fluctuation integral is now

$$\Xi_k(\mathcal{A}, \tilde{\psi}_{k+1}(\mathcal{A})) = \int \exp \left(\sum_X E''_k(X, \mathcal{A}, \tilde{Z}_k, \tilde{\psi}_{k+1}(\mathcal{A}), W_k) \right) \hat{\chi}(\tilde{Z}_k) d\mu_I(\tilde{Z}_k) d\mu_I(W_k) \quad (277)$$

Step V: cluster expansion. The fluctuation integral is estimated by a cluster expansion. We have set it up as an ultra-local version of this standard technique. The Gaussian integrals are strictly local since the covariance is the identity. The integrand is local since it is expressed as a polymer expansion in the field \tilde{Z}_k, W_k . Furthermore the fluctuation gauge field is bounded by $|\tilde{Z}_k| \leq p_0(e_k)$. Things are as nice as they could be and the cluster expansion is essentially a combinatoric problem. The result is that there are polymer functions $E_k^\# = \sum_X E_k^\#(X)$ such that

$$\Xi_k(\mathcal{A}, \tilde{\psi}_{k+1}(\mathcal{A})) = \exp \left(\sum_X E_k^\#(X, \mathcal{A}, \tilde{\psi}_{k+1}(\mathcal{A})) \right) \quad (278)$$

and we still have the bound with a new smaller $\kappa' < \kappa$

$$\|E_k^\#(Y, \mathcal{A})\|_{\frac{1}{4}h_k} \leq C e_k^{\frac{1}{4}-\epsilon} e^{-\kappa' d_M(Y)} \quad (279)$$

For details see for example [22].

Step VI: determinants. We still have to deal with the \mathcal{A} dependence in

$$\delta Z_k(\mathcal{A}) = \det \Gamma_k(\mathcal{A})^{-1} = \det \left(D_k(\mathcal{A}) + bL^{-1}Q^T(-\mathcal{A})Q(\mathcal{A}) \right) \quad (280)$$

We use the following result [16], [27]. Let T be an invertible self-adjoint matrix, and consider $\det T = \exp(\text{Tr} \log T)$. We take the branch of the logarithm with the cut on the negative imaginary axis so the log of negative numbers is defined. Then

$$\log T = \frac{1}{2\pi i} \int_{\Gamma} dz \log z (z - T)^{-1} \quad (281)$$

where Γ is any closed contour which avoids the negative imaginary axis and encloses the spectrum of T , a finite non zero subset of the real axis. For the contour take an inner circle of small radius r , and outer circle of large radius R and join them with rays which make an angle ϵ with the negative imaginary axis. See figure 2.

For any R_0 we can assume $r < R_0 < R$. If we take the limits $\epsilon \rightarrow 0$, $R \rightarrow \infty$, $r \rightarrow 0$ we find

$$\log T = T \int_{R_0}^{\infty} \frac{dy}{y} (T + iy)^{-1} - i \int_0^{R_0} dy (T + iy)^{-1} + \log R_0 + \frac{i\pi}{2} \quad (282)$$

Now the Dirac operator $\mathfrak{D}_{\mathcal{A}}$ is not self-adjoint and neither are the operators $S_k(\mathcal{A}) = \left(\mathfrak{D}_{\mathcal{A}} + \bar{m} + b_k P_k(\mathcal{A}) \right)^{-1}$ or $D_k(\mathcal{A}) = b_k - b_k^2 Q_k(\mathcal{A}) S_k(\mathcal{A}) Q_k^T(-\mathcal{A})$. But we can suppose that our representation of the Dirac matrices $\gamma_0, \gamma_1, \gamma_2$ in three dimensions is obtained from a collection $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ in four dimensions. Since γ_3 anti-commutes with the others $\mathfrak{D}_{\mathcal{A}} \gamma_3$ is self-adjoint as are $S_k(\mathcal{A}) \gamma_3$ and $D_k(\mathcal{A}) \gamma_3$. This does not change the determinant and we can take.

$$\delta Z_k(\mathcal{A}) = \det \left(\left(D_k(\mathcal{A}) + bL^{-1}Q^T(-\mathcal{A})Q(\mathcal{A}) \right) \gamma_3 \right) \quad (283)$$

Now apply the formula (282). It turns out one can take the limit $R_0 \rightarrow \infty$ and one finds for some explicit constant c_k

$$\delta Z_k(\mathcal{A}) = c_k \exp \left(-i\gamma_3 b_k^2 \int_0^{\infty} \text{Tr} \left[\mathbf{B}_{k,y}(\mathcal{A}) Q_k(\mathcal{A}) S_{k,y}(\mathcal{A}) Q_k^T(-\mathcal{A}) \mathbf{B}_{k,y}(\mathcal{A}) \right] dy \right) \quad (284)$$

Here $\mathbf{B}_{k,y}(\mathcal{A})$ is local operator which is a linear combination of the identity I and $Q^T(-\mathcal{A})Q(\mathcal{A})$ and is $\mathcal{O}(y^{-1})$ as $y \rightarrow \infty$. The $S_{k,y}(\mathcal{A})$ interpolates between $S_k(\mathcal{A})$ at $y = \infty$ and $\tilde{S}_{k+1}(\mathcal{A})$ at $y = 0$. ($\tilde{S}_{k+1}(\mathcal{A})$ scales to $S_{k+1}(\mathcal{A})$).

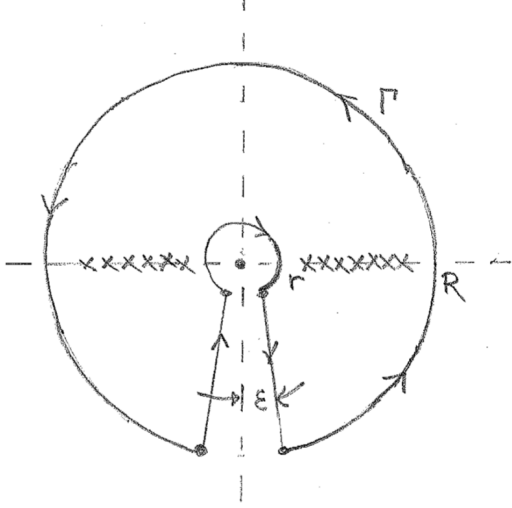


Figure 2: The contour Γ

Now as for $S_k(\mathcal{A})$ we have a polymer expansion $S_{k,y}(\mathcal{A}) = \sum_X S_{k,y}(X, \mathcal{A})$. The other terms under the trace are local, so this generates an expansion

$$\delta Z_k(\mathcal{A}) = c_k \exp \left(\sum_X E^d(X, \mathcal{A}) \right) \quad (285)$$

where $E^d(X, \mathcal{A})$ is gauge invariant and satisfies $|E^d(X, \mathcal{A})| \leq CM^3 e^{-\kappa d_M(X)}$.

Next define $E^{\det}(X, \mathcal{A}) = E^d(X, \mathcal{A}) - E^d(X, 0)$ and we have

$$Z_k(\mathcal{A}) = Z_k(0) \exp \left(\sum_X E^{\det}(X, \mathcal{A}) \right) \quad (286)$$

We are assuming $\mathcal{A} \in \mathcal{R}_k$ but in fact $E^d(X, \mathcal{A})$ is defined and analytic in the larger domain $e_k^{-\frac{1}{4}+\epsilon} \mathcal{R}_k$. Thus we can write

$$E^{\det}(X, \mathcal{A}) = \frac{1}{2\pi i} \int_{|t|=e_k^{-\frac{1}{4}+\epsilon}} \frac{dt}{t(t-1)} E_k^d(X, t\mathcal{A}) \quad (287)$$

and obtain the bound

$$|E_k^{\det}(X, \mathcal{A})| \leq e_k^{\frac{1}{4}-\epsilon} e^{-\kappa' d_M(X)} \quad (288)$$

Step VII: scaling We define $E_k^* = E_k^\# + E_k^{\det}$. This inherits a polymer expansion with

$$\|E_k^*(X, \mathcal{A}, \tilde{\psi}_{k+1}(\mathcal{A}))\|_{\frac{1}{4}h_k} \leq C e_k^{\frac{1}{4}} e^{-\kappa' d_M(X)} \quad (289)$$

Now we have

$$\begin{aligned} \tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) &= (Z_k \delta Z_k) (\mathcal{N}_k N_k Z_k(0) \delta Z_k(0)) \exp \left(-\frac{1}{2} \|d\mathcal{A}\|^2 - \tilde{\mathfrak{G}}_{k+1}(\mathcal{A}, \Psi_{k+1}, \tilde{\psi}_{k+1}(\mathcal{A})) \right) \\ &\exp \left(-\varepsilon'_k \text{Vol}(\mathbb{T}_{N-k}) - m'_k \left\langle \overline{\tilde{\psi}_{k+1}}(\mathcal{A}), \tilde{\psi}_{k+1}(\mathcal{A}) \right\rangle + (\mathcal{R}E_k + E_k^*)(X, \mathcal{A}, \tilde{\psi}_{k+1}(\mathcal{A})) \right) \Big|_{\mathcal{A}=\tilde{A}_{k+1}} \end{aligned} \quad (290)$$

This scales to the desired expression for ρ_{k+1} . In particular $\varepsilon'_k = \varepsilon_k + \varepsilon_k^*$ scales to $\varepsilon_{k+1} = L^3(\varepsilon_k + \varepsilon_k^*)$ and $m'_k = m_k + m_k^*$ scales to $m_{k+1} = L(m_k + m_k^*)$. Furthermore $\mathcal{R}E_k + E_k^*$ scales to $E_{k+1} = \mathcal{L}(\mathcal{R}E_k + E_k^*)$. This completes the proof of theorem 1.

Note: Let us check that the polymer function is not growing from its initial $\|E_k\|_k \leq 1$. We have from (217) and (289)

$$\begin{aligned} \|\mathcal{L}E_k^*\|_{\mathcal{R}_{k+1}, h_{k+1}\kappa} &\leq \mathcal{O}(1)L^3 \|E_k^*\|_{\mathcal{R}_k, \frac{1}{4}h_k, \kappa_0 + 2L^{-1}\kappa} \\ &\leq \mathcal{O}(1)L^3 \|E_k^*\|_{\mathcal{R}_k, \frac{1}{4}h_k, \kappa'} \\ &\leq CL^3 e_k^{\frac{1}{4}} < \frac{1}{2} \end{aligned} \quad (291)$$

Here we have used $\kappa_0 + 2L^{-1}\kappa \leq \kappa'$. The κ' has been shrinking throughtout the proof, but if you trace it through you find that you can take $\kappa' = \kappa - 10\kappa_0 - 10$. Then the inequality says $11\kappa_0 + 10 \leq (1 - 2L^{-1})\kappa$ which holds if κ is a sufficiently large multiple of κ_0 .

We also have from (217) and $\|\mathcal{R}E_k\|_k \leq \mathcal{O}(1)\|E_k\|_k$

$$\begin{aligned} \|\mathcal{L}\mathcal{R}E_k\|_{\mathcal{R}_{k+1}, h_{k+1}\kappa} &\leq \mathcal{O}(1)L^3 \|\mathcal{R}E_k\|_{L^{-\frac{7}{8}}\mathcal{R}_k, L^{-\frac{9}{8}}h_k, \kappa} \\ &\leq \mathcal{O}(1)L^{-1/4} \|\mathcal{R}E_k\|_k \\ &\leq \mathcal{O}(1)L^{-1/4} \|E_k\|_k < \frac{1}{2} \end{aligned} \quad (292)$$

Here we have used the fact that $\mathcal{R}E_k$ has relevant terms removed to get the improved scaling

$$\|\mathcal{R}E_k\|_{L^{-\frac{7}{8}}\mathcal{R}_k, L^{-\frac{9}{8}}h_k, \kappa} \leq L^{-\frac{13}{4}} \|\mathcal{R}E_k\|_{\mathcal{R}_k, h_k, \kappa} \quad (293)$$

To give the idea for this bound consider that constants and local $\bar{\psi}\psi$ which scale like $L^{-\frac{9}{4}}$ are gone, but they may have left terms $\bar{\psi}\partial\psi$ which scale like $L^{-13/4}$. Or consider that local $|\mathcal{A}|^2$ term are forbidden by gauge invariance, but there might be $|d\mathcal{A}|^2$ which scale like $L^{-\frac{15}{4}}$. Actually this kind of argument can only be made for small polymers. For large polymers one uses the scaling of $e^{-d_M(X)}$ to get extra powers of L^{-1} . For details see [27].

Altogether then

$$\|E_{k+1}\|_{k+1} \leq \|\mathcal{L}\mathcal{R}E_k\|_{k+1} + \|\mathcal{L}E_k^*\|_{k+1} < 1 \quad (294)$$

7 Renormalization

We have seen that under a single renormalization group step the parameters of the theory are changed by $e_{k+1} = L^{\frac{1}{2}}e_k$ and by the following equations

$$\begin{aligned}\varepsilon_{k+1} &= L^3(\varepsilon_k + \varepsilon_k^*) \\ m_{k+1} &= L(m_k + m_k^*) \\ E_{k+1} &= \mathcal{L}(\mathcal{R}E_k + E_k^*)\end{aligned}\tag{295}$$

We want to iterate these transformations and study the flow. The main difficulty is keeping growth of the mass m_k under control (and also the energy density ε_k , but this is less serious since it does not affect the other quantities). We have seen that if this is accomplished then the flow of the polymer functions E_k is also controlled. The idea is to pick initial conditions ε_0, m_0 for the counter terms such that the final values take prescribed values, which for simplicity we take to be zero. The final values come at $k = K \equiv N - m$ since at this point we are on the torus $\mathbb{T}_{N-K}^0 = \mathbb{T}_m^0$ which consists of a unit lattice and a single $M = L^m$ cube. This accomplishes the goal a reducing our problems to a fixed finite number of degrees of freedom. Nevertheless our smeared fields which we are using to track the action are on the L^{-K} lattice \mathbb{T}_{N-K}^{-K} which is close to the original \mathbb{T}_0^{-N} .

Now we can state:

Theorem 2. *Let L be sufficiently large and e be sufficiently small. Then for each N there is a unique sequence ε_k, m_k, E_k for $k = 0, 1, 2, \dots, K$ satisfying of the dynamical equation (295), the boundary conditions*

$$\varepsilon_K = 0 \quad m_K = 0 \quad E_0 = 0\tag{296}$$

It satisfies the bounds

$$\begin{aligned}|\varepsilon_k| &\leq e_k^{\frac{1}{4}-\epsilon} \\ |m_k| &\leq e_k^{\frac{3}{4}-2\epsilon} \\ \|E_k\|_k &\leq e_k^{\frac{1}{4}-\epsilon}\end{aligned}\tag{297}$$

We sketch the proof. It suffices to consider only m_k, E_k , the ε_k can be treated separately. Let ξ_k be the pair $\xi_k = (m_k, E_k)$ and consider the space of all sequences $\xi = (\xi_0, \dots, \xi_K)$ which satisfy the boundary conditions. Supplied with the norm

$$\|\xi\| = \sup_{0 \leq k \leq K} \{e_k^{-\frac{3}{4}+2\epsilon}|m_k|, e_k^{-\frac{1}{4}+\epsilon}\|E_k\|_k\}\tag{298}$$

we have a Banach space \mathcal{B} . Our problem is to find elements of the unit ball

$$\mathcal{B}_1 = \{\xi \in \mathcal{B} : \|\xi\| \leq 1\} \quad (299)$$

which satisfy the dynamical equations (297). Elements of the unit ball satisfy the bounds $|m_k| \leq e_k^{\frac{3}{4}-2\epsilon}$ and $\|E_k\|_k \leq e_k^{\frac{1}{4}-\epsilon}$. These are stronger than the hypotheses of theorem 1 so the consequences of that theorem hold.

We reformulate into a contractive setting as follows. Define an mapping $\xi' = T\xi$ on \mathcal{B}_1 by

$$\begin{aligned} m'_k &= L^{-1}m_{k+1} - m^*(E_k) \\ E'_k &= \mathcal{L}\left(\mathcal{R}E_{k-1} + E_{k-1}^*(m_{k-1}, E_{k-1})\right) \end{aligned} \quad (300)$$

Then ξ satisfies the dynamical equations and the boundary conditions if and only if it is a fixed point for T on \mathcal{B}_1 .

First we need to know that T maps \mathcal{B}_1 to itself. This is true since $e_{k+1} = L^{\frac{1}{2}}e_k$ and we know $|m^*(E_k)| \leq \mathcal{O}(1)e_k^{\frac{1}{2}}\|E_k\|_{h_k}$ and so

$$\begin{aligned} e_k^{-\frac{3}{4}+2\epsilon}|m'_k| &\leq e_k^{-\frac{3}{4}+2\epsilon}\left(L^{-1}|m_{k+1}| + \mathcal{O}(1)e_k^{\frac{1}{2}}\|E_k\|_k\right) \\ &\leq L^{-\frac{5}{8}}\left[e_{k+1}^{-\frac{3}{4}+2\epsilon}|m_{k+1}|\right] + \mathcal{O}(1)e_k^\epsilon\left[e_k^{-\frac{1}{4}+\epsilon}\|E_k\|_k\right] \\ &< \left(L^{-\frac{5}{8}} + \mathcal{O}(1)e_k^\epsilon\right)\|\xi\| < \frac{1}{2}\|\xi\| \end{aligned} \quad (301)$$

We also have from (291), (292) and $e_{k-1} < e_k$

$$\begin{aligned} e_k^{-\frac{1}{4}+\epsilon}\|E'_k\|_k &\leq e_k^{-\frac{1}{4}+\epsilon}\left(\|\mathcal{L}\mathcal{R}E_{k-1}\|_k + \|\mathcal{L}E_{k-1}^*\|_k\right) \\ &\leq e_{k-1}^{-\frac{1}{4}+\epsilon}\left(\mathcal{O}(1)L^{-\frac{1}{4}}\|E_{k-1}\|_{k-1} + CL^3e_{k-1}^{\frac{1}{4}}\right) \\ &\leq \mathcal{O}(1)L^{-\frac{1}{4}}\left[e_{k-1}^{-\frac{1}{4}+\epsilon}\|E_{k-1}\|_{k-1}\right] + CL^3e_{k-1}^\epsilon \\ &< \frac{1}{2}\|\xi\| + \frac{1}{2} \end{aligned} \quad (302)$$

Together these imply $\|\xi'\| = \|T\xi\| \leq \frac{1}{2}\|\xi\| + \frac{1}{2} < 1$.

Now we claim that the map is a contraction and assert that for any two sequences ξ_1, ξ_2 in \mathcal{B}_1 we have

$$\|\xi'_1 - \xi'_2\| = \|T\xi_1 - T\xi_2\| \leq \frac{1}{2}\|\xi_1 - \xi_2\| \quad (303)$$

Indeed since $m^*(E_k)$ is linear we have $|m^*(E_{1,k}) - m^*(E_{2,k})| \leq \mathcal{O}(1)e_k^{\frac{1}{2}}\|E_{1,k} - E_{2,k}\|_k$ and so as in (301)

$$\begin{aligned}
e_k^{-\frac{3}{4}+2\epsilon}|m'_{1,k} - m'_{2,k}| &\leq e_k^{-\frac{3}{4}+2\epsilon} \left(L^{-1}|m_{1,k+1} - m_{2,k+1}| + \mathcal{O}(1)e_k^{\frac{1}{2}}\|E_{1,k} - E_{2,k}\|_k \right) \\
&\leq L^{-\frac{5}{8}} \left[e_{k+1}^{-\frac{3}{4}+2\epsilon}|m_{1,k+1} - m_{2,k+1}| \right] \mathcal{O}(1)e_k^\epsilon \left[e_k^{-\frac{1}{4}+\epsilon}\|E_{1,k} - E_{2,k}\|_k \right] \\
&\leq \left(L^{-\frac{5}{8}} + \mathcal{O}(1)e_k^\epsilon \right) \|\xi_1 - \xi_2\| \leq \frac{1}{2} \|\xi_1 - \xi_2\|
\end{aligned} \tag{304}$$

Similarly as in (302) we have

$$e_k^{-\frac{1}{4}+\epsilon}\|E'_{1,k} - E'_{2,k}\|_k < \frac{1}{2}\|\xi_1 - \xi_2\| \tag{305}$$

In this case E_k^* is not a linear function of (m_k, E_k) so we need estimates on the derivatives with respect to these variables. These follow by Cauchy bounds since E_k^* is in fact an analytic function of m_k, E_k . Applying these remarks to $E_{k-1}^*(m_{k-1}, E_{k-1})$ gives the result. The last two estimates give (303).

Now we have a contraction mapping on a complete metric space, so by a standard theorem there is a unique fixed point. This proves theorem 2.

The above analysis is a kind of non-perturbative renormalization. The technique should work for any super-renormalizable quantum field theory, i.e. models in which the coupling constant has positive dimension (in mass units). But it probably does not work for strictly renormalizable models for which the coupling constant is dimensionless.

8 The full expansion

8.1 the full RG

The previous discussion focussed on the special case where small field regions are the whole torus. Now we discuss the general case where we have both large and small field regions. We have previously discussed the first step. After k steps we have a sequence of small field regions

$$\Omega = (\Omega_1, \dots, \Omega_k) \quad (306)$$

which are decreasing

$$\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_k \quad (307)$$

Each Ω_j is a union of $L^{-(k-j)}M$ cubes in \mathbb{T}_{N-k}^{-k} . On Ω_j we have increasingly tight small field bounds.

The small field regions Ω are introduced as follows. After k steps the density will have the form

$$\rho_k(A_k, \Psi_k) = \sum_{\Omega} \rho_{k,\Omega}(A_k, \Psi_k) \quad (308)$$

Insert this into the general RG transformation and take the sum outside the integral to obtain

$$\begin{aligned} \tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) = \\ \sum_{\Omega} N_k \int \delta(A_{k+1} - \mathcal{Q}A_k) \delta(\tau A_k) \exp\left(-\frac{b}{L} |[\Psi_{k+1} - Q(A_{k+1})\Psi_k]|^2\right) \\ \rho_{k,\Omega}(A_k, \Psi_k) D\Psi_k D A_k \end{aligned} \quad (309)$$

Now introduce a new small field region. Ω_{k+1} by inserting under the integral

$$1 = \sum_{\Omega_{k+1} \subset \Omega_k} C_{k+1,\Omega} \chi_{k+1}(\Omega_{k+1}) \hat{\chi}_k(\Omega_{k+1}) \quad (310)$$

Let $\Omega^+ = (\Omega, \Omega_{k+1})$. then $\chi_{k+1}(\Omega_{k+1})$ enforces (a local version of)

$$|d\tilde{\mathcal{A}}_{k+1,\Omega^+}| \leq L^{-\frac{3}{2}} p(e_{k+1}) \quad \text{on } \Omega_{k+1} \quad (311)$$

and the $\hat{\chi}_k(\Omega_k)$ enforces (a local version of)

$$|A_k - A_{k,\Omega^+}^{\min}| \leq p_0(e_k) \quad \text{on } \Omega_{k+1} \quad (312)$$

for a certain minimizer A_{k,Ω^+}^{\min} . One can show that these bounds imply (a local version of) $|d\mathcal{A}_{k,\Omega}| \leq p(e_k)$. The term $C_{k+1,\Omega}(A_k, A_{k+1})$ collects various large and small field characteristic functions, but has at least one large field characteristic function in in block $\square \subset \Omega_{k+1}^c$.

Then we again get something of the form

$$\tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) = \sum_{\Omega^+} \tilde{\rho}_{k,\Omega^+}(A_{k+1}, \Psi_{k+1}) \quad (313)$$

where

$$\begin{aligned} \tilde{\rho}_{k+1,\Omega^+}(A_{k+1}, \Psi_{k+1}) &= N_{k+1} \int C_{k+1,\Omega} \chi_{k+1}(\Omega_{k+1}) \hat{\chi}_k(\Omega_k) \\ &\delta\left(A_{k+1} - \mathcal{Q}A_k\right) \delta(\tau A_k) \exp\left(-\frac{b}{L} \|[\Psi_{k+1} - \mathcal{Q}(\mathcal{A}_{k+1})\Psi_k]\|^2\right) \rho_{k,\Omega}(A_k, \Psi_k) D\Psi_k D A_k \end{aligned} \quad (314)$$

In the new small field region Ω_{k+1} we expand around the critical points of the quadratic terms and process the result as a bounded fluctuation integral just as we did in section 6. Finally we scale down to get $\rho_{k+1,\Omega^+}(A_{k+1}, \Psi_{k+1})$.

After k steps the claim is that density can be represented on the lattice \mathbb{T}_{N-k}^0 in the form which is something like

$$\begin{aligned} \rho_k(A_k, \Psi_k) &= \sum_{\Omega} Z_{k,\Omega}(0) Z_{k,\Omega} \int Dm_{k,\Omega}(A) Dm_{k,\Omega}(\Psi) \mathcal{C}_{k,\Omega} \chi_k(\Omega_k) \\ &\exp\left(-\frac{1}{2} \|d\mathcal{A}\|^2 - \mathfrak{S}_{k,\Omega}\left(\mathcal{A}, \Psi_{k,\Omega}, \psi_{k,\Omega}(\mathcal{A})\right) - m_k \langle \bar{\psi}_{k,\Omega}(\mathcal{A}), \psi_{k,\Omega}(\mathcal{A}) \rangle_{\Omega_k} \right. \\ &\left. - \varepsilon_k \text{Vol}(\Omega_k) + E_k(\Omega_k, \mathcal{A}, \psi_{k,\Omega}(\mathcal{A})) + B_{k,\Omega}(\mathcal{A}, \psi_{k,\Omega}(\mathcal{A}))\right) \Big|_{\mathcal{A}=A_k, \Omega} \end{aligned} \quad (315)$$

The parameters ε_k, m_k are tuned as in the previous section and various terms are defined as follows:

- The factors $Z_{k,\Omega}(0), Z_{k,\Omega}$ contain normalization factors for fermions and boson fluctuation integrals respectively, and also include normalization factors for fermion block averaging.
- $Dm_{k,\Omega}(A)$ is a measure on large gauge fields which have not been processed. It roughly has the form

$$Dm_{k,\Omega}(A) = \prod_{j=0}^{k-1} \delta_{\Omega_j^c}(A_{j+1} - \mathcal{Q}A_j) \delta_{\Omega_j^c}(\tau A_j) D A_{j,\Omega_j^c} \quad (316)$$

- $Dm_{k,\Omega}(\Psi)$ is a "measure" on fermi fields which have not been processed. It has the form up to normalization factors

$$Dm_{k,\Omega}(\Psi) = \prod_{j=0}^{k-1} N_{j+1,\Omega_{j+1}^c} \exp\left(-\|[\Psi_{j+1} - \mathcal{Q}(0)\Psi_j]\|_{\Omega_j^c}^2\right) D\Psi_{j,\Omega_j^c} \quad (317)$$

- Gauge fields which do not appear in $Dm_{k,\Omega}(A)$ and are not integrated out in the fluctuation integral have the form $A_{j,\delta\Omega_j} = A_{k,\Omega_j - \Omega_{j+1}}$. These are active fields and after k steps are collected into

$$A_{k,\Omega} = (A_{0,\Omega_1^c}, A_{1,\delta\Omega_1}, \dots, A_{k-1,\delta\Omega_{k-1}}, A_{k,\Omega_k}) \quad \delta\Omega_j = \Omega_j - \Omega_{j+1} \quad (318)$$

where A_j is defined on $\mathbb{T}_{N-k}^{-(k-j)}$. The k -step axial gauge minimizer is the minimizer of $\|d\mathcal{A}\|^2$ on \mathbb{T}_{N-k}^{-k} subject to the constraints $\mathcal{Q}_{k,\Omega}\mathcal{A} = A_{k,\Omega}$ and $\tau_{k,\Omega}\mathcal{A} = 0$. Here $\mathcal{Q}_{k,\Omega} = \mathcal{Q}_j$ on $\delta\Omega_j$ and $\tau_{k,\Omega}$ is j -step axial gauge fixing on $\delta\Omega_j$. The minimizer has the form $\mathcal{A}_{k,\Omega}^x = \mathcal{H}_{k,\Omega}^x A_{k,\Omega}$. There is also a gauge equivalent Landau gauge minimizer which minimizes $\|d\mathcal{A}\|^2$ subject to the constraints $\mathcal{Q}_{k,\Omega}\mathcal{A} = A_{k,\Omega}$ and $R_{k,\Omega}\delta\mathcal{A} = 0$. It has the form

$$\mathcal{A}_{k,\Omega} = \mathcal{H}_{k,\Omega} A_{k,\Omega} \quad (319)$$

and this is what appears in the action.

The $A_{k,\Omega}^{\min}$ is essentially the minimizer of $\|d\mathcal{A}_{k,\Omega}^x\|^2$ in A_k subject to the constraints $\mathcal{Q}A_k = A_{k+1}$, $\tau A_k = 0$ on Ω_{k+1} .

- The free fermi action depends on the active fields

$$\Psi_{k,\Omega} = (\Psi_{0,\Omega_1^c}, \Psi_{1,\delta\Omega_1}, \dots, \Psi_{k-1,\delta\Omega_{k-1}}, \Psi_{k,\Omega_k}) \quad (320)$$

and has the form

$$\mathfrak{S}_{k,\Omega}(\mathcal{A}, \Psi_{k,\Omega}, \psi) = \|[\Psi_{k,\Omega} - Q_{k,\Omega}(\mathcal{A})\psi]\|_{\Omega_1}^2 + \left\langle \bar{\psi}, \left(\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k \right) \psi \right\rangle \quad (321)$$

It is evaluated at the critical point in $\psi = \psi_{k,\Omega}(\mathcal{A})$ on \mathbb{T}_{N-k}^{-k} which has the form

$$\psi_{k,\Omega}(\mathcal{A}) = \mathcal{H}_{k,\Omega}(\mathcal{A})\Psi_{k,\Omega} \quad (322)$$

- The main higher order interactions are contained in the polymer function

$$E_k(\Omega_k, \mathcal{A}, \psi) = \sum_{X \subset \Omega_k} E_k(X, \mathcal{A}, \psi) \quad (323)$$

where $E_k(X, \mathcal{A}, \psi)$ is identical with the global small field polymer function. In particular it is $\mathcal{O}(e^{\frac{1}{k}})$, has all the symmetries, and is independent of the history Ω . However it is evaluated at $\mathcal{A} = \mathcal{A}_{k,\Omega}$ and $\psi = \psi_{k,\Omega}(\mathcal{A}_{k,\Omega})$ which remember the history. But in the current small field region Ω_k we do have $\mathcal{A}_{k,\Omega} \approx \mathcal{A}_k$ and $\psi_{k,\Omega}(\mathcal{A}_{k,\Omega}) \approx \psi_k(\mathcal{A}_k)$

- The term $B_{k,\Omega}(\mathcal{A}, \psi)$ is a boundary term which plays no role in renormalization. It has the form

$$B_{k,\Omega} = \sum_j B_{k,j,\Omega}$$

$$B_{k,j,\Omega} = \sum_X B_{k,j,\Omega}(X)$$
(324)

Here in $B_{k,j,\Omega}$ the sum is over polymers X made up of $L^{-(k-j)}M$ cubes which intersect both Ω_j^c, Ω_j . In a suitable norm $B_{k,\Omega}$ is $\mathcal{O}(e_k^{\frac{1}{4}})$.

- The characteristic function $\chi_k(\Omega_k)$ is just a record of the bound $|dA_{k,\Omega}| \leq p(e_k)$ in Ω_k . Since $A_k = \mathcal{Q}_k \tilde{\mathcal{A}}_{k,\Omega}$ in Ω_k it also implies $|dA_k| \leq p(e_k)$ in Ω_k . The $\mathcal{C}_{k,\Omega}$ collects the large field characteristic functions $C_{j,\Omega}$ from each single step.

In fact at this point we have to admit that the expansion is not a simple as stated. For technical reasons we have to introduce more characteristic functions and get more sums over small field regions. There is not just a sum over Ω also over several small field regions.

Another point is that although the characteristic functions start out local, when we translate to the minimizers of the action the locality is again compromised. Restoring locality is a difficult technical problem. It is more difficult than the induced non-locality in the action. There we could take advantage of analyticity in the fields, but here there is no analyticity.

8.2 multiscale analysis

As in the global small field case a key role is played by the operators $\mathcal{H}_{k,\Omega}$ and $\mathcal{H}_{k,\Omega}(\mathcal{A})$ which select the critical points of the action. Let us discuss in particular the fermion case $\mathcal{H}_{k,\Omega}(\mathcal{A})$. It has the form

$$\mathcal{H}_{k,\Omega}(\mathcal{A})\Psi_{k,\Omega} = S_{k,\Omega}(\mathcal{A})Q_{k,\Omega}^T(-\mathcal{A})b_{k,\Omega} \Psi_{k,\Omega}$$
(325)

where the Greens functions is

$$S_{k,\Omega}(\mathcal{A}) = \left[\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k + Q_{k,\Omega}^T(-\mathcal{A})b_{k,\Omega}Q_{k,\Omega}(\mathcal{A}) \right]_{\Omega_1}^{-1}$$
(326)

and $b_{k,\Omega} = b_j L^{k-j}$ on $\delta\Omega_j$. Again the term $Q_{k,\Omega}^T(-\mathcal{A})b_{k,\Omega}Q_{k,\Omega}(\mathcal{A})$ provides an effective mass which is now $\mathcal{O}(L^{k-j})$ in $\delta\Omega_j$. Thus it increases the farther back we go in the history, or the farther away we are from the current small field region Ω_k . It provides a kind of soft boundary conditions, with the result that $S_{k,\Omega}(\mathcal{A})$ is going to have better estimates than say $[\mathfrak{D}_{e_k}(\mathcal{A}) + \bar{m}_k + Q_{k,\Omega}^T(-\mathcal{A})b_k Q_k(\mathcal{A})]_{\Omega_1}^{-1}$

We need exponential decay estimates on the kernels of $S_{k,\Omega}(\mathcal{A})$ and hence $\mathcal{H}_{k,\Omega}(\mathcal{A})$, and we also need to be able to break it up into local pieces. Both these are accomplished

by random walk expansions. In $\delta\Omega_j$ the operator $Q_{k,\Omega}(\mathcal{A})$ averages over blocks with width L^j -sites in an L^{-k} lattice, so we are averaging over blocks of size $L^{-(k-j)}$. Correspondingly our random walk expansion should be based on blocks of size $L^{-(k-j)}3M$ in $\delta\Omega_j$ (which is itself a union of $L^{-(k-j)}M$ blocks).

A random walk is then a sequence

$$\omega = \left(\square_0, \square_1, \square_2, \dots, \square_n \right) \quad (327)$$

such that \square_i is an $L^{-(k-j)}3M$ block if $\square_i \in \delta\Omega_j$ and such that \square_i, \square_{i+1} overlap. We have a multi-scale random walk expansion (for sufficiently regular background field \mathcal{A})

$$S_{k,\Omega}(\mathcal{A}) = \sum_{\omega} S_{k,\Omega,\omega}(\mathcal{A}) \quad (328)$$

where $S_{k,\Omega,\omega}(\mathcal{A})$ only depends on \mathcal{A} in ω . This leads to an exponential decay estimate of the form

$$|1_{\Delta_x} S_{k,\Omega}(\mathcal{A}) 1_{\Delta_{x'}} f| \leq CL^{-(k-j')} e^{-\gamma d_{\Omega}(x,x')} \|f\|_{\infty} \quad (329)$$

where Δ_x is an $L^{-(k-j)}$ block centered on $x \in \delta\Omega_j$, and where $d_{\Omega}(x, x')$ is the length of the shortest path from x to x' with paths weighted by L^{k-j} when they pass through $\delta\Omega_j$.

Similarly the gauge operators $\mathcal{G}_{k,\Omega}$ and $\mathcal{H}_{k,\Omega}$ admit multiscale random walk expansions.

These random walk expansion again permit the introduction of weakening parameters. This is critical for preserving the localization of our effective actions as previously discussed.

8.3 UV stability

The representation (315) now yields a stability result. Stopping the iteration at K with $N - K = \mathcal{O}(1)$ the effective density is $\rho_K(A_k, \psi_K)$. We make a final integral over A_k, Ψ_k with axial gauge fixing and have the representation of the partition function

$$Z_N(e) = \int \delta(\mathcal{Q}^* A_k) \delta(\tau^* A_k) \rho_K(A_k, \psi_K) D\Psi_k D A_K \quad (330)$$

Here \mathcal{Q}^*, τ^* are defined with M -blocks rather than L -blocks. The density $\rho_K(A_k, \psi_K)$ is expressed as a sum

$$\rho_K(A_k, \psi_K) = \sum_{\Omega} \rho_{K,\Omega}(A_k, \psi_K) \quad (331)$$

which gives

$$Z_N(e) = \sum_{\Omega} \int \delta(\mathcal{Q}^* A_k) \delta(\tau^* A_k) \rho_{K,\Omega}(A_k, \psi_K) D\Psi_k D A_K \quad (332)$$

In this sum the final small field region Ω_K is either the whole reduced torus \mathbb{T}_{N-K}^{-K} or it is not. In the first case the small field region is always maximal and we are in the situation discussed in sections 6 and 7. In this case we have a small fixed number of field variables and bounds independent of N and we can get a good estimate. In the other case at some point in the iteration the large field region is not empty, and then the same is true for subsequent iterations. These terms sum to an expression which is $\mathcal{O}(e_K^{n_0})$ for any n_0 . The contribution is much smaller than the global small field term. Now dividing by the free partition function $Z_N(0)$ we have:

Theorem 3. (*UV stability bound*) *Let the coupling constant e be sufficiently small and choose counterterms as in theorem 2. Then for all N*

$$\frac{1}{2} \leq \left| \frac{Z(N, e)}{Z(N, 0)} \right| \leq \frac{3}{2} \tag{333}$$

This is a bound on the unit cube, but we could get a similar result starting with any finite volume. We have assumed that e is sufficiently small but this could also be relaxed. The important thing is that the running coupling constant e_k be small and this can always be arranged if we stop the iteration earlier. Finally it should be possible to include source terms in the partition function and thereby generate results for correlation functions.

Remark. In the following expository papers are [17], [25], [30],[33].

References

- [1] T. Balaban (Higgs)_{2,3} Quantum fields in a finite volume- I *Commun. Math. Phys.*, 85: 603-636, 1982.
- [2] T. Balaban (Higgs)_{2,3} Quantum fields in a finite volume- II *Commun. Math. Phys.*, 86: 555-594, 1982.
- [3] T. Balaban (Higgs)_{2,3} Quantum fields in a finite volume- III *Commun. Math. Phys.*, 88: 411-445, 1983.
- [4] T. Balaban, Regularity and decay of lattice Green's functions, *Commun. Math. Phys.*, 89: 571-597, 1983.
- [5] T. Balaban, Propagators and renormalization transformations for lattice gauge field theories -I, *Commun. Math. Phys.* 95 (1984) 17-40.
- [6] T. Balaban, Propagators and renormalization transformations for lattice gauge field theories- II, *Commun. Math. Phys.* 96 (1984) 223-250.
- [7] T. Balaban, Averaging operators for lattice gauge field theories. *Commun. Math. Phys.* 98 (1985) 17-51.
- [8] T. Balaban, Propagators for lattice gauge field theories in a background field, *Commun. Math. Phys.* 99 (1985) 389-434.
- [9] T. Balaban, Ultraviolet stability of three-dimensional lattice pure gauge field theories. *Commun. Math. Phys.* 102 (1985) 255-275.
- [10] T. Balaban, Variational problem and background field in renormalization group method for lattice gauge field theories. *Commun. Math. Phys.* 102 (1985) 277-309.
- [11] T. Balaban, Renormalization group approach to lattice gauge field theories- I, *Commun. Math. Phys.* 109 (1987) 249-301.
- [12] T. Balaban, Renormalization group approach to lattice gauge field theories- II, *Commun. Math. Phys.* 116 (1988) 1-22.
- [13] T. Balaban, Convergent renormalization expansions for lattice gauge field theories, *Commun. Math. Phys.* 119 (1988) 243-285.
- [14] T. Balaban, Large field renormalization-I, *Commun. Math. Phys.* 122 (1989) 175-202.

- [15] T. Balaban, Large field renormalization-II, *Commun. Math. Phys.* 122 (1989) 355-392.
- [16] T. Balaban, Localization expansions I. function of the background configurations. *Commun. Math. Phys.* 182: 33-82, 1996.
- [17] T. Balaban, A. Jaffe, Constructive gauge theory. In *Fundamental problems of gauge field theory, Erice, 1985*, G. Velo and A. Wightman, eds., Plenum Press, 1986.
- [18] T. Balaban, J. Imbrie, A. Jaffe, Renormalization of the Higgs model: minimizer, propagators, and the stability of mean field theory, *Commun. Math. Phys.* 97: 299-329, 1985.
- [19] T. Balaban, J. Imbrie, A. Jaffe, Effective action and cluster properties of the abelian Higgs model, *Commun. Math. Phys.* 114: 257-315 , 1988.
- [20] T. Balaban, M. O'Carroll, R. Schor, Block renormalization group for Euclidean fermions, *Commun. Math. Phys.* 122: 233-247, 1989.
- [21] T. Balaban, M. O'Carroll, R. Schor, Properties of block renormalization group operators for Euclidean fermions in an external field, *J. Math Phys* 32: 3199-3208, 1991.
- [22] J. Dimock, The renormalization group according to Balaban - I. small fields, *Rev. Math. Phys.* 25, 1330010, 1-64, 2013.
- [23] J. Dimock, The renormalization group according to Balaban - II. large fields, *J. Math. Phys.* 54, 092301, 1-85, 2013.
- [24] J. Dimock, The renormalization group according to Balaban - III. convergence, *Annales Henri Poincaré* 15, 2133-2175, 2014.
- [25] J. Dimock, Covariant axial gauge, *Letters in Mathematical Physics* 105, 959-987, 2015.
- [26] J. Dimock, Nonperturbative renormalization of scalar QED in $d=3$, *J. Math. Phys.* 56, 102304, 1-78, 2015.
- [27] J. Dimock, Ultraviolet regularity for QED in $d=3$, *Journal of Mathematical Physics* 59, 012301, (2018).
- [28] J. Dimock, Multiscale block averaging for QED in $d=3$, *Journal of Mathematical Physics* 61, 032302, (2020) .
- [29] J. Dimock, Ultraviolet stability for QED in $d=3$, *Annales Henri Poincare*, published online, awaiting assignment to an issue (2022).

- [30] J. Dimock, Stability for QED in $d=3$: an overview, *Journal of Mathematical Physics* 63, 042305 (2022).
- [31] J. Dimock, T. Hurd, A renormalization group analysis of infrared QED, *J. Math. Phys.* 33, 814-821, 1992.
- [32] J. Feldman, H. Knörrer, E. Trubowitz, *Fermion functional integrals and the renormalization group*, American Mathematical Society, 2002.
- [33] J. Imbrie, Renormalization group methods in gauge field theories. In K. Osterwalder and R. Stora, editors, *Critical phenomena, random systems, gauge theories*. North-Holland, 1986.