12.10 The Minimum-Variance Hedge
Basis risk = spot price of asset to be hedged
- futures price of contract used to hedge

is a bin of fire.

Perfect hedge: ideal financial position;
in which investor eliminates all uncertainty.

Usually, imperfect

ex/ small quantities or non-integer multiples.
ex/ 1 unit of 5000 bu of corn.

ex/ may not be a futures contract for our delivery date.
ex/ commodity might not be on a futures market.
ex/ commodity might not be in right location.

locational risk.
If can't find a perfect hedge, what to do?

One scheme - minimum variance hedging.

Let \( x \) = cash flow at time \( T \),

\( \) plan to purchase \( W \) units of a commodity

at time \( T \); spot price \( S(T) \).

Then: \( x = -WS(T) \) (neg. for expenditure).

To hedge against unexpected changes in \( S \):

use \( h \) units of a futures contract.

let \( F(0) \) = price/unit now, \( F(T) \) = price/unit at \( T \).

Recall value at time \( t \) of holding a forwards contract

\( \) is \( f(t) = \int_{t}^{T} (F(t) - F(0)) \text{d}t \).

At \( t = T \),

\( f(T) = \int_{T}^{T} (F(T) - F(0)) \text{d}(T, T) = F(T) - F(0) \).
To hedge the expenditure \( x \), suppose at \( t = 0 \), go long \( h \) units of a future.

Cash flow at \( T \) is

\[
y = x + (F(T) - F(0)) \cdot 1 \cdot h
\]

where: \( x = -WS(T) \) is a random var.,

\[
F(T) - F(0) \quad \text{[gain or loss from 1 unit of future]}
\]

if available on futures market, then \( F(T) = S(T) \) exactly the right contract.

In general, \( F(T) \neq S(T) \):

\[
\text{instead, } F(T) \text{ is different random var.}
\]

Consider:

\[
y(h) = -WS(T) + h(F(T) - F(0))
\]

\( \text{r.v., random var., } \text{const}. \)
At $t = T$, each futures contract is worth $F(T) - F(0)$.

Costs nothing to arrange futures contracts at $t = 0$.

Cash flow at $t = T$, from both $-wS(T)$ (ex

and $\Lambda (F(T) - F(0))$

is:

$$y = -wS(T) + \Lambda (F(T) - F(0)).$$

$x$ to be hedged.

In general, $S(T) \neq F(T)$, however if available

on futures market at right time, place, and quantity,

can have $S(T) = F(T)$.

Consider $y(x) = -wS(T) + \Lambda (F(T) - F(0))$.

↑ random vars.

↑ const.
\[ \text{var } y(h) = \text{var } (-WS(T) + h F(T) - \frac{h \var F(T)}{\text{const}}) \]

\[ = \text{var } (-WS(T) + h F(T)) \]

\[ = \text{cov } (-WS(T) + h F(T), -WS(T) + h F(T)) \]

\[ = \text{cov } (-WS(T), -WS(T)) \]

\[ + 2 \text{cov } (-WS(T), h F(T)) + \text{cov } (h F(T), h F(T)) \]

\[ = w^2 \text{var } (s(T)) - 2 w h \text{cov } (s(T), F(T)) + h^2 \frac{\text{var } (F(T))}{\text{const}} \]

To minimize, set \( \frac{\partial}{\partial h} \text{var } y(h) = 0 \)

\[ \frac{\partial}{\partial h} \text{var } y(h) = 0 - 2w \text{cov } (s(T), F(T)) + 2h \text{var } (F(T)) \]

\[ \Rightarrow h = h_{\text{min}} = \frac{w \text{cov } (s(T), F(T))}{\text{var } (F(T))}. \]

Claim:

\[ \text{var } (y(h_{\text{min}})) = w^2 \left[ \text{var } (s(T)) - \frac{(\text{cov } (s(T), F(T)))^2}{\text{var } (F(T))} \right] \]
Also, \[
\text{cov} (y \left( h_{\min} \right), F(T)) = 0, \text{ similar to orthogonality in linear algebra.}
\]

Also,

\[ F(T) = S(T), \quad h_{\min} = \frac{w \cdot \text{cov}(S(T), S(T))}{\text{var}(s(T))} = w; \]

\[
\text{var}(y \left( h_{\min} \right)) = w^2 \left[ \frac{\text{var}(s(T))}{\text{var}(s(T))} - \frac{\left( \frac{\text{var}(s(T))}{\text{var}(s(T))} \right)^2}{\text{var}(s(T))} \right] = 0.
\]

A perfect hedge!
In terms of symbols \( b_{s,F} = \cos (S(T), F(T)) \)
\[ b_s^2 = \text{var} (S(T)), \]
\[ b_F^2 = \text{var} (F(T)), \]

\[ h_{\min} = \frac{W \cdot b_{s,F}^2}{b_F^2} \quad \text{and} \quad \text{var} (y(h_{\min})) = W^2 \left[ b_s^2 - \frac{(b_{s,F})^2}{b_F^2} \right] \]

Let \( b_{s,F} = \text{correlation between } S(T) \text{ and } F(T) \),

then \( h_{s,F} = \frac{b_{s,F}}{b_s b_F} \) and \( b_{s,F} = b_{s,F} b_s b_F \).

In terms of correlation instead of covariance,

\[ h_{\min} = \frac{W \cdot b_{s,F} b_s}{b_F} = W h_{s,F} b_s \quad \text{and} \quad \text{var} (y(h_{\min})) = W^2 \left[ b_s^2 - h_{s,F}^2 \frac{b_s^2 b_F^2}{b_F^2} \right] \]

\[ \text{var} (y(h_{\min})) = W^2 b_s^2 \left[ 1 - h_{s,F}^2 \right]. \]

If it can be arranged that

\[ h_{s,F} = 1, \] or if \( F(T) = S(T) \), then \( \text{var} (y(h_{\min})) = W^2 b_s^2 [1 - 1] = 0 \)

perfect hedge
for film venture \( r_i = \begin{cases} 2 & \text{with prob. } \frac{1}{2} \\ -1 & \text{with prob. } \frac{1}{2} \end{cases} \)

Portfolio has fraction \( 1 - \alpha \) in film venture, fraction \( \alpha \) in risk-free

Return rate \( r \) for portfolio is

\[
r = (1 - \alpha) r_i + \alpha r_f,
\]

so

\[
r = \begin{cases} (1 - \alpha) 2 + \alpha r_f & \text{with prob. } \frac{1}{2} \\ (1 - \alpha) (-1) + \alpha r_f & \text{with prob. } \frac{1}{2} \end{cases}
\]

Wealth \( w_0 = 1 \) grows by factor \( 1 + r \):

After 1 yr, \( w_t = w_0 (1 + r) = 1 + r = \begin{cases} 1 + (1 - \alpha) 2 + \alpha \cdot \frac{1}{50} & \text{w/ prob. } \frac{1}{2} \\ 1 + (1 - \alpha) (-1) + \alpha \cdot \frac{1}{50} & \text{w/ prob. } \frac{1}{2} \end{cases} \)

is two-piece formula for \( w_t \).
b) \[ U(w_i) = \ln(w_i) = \ln(1 + \lambda) = \begin{cases} \ln(1 + (1-\alpha)2 + \frac{\alpha}{50}) & \text{w/prob. } \frac{1}{2} \\ \ln(1 + ((1-\alpha)(-1) + \frac{\alpha}{50}) & \text{w/prob. } \frac{1}{2} \end{cases} \]

\[ E[U(w_i)] = \frac{1}{2} \ln(1 + (1-\alpha)2 + \frac{\alpha}{50}) + \frac{1}{2} \ln(1 + ((1-\alpha)(-1) + \frac{\alpha}{50}) \]

\[ = \frac{1}{2} \ln(3 - 2\alpha + \frac{\alpha}{50}) + \frac{1}{2} \ln(\alpha + \frac{\alpha}{50}) \]

\[ = \frac{1}{2} \ln\left[3 - \frac{100}{50} \alpha + \frac{\alpha}{50}\right] + \frac{1}{2} \ln\left[\frac{51}{50} \alpha\right] \]

\[ = \frac{1}{2} \ln\left(3 - \frac{99}{50} \alpha\right) + \ln\left[\frac{51}{50} \alpha\right] \]

\[ = \frac{1}{2} \ln(3 - \frac{99}{50} \alpha) + \ln\left[\frac{51}{50} \alpha\right] \]

\[ \frac{99}{50} = 1.98 \]

\[ \frac{51}{50} = 1.02 \]

c) max \[ E[U(w_i)] \text{: take } \frac{d}{d\alpha}, \text{ set } = 0. \]

\[ \frac{dE}{d\alpha} = \frac{1}{2} \left[ \frac{1}{3 - \frac{99}{50} \alpha} \cdot (-99) + \frac{1}{\frac{51}{50} \alpha} \cdot \frac{51}{50} \right] = 0 \]

\[ = \frac{1}{2} \text{ Solve for } \alpha: \quad \alpha = \frac{35}{33} = 1.0575 \]
12.10 The Minimum-Variance Hedge
A model for hedging grapefruit juice using orange juice futures.

Suppose growing season: \( z_1 \) for oranges, \( z_2 \) for grapefruit.

Let \( z_t \) represent random weather effects, first period \( t = 1 \), second period \( t = 2 \).

Suppose \( S = \) grapefruit price at \( t = T \),

\[ F = \text{orange price at } t = T \]

Assume: \( S = a_0 + a_1 z_1 + a_2 z_2 \) \( z_1, z_2 \) random;

\[ F = b_0 + b_1 z_1. \]

\[ E[z_1] = \bar{z}_1 = 0 \]
\[ E[z_2] = \bar{z}_2 = 0 \]
\[ \text{var}(z_1) = \text{var}(z_2) = 1 \]
\[ \text{cov}(z_1, z_2) = 0. \]
\[
\begin{align*}
\text{var}(F) &= \text{var}(\beta' + b_1 z_1) = b_1^2 \text{var}(z_1) = b_1^2 \\
\text{var}(S) &= \text{cov}(\beta' + a_1 z_1 + a_2 s_2, \beta' + a_1 z_1 + a_2 s_2) \\
&= a_1^2 \text{cov}(z_1, z_1) + 2a_1 a_2 \text{cov}(z_1, s_2) + a_2^2 \text{cov}(s_2, s_2) \\
&= a_1^2 + a_2^2 \\
\text{cov}(S, F) &= \text{cov}(\beta' + a_1 z_1 + a_2 s_2, \beta' + b_1 z_1) \\
&= a_1 b_1 \text{cov}(z_1, z_1) + a_2 b_1 \text{cov}(s_2, z_1) = a_1 b_1 \\
\text{var}(y(\hat{h}_{\text{min}})) &= \mathbf{W}^2 \left[ \text{var}(S(T)) - \frac{\text{cov}(S(T), F(T))^2}{\text{var}(F(T))} \right] \\
&= \mathbf{W}^2 \left[ a_1^2 + a_2^2 - \frac{(a_1 b_1)^2}{b_1^2} \right] = \mathbf{W}^2 \left[ a_1^2 + a_2^2 - \frac{(a_1 b_1)^2}{b_1^2} \right] \\
&= \mathbf{W}^2 \left[ a_1^2 + a_2^2 - \frac{(a_1 b_1)^2}{b_1^2} \right] = \mathbf{W}^2 \left[ a_1^2 + a_2^2 - \frac{(a_1 b_1)^2}{b_1^2} \right].
\end{align*}
\]
12.12 Hedging Nonlinear Risk
Nonlinear risk

"Corn" is grown by large number (say 100) farmers. Amount grown by each, highly correlated with all the others.

Suppose each produces $C$ bushels; total $100C$ bushels.

\[ P = P_d(D) \]
\[ P_d(D) = 10 - \frac{D}{10^5} \]

If total demand = total production, then $D = 100C$.

Revenue (each farmer) is

\[ R_p(C) = P_d(100C) \cdot C = (10 - \frac{C}{10^3})C \]

price & bu.
Suppose expected production is $\bar{c} = 3,000$.

For $c$ near $\bar{c}$, $R_p'(c) > 0$.

Farmer has "natural hedge" against decreasing prices provided $c < 5,000$:

increasing production $\Rightarrow$ revenue increase.

If, however, $c > 5,000$, if prices decrease then despite increasing production, revenue is decreasing.
Consider an effort to counteract these effects by not using futures contracts to hedge.

Suppose each future is contract to buy 1 bu of corn at price $\bar{p}$, at time $\bar{t}$: $\bar{P} = \bar{P}(100\bar{C})$

Suppose each farmer arranges $\lambda$ such futures.

$(\lambda > 0$ : buy $\lambda$ bushels for $\bar{P}$ at $\bar{t}$)

"< 0" : sell "

Payoff from futures: $\lambda(P - \bar{P})$, $P =$ spot price at $\bar{t}$.

Total revenue $R_{tot} = R_p(C) + \lambda(P - \bar{P})$

$= 10C - \frac{C^2}{10^3} + \lambda(P - \bar{P})$.

But $P(T) = Pd(100C)$ and $\bar{P} = Pd(100\bar{C})$

recall $Pd = 10 - \frac{D}{105}$

$\Rightarrow R_{tot} = 10C - \frac{C^2}{10^3} + \frac{\lambda}{10^3} (\bar{C} - C)$.

Pick $\lambda$ such that $\frac{dR_{tot}}{dC} \bigg|_{C = \bar{C}} = 0$. 

Recall revenue to farmer, both from selling crops and from investment in futures:

\[ R_{\text{tot}}(C, h) = 10C - \frac{C^2}{10^3} + h(P - \overline{P}) \]

expected price, for \( C = \overline{C} \):
\[ \overline{P} = P_d(100\overline{C}) \]
also:
\[ P = P_d(100C) \]

\[ R_{\text{tot}} = 10C - \frac{C^2}{10^3} + \frac{h}{10^3}(\overline{C} - C) \]

To minimize the sensitivity of \( R_{\text{tot}} \) with respect to changes in \# of bushels \( C \), at (near) expected \( \overline{C} = 3000 \), choose \( h \) such that

\[ \frac{\partial R_{\text{tot}}(C, h)}{\partial C} = 0. \]

\[ \frac{\partial^2 R_{\text{tot}}}{\partial C^2} = 10 - \frac{2C}{10^3} + \frac{h}{10^3}(-1) = 0 \quad \text{for} \quad h = 10^3 \left(10 - \frac{23000}{10^3}\right). \]

For \( \overline{C} = 3000 \), \( \frac{\partial R_{\text{tot}}}{\partial C} = 0 \) when \( h = 10^3 \left(10 - \frac{2 \times 3000}{10^3}\right) = 4000. \)

Strategy: buy 4000 bushels futures contract.
\[ R_{\text{tot}}(c, h) = 10c - \frac{c^2}{10^3} + k(p - \bar{p}) \]

But \( P(T) = P_{d}(100c) \) and \( \bar{p} = P_{d}(100\bar{c}) \),

\[ \Rightarrow P(T) - \bar{p} = P_{d}(100c) - P_{d}(100\bar{c}) \]

\[ = n_{0} - \frac{100c}{10^3} - (n_{0} - \frac{100\bar{c}}{10^3}) \]

\[ = \frac{1}{10^3}(\bar{c} - c). \]

\[ R_{\text{tot}} = 10c - \frac{c^2}{10^3} + \frac{k}{10^3}(\bar{c} - c). \]

\[ \frac{2R_{\text{tot}}(c, h)}{dc} = 10 - \frac{2c}{10^3} + \frac{k}{10^3}(-1) = 0 \text{ for } h = 10^3\left(10 - \frac{2c}{10^3}\right). \]

For \( \bar{c} = 300c \),

\[ \frac{2}{dc} R_{\text{tot}}(c, h) \bigg| = 0 \text{ when } h = 10^3\left(10 - \frac{2\times 300c}{10^3}\right) = 4000. \]

\[ c = \bar{c} \]

\[ \text{Buy 4000 bushels!} \]
Links visited in class

Hedging nonlinear risk
2.5 Put-call parity
Recall definition of Call, Put Options from 458
Relationship between (Euro) calls & puts:

**Put-Call Parity**

Let $\Pi_A$: portfolio consisting of
one call option (strike $K$, expiration $T$)
+ P.V. of strike, invested in ideal bank.

Let $d_{0,T} = $ discount factor for interval $(0, T)$
Use continuous compounding: $d_{0,T} = e^{-rT}$, $r =$ rate.
P.V. of strike $= K e^{-rT}$

Let $\Pi_B$: portfolio consisting of
one put option (strike $K$, expiration $T$)
+ one unit of asset $S$. 
At $t=T$, the value of $\Pi_A$ is

$$\Pi_A(T) = \max(S(T) - K, 0) + (K - \delta \nu_{u,T}) \cdot \frac{1}{\delta_{u,T}} = \begin{cases} S(T), & \text{if } S(T) \geq K \\ K, & \text{if } S(T) < K. \end{cases}$$

At $t=T$, the value of $\Pi_B$ is

$$\Pi_B(T) = \max(K - S(T), 0) + S(T) = \begin{cases} S(T), & \text{if } S(T) \geq K \\ K, & \text{if } S(T) < K. \end{cases}$$

Since portfolios produce same outcome at $t=T$, expect values at $t=0$ of portfolios, should be equal.
2.6 Upper and lower bounds on option values
negative of a portfolio \( \Pi \)

Suppose \( \Pi \) is made of:

i) funds deposited in or borrowed from, an ideal bank.

ii) units of an asset, either bought or sold (short).

iii) units of options on the asset.

<table>
<thead>
<tr>
<th>in ( \Pi )</th>
<th>in ( -\Pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) amount deposited</td>
<td>i) same amount borrowed</td>
</tr>
<tr>
<td>&quot; borrowed</td>
<td>&quot; deposited</td>
</tr>
<tr>
<td>ii) units of asset bought</td>
<td>ii) same # of units sold (short)</td>
</tr>
<tr>
<td>&quot; sold</td>
<td>&quot; &quot; &quot; bought</td>
</tr>
<tr>
<td>iii) units of option bought</td>
<td>iii) same # of units of option written</td>
</tr>
<tr>
<td>&quot; written</td>
<td>&quot; &quot; &quot; &quot; &quot; bought</td>
</tr>
</tbody>
</table>
Component-level (or detailed) arbitrage arguments describe each transaction needed to construct the example of arbitrage, (which violates no-arbitrage).

Portfolio-level no-arbitrage arguments:

Assume portfolios $\Pi_A, \Pi_B$, where each may be negated.

If $\Pi_A(T) \leq \Pi_B(T)$ in all cases ($S(T)$ is random variable)

then $\Pi_A(0) \leq \Pi_B(0)$.

Also, if $\Pi_A(T) = \Pi_B(T)$,

then $\Pi_A(0) = \Pi_B(0)$. 
Portfolio level argument: at \( t=0 \)

\[ \Pi_A: \text{call option} + \text{P.V. of strike in ideal bank} \]

\[ \Pi_B: \text{put option} + \text{unit of asset} \]

At \( t = T \), either \( S(T) \geq K \) or \( S(T) < K \).

Show: \( \Pi_A(T) = \Pi_B(T) \) in both cases.

\[ \therefore \quad \Pi_A(0) = \Pi_B(0) \]

\[ C(0; K) + Ke^{-rT} = P(0; K) + S(0) \]

The "\( t=0 \) is not special" argument:

may replace \( t=0 \) with general \( t < T \),

provided also replace \( d(0, T) \) with \( d(t, T) \)

\[ = e^{-rT} \cdot e^{-r(T-t)} \]

By "\( t=0 \) not special",

\[ C(t; K) + Ke^{-r(T-t)} = P(t; K) + S(t) \]
"Easy" bounds: $ P$ is an option, so $ P \geq 0$.

\[ C(L; K) + Ke^{-\lambda(t-t)} \geq S \]
\[ C(L; K) \geq S - Ke^{-\lambda(t-t)} \]

Also, $ C < S$:

$\hat{\Pi}_A$: one call option; $\hat{\Pi}_B$: one unit of asset $S$

\[ \hat{\Pi}_A(t) = C(L; K) = \max(S(t) - K, 0) \]
\[ \hat{\Pi}_B(t) = S(t) \]

\[ \hat{\Pi}_A(t) = \begin{cases} S(t) - K & S(t) > K \\ 0 & S(t) \leq K \end{cases} \]
\[ \hat{\Pi}_B(t) = \begin{cases} S(t) & S(t) > K \\ S(t) & S(t) \leq K \end{cases} \]

In all cases, $\hat{\Pi}_A(t) < \hat{\Pi}_B(t)$,

so: $\hat{\Pi}_A(0) < \hat{\Pi}_B(0)$

\[ C(0; K) < S(0) \]

by "$t = 0$ is not special",

\[ C(L; K) < S(t) \]
for $t = 0$

$C(0)$

$C = S$

$S - Ke^{-\alpha T}$

$Ke^{-\alpha (T-t)}$ $S(0)$

Can change to

$C(t) = S(t)$

$S(t) - Ke^{-\alpha (T-t)}$

$Ke^{-\alpha (T-t)}$ $S(t)$
2.6 Upper and lower bounds on option values
Claim: A Euro Call is non-decreasing in $T$.

Let $C(t, T, K)$ denote call option value.

Consider $C(0, T_1, K)$ and $C(0, T_2, K)$, where

\[
\begin{array}{c}
\text{K} \\
0 \\
T_1 \\
T_2 \\
\end{array}
\]

We will show: $C(0, T_2, K) \geq C(0, T_1, K)$.

At $T_1$, $C(0, T_1, K) = \begin{cases} S(T_1) - K & \text{if } S(T_1) > K \\ 0 & \text{if } S(T_1) \leq K. \end{cases}$

Case 1: If $S(T_1) \leq K$, $C(0, T_2, K) \geq 0 = C(0, T_1, K)$: claim true.

Case 2: If $S(T_1) > K$, (short)sell 1 unit of asset, invest $S(T_1)$ in ideal bank.

At $T_2$, take $S(T_1) e^{r(T_2-T_1)}$ from bank.

Buy asset on spot market for $S(T_2)$ & return to owner to satisfy short.
Payoff in case 2 at \( T_2 \)

\[
= S(T_1) e^{r(T_2 - T_1)} - S(T_2) + C(T_2, T_2, K) \left\{ \max(S(T_2), 0) - K \right\}
\]

\[
= \begin{cases} 
S(T_1) e^{r(T_2 - T_1)} - S(T_2) + S(T_2) - K & \text{if } S(T_2) > K \\
S(T_1) e^{r(T_2 - T_1)} - S(T_2) + 0 & \text{if } S(T_2) \leq K
\end{cases}
\]

but: if \( S(T_2) \leq K \),

\[-S(T_2) \geq -K.\]

Payoff in case 2 at \( T_2 \)

\[
\geq S(T_1) e^{r(T_2 - T_1)} - K.
\]

but: \( e^{r(T_2 - T_1)} > 1 \)

\[\Rightarrow S(T_1) e^{r(T_2 - T_1)} > S(T_1) - K.\]

Payoff in case 2 at \( T_2 \)

\[
> S(T_1) e^{r(T_2 - T_1)} - K e^{r(T_2 - T_1)}
\]

\[= e^{r(T_2 - T_1)} (S(T_1) - K).\]

Value at \( T_1 \), of payoff in case 2

\[> S(T_1) - K, \text{ payoff for } T_1 \text{- expiration call.}\]

\[
\therefore C(0, T_2, K) \geq C(0, T_1, K).
\]

"\( t=0 \) was not special" argument:\n
\[C(t, T_2, K) \geq C(t, T_1, K), 0 < t < T_2.\]
3.2 Random variables, probability and mean
Review of some probability: uniform, normal & lognormal

Prob. density for f has properties, \( b \geq 0 \) and \( \int_{-\infty}^{\infty} f \, dx = 1 \).

\[ P(A \leq X \leq B) = \int_{A}^{B} f(x) \, dx \]

Let \( X \) be (cts) random variable with p. density \( f \)

Then:

\[ E[X] = \int_{-\infty}^{\infty} x \cdot f(x) \, dx \]

Let \( \bar{X} = E[X] \). Then

\[ E[X - \bar{X}] = \int_{-\infty}^{\infty} (x - \bar{X}) \cdot f(x) \, dx \]

\[ = \int_{-\infty}^{\infty} x \cdot f(x) \, dx - \bar{X} \cdot \int_{-\infty}^{\infty} f(x) \, dx \]

\[ = 0. \]
Also:
\[ \text{var}(x) = E[(x - \bar{x})^2] = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x) \, dx. \]

Let \( 6_x^2 = \text{var}(x) \): \( 6_x = \sqrt{\text{var}(x)} \geq 0 \).

Consider \( \frac{x - \bar{x}}{6_x} \).

This is a random variable with mean (or expected value)
\[ E\left[ \frac{x - \bar{x}}{6_x} \right] = \frac{1}{6_x} E[ x - \bar{x}] = 0. \]

and variance
\[ \text{var}\left[ \frac{x - \bar{x}}{6_x} \right] = E\left[ \frac{(x - \bar{x})^2}{6_x^2} \right] \]
\[ = \frac{1}{6_x^2} E[(x - \bar{x})^2] = \frac{\text{var}(x)}{6_x^2} = \frac{6_x^2}{6_x} = 1. \]
Consider \( f = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases} \). Let \( X \) be r.v. with p.d.f. \( f \):

\[
\text{Then: } P(A \leq X \leq B) = \int_A^B f(x) \, dx.
\]

Let \( Y \) be the r.v. such that \( Y = F(X) \). Let \( a = F^{-1}(A) \), \( b = F^{-1}(B) \):

\[
A = F^{-1}(a), \quad B = F^{-1}(b).
\]

\[
P(A \leq X \leq B) = P(a \leq Y \leq b)
\]

\[
\int_A^B f(x) \, dx = F(B) - F(A)
\]

\[
= b - a.
\]

\[
P(a \leq Y \leq b) = b - a,
\]

means \( Y \) is a uniform random variable.
3.6 Central Limit Theorem
why does normal dist. keep cropping up?

Central Limit Theorem:

Let $X_1, X_2, X_3, ...$ be a sequence of independent, identically distributed r.v.'s, each with mean $\mu$ and std. dev. $\sigma$.

Let $S_n = X_1 + X_2 + \ldots + X_n$.

Then for $S_n$, mean $n\mu$ and std. dev. is $\sqrt{n}\sigma$.

$$P \left[ \frac{S_n - n\mu}{\sqrt{n}\sigma} < x \right] \rightarrow N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} \, ds,$$

cumulative density

for std. normal.

The central limit theorem

can be used to show binomial tree pricing, in limit $\Delta t \to 0$,
gives same Black-Scholes result.
 Lehmer random number generator using:

\[ X_{k+1} = a \times X_k \mod m, \quad m = 2^{31} - 1 \]

\[ a = 48271 \]

to generate integers. \( 0 < X_k < 2^{31} \)

to generate pseudo-random uniform \((0, 1)\) values,

\[ \{ \frac{X_k}{2^{31}} \}, \quad k = 1, 2, 3, \ldots \]

Issue: seed \( X_1 = ? \)

possible: use some 32-bit integer

derived from "unpredictable" source

or use random.org data.
Links visited in class

Googled the number 48271: found
Lehmer random number generator,
random.org,
random.org/decimal-fractions/,
generated 100 random decimal fractions,
nsm.buffalo.edu/~hassard/459/py,
nsm.buffalo.edu/~hassard/459/py/uniform.html,
nsm.buffalo.edu/~hassard/459/py/uniform_to_others.html
4.3 Statistical tests
ppf: percentage point function for a prob. density fn $f$:

given $f$: $f \geq 0$, $\int_{-\infty}^{\infty} f(x) \, dx = 1$.

Defn: given $p$, $0 < p < 1$, the $p^{th}$ quantile of $f$

is $z(p)$ such that $\int_{-\infty}^{z(p)} f(x) \, dx = p$ (i.e. ch 4).

Using previous defs:

had $F(x) = \int_{-\infty}^{x} f(s) \, ds$;

solve $F(z(p)) = \int_{-\infty}^{z(p)} f(s) \, ds = p$ by find $z(p)$:

:. $z(p) = F^{-1}(p)$, where $F^{-1}$ is inverse of the cumulative distribution fn.

Different names: percentage point fn,

$p^{th}$ quantile,

inverse of cumulative distrib. fn,

for same expression.
Quantiles:

Example Deciles

Choose $p = 0.1, 0.2, 0.3, \ldots, 0.9$.

Then values $x = z(p) = z(0.1), z(0.2), z(0.3), \ldots, z(0.9)$ divide $x$-axis.

Fig. 4.3. Asterisks on the $x$-axis mark the quantiles $z(k/(M + 1))$ in (4.6) for an $N(0, 1)$ distribution using $M = 9$. Upper picture: the quantiles break the $x$-axis into regions where $f(x)$ has equal area. Lower picture: equivalently, the quantiles break the $x$-axis into regions where $N(x)$ has equal increments.
The quantile-quantile idea:

to see if random samples obey p.d.f. \( f \):

Given points \( \hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n \)

a) sort into increasing order

\( \hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n \)

b) plot \( \hat{r}_k \) versus \( \hat{z}(\frac{k}{n+1}) \), \( k = 1, 2, \ldots, n \)

idea:

evenly spaced values of \( p \).

if \( \hat{r}_k \) are random with p.d.f. \( f \),

expect linear-looking plot.

\[ 
\begin{array}{c|c}
\text{N}(0,1) \text{ samples and } \text{N}(0,1) \text{ quantiles} & \text{N}(0,1) \text{ samples and } \text{U}(0,1) \text{ quantiles} \\
\hline
\text{U}(0,1) \text{ samples and } \text{N}(0,1) \text{ quantiles} & \text{U}(0,1) \text{ samples and } \text{U}(0,1) \text{ quantiles}
\end{array}
\]
Links visited in class

Lehmer Park-Miller pseudo-random numbers seed based on letter E
scipy reference stats norm for vocabulary ppf (percent point function), pdf, cdf
5.0 Asset price movement
Ch. 5 Asset price movement

Idea: Efficient Market Hypothesis.

the current asset price reflects all past information.

so: the knowing past history of asset price gives no advantage

over just knowing current price

No edge to 'reading the charts'.

In a model to describe evolution of asset from \( t \) to \( t + \Delta t \)

need only consider asset at \( t \), not earlier times.

Daily return: 
\[
r_i = \frac{S(t_{i+1}) - S(t_i)}{S(t_i)}
\]
daily

Log returns:  
\[
\log \left( \frac{S(t_{i+1})}{S(t_i)} \right) = \ln (1 + r_i) = r_i - \frac{r_i^2}{2} + \ldots
\]
Assumptions: \( S(t) \geq 0 \) (normally \( S(t) > 0 \)).

- Buying and selling any time \( t, \ 0 \leq t \leq T \).
- may buy & sell any amount.
- bid-ask spread is 0.
- no transaction costs
- no dividends (for now) or splits.
- short selling allowed
- there is a single, constant risk-free interest rate for any amount, deposited in/borrowed from "ideal" bank
6.0 Asset price model: Part I
Asset models: I.

A risk-free investment: \( E \) to \( t + \Delta t \) \( \text{(Text: \( s \))} \)

\[
P(t + \Delta t) = P(t) + (\gamma \Delta t) P(t)
\]

An asset with an unpredictable component of growth:

\[
S(t + \Delta t) = S(t) + (\gamma \Delta t) S(t) + (b \sqrt{\Delta t} \gamma_i) S(t)
\]

\[
= S(t)(1 + \gamma \Delta t + b \sqrt{\Delta t} \gamma_i)
\]

\( \gamma_i \) random var;

\[
E[\gamma_i] = 0 \quad \text{and} \quad \text{var}(\gamma_i) = 1
\]

where: \( \gamma \) represents "drift" of asset;

\( b \geq 0 \) controls strength of fluctuation: volatility.

\( \sqrt{\Delta t} \) has \( J \) because only sensible power:

\[
S(t) = S(0) \prod_{i=0}^{M-1} (1 + \gamma \Delta t + b \sqrt{\Delta t} \gamma_i) \quad t = M \Delta t
\]

\[
\text{log:} \quad \log(s(t)/s(0)) = \sum_{i=0}^{M-1} \log(1 + \gamma \Delta t + b \sqrt{\Delta t} \gamma_i)
\]

\[
\log = \ln \quad \text{natural log}
\]
For small \( \Delta t \), \( \mu \Delta t + 6 \sqrt{\Delta t} \tilde{\varepsilon}_i \) is small:

\[
\text{but } \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
\]

\[
\log \left( \frac{s(t)}{s(0)} \right) \approx \sum_{i=0}^{M-1} \mu \Delta t + 6 \sqrt{\Delta t} \tilde{\varepsilon}_i - \frac{1}{2} \left( 6 \Delta t \tilde{\varepsilon}_i^2 + O((\Delta t)^{3/2}) \right)
\]

\[
= \sum_{i=0}^{M-1} \mu \Delta t + 6 \sqrt{\Delta t} \tilde{\varepsilon}_i - \frac{1}{2} 6^2 \Delta t \tilde{\varepsilon}_i^2.
\]

\[
\mathbb{E} \left[ \mu \Delta t + 6 \sqrt{\Delta t} \tilde{\varepsilon}_i - \frac{1}{2} 6^2 \Delta t \tilde{\varepsilon}_i^2 \right] = \mu \Delta t + 6 \sqrt{\Delta t} \mathbb{E} [\tilde{\varepsilon}_i] - \frac{1}{2} 6^2 \Delta t \mathbb{E} [\tilde{\varepsilon}_i^2]
\]

\[
= \mu \Delta t - \frac{1}{2} 6^2 \Delta t.
\]

Also:

\[
\text{var} \left( \mu \Delta t + 6 \sqrt{\Delta t} \tilde{\varepsilon}_i - \frac{1}{2} 6^2 \Delta t \tilde{\varepsilon}_i^2 \right) = 6^2 \Delta t + O((\Delta t)^{3/2})
\]

\[
\text{Sum} = \sum_{i=0}^{M-1} \mu \Delta t + 6 \sqrt{\Delta t} \tilde{\varepsilon}_i - \frac{1}{2} 6^2 \Delta t \tilde{\varepsilon}_i^2 \] should behave like a normal distribution.

Central limit theorem.
\[ E \left[ \text{Sum} \right] = M(\mu + \frac{1}{2} \sigma^2 \Delta t) = (\mu - \frac{1}{2} \sigma^2)T \]

\[ \text{var} \left[ \text{Sum} \right] = M \left( \sigma^2 \Delta t + O(\Delta t^{\frac{3}{2}}) \right) = \sigma^2 T + O(\Delta t^{\frac{1}{2}}) \]

\[ \log \left( \frac{S(T)}{S(0)} \right) \text{ should behave like:} \]

Normal dist., mean \((\mu - \frac{1}{2} \sigma^2)T\), var \(\sigma^2 T\). ←

Let \( Z \) be a. n., std. normal.

Then: \((\mu - \frac{1}{2} \sigma^2)T + \sigma \sqrt{T}Z\) has these properties.

We may adopt model

\[ \log \left( \frac{S(T)}{S(0)} \right) = (\mu - \frac{1}{2} \sigma^2)T + \sigma \sqrt{T}Z, \] where \( Z \) is std. normal.

Take exp.:

\[ \frac{S(T)}{S(0)} = e^{(\mu - \frac{1}{2} \sigma^2)T + \sigma \sqrt{T}Z} \]

By "\( T = 0 \) is not special" argument,

\[ \frac{S(t_2)}{S(t_1)} = e^{(\mu - \frac{1}{2} \sigma^2)(t_2 - t_1) + \sigma \sqrt{t_2 - t_1}Z} \quad \text{exp} \quad t_2 = T, \ t_1 = t. \]

\( T - t = \text{time to expiration} \)
Model $S(T) = S(0) e^{\mu T + \sigma \sqrt{T} Z}$ where $Z$ std. normal r.v., $\sigma = \mu - \frac{\sigma^2}{2}$ const.

Then

$$P(S(T) \leq B) = P(Z \leq b)$$

where $B$ and $b$ are related by

$$B = S(0) e^{\mu T + \sigma \sqrt{T} b}$$

But $Z$ is a std. normal r.v., so $P(Z \leq b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{x^2}{2}} dx$.

\therefore \quad P(S(T) \leq B) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{x^2}{2}} dx.$$

Change integration variables: let

$$y = S(0) e^{\mu T + \sigma \sqrt{T} x},$$

solving for $x$

$$x = \frac{\ln(y/S(0)) - \mu T}{\sigma \sqrt{T}}.$$ 

As $x \to -\infty$, $y \to 0$, and (after some work),

$$P(S(T) \leq B) = \int_{y = 0}^{B} \sqrt{2\pi} e^{-\frac{1}{2} \left( \ln\left( \frac{y}{S(0)} - \mu T \right) \right)^2 / 6\sigma^2} dy$$

provides density for a lognormal distribution.
8.5 Black-Scholes formulas
Recall last dy.

Model \( S(t) = S(0) e^{\mu t} + 6\sqrt{\mu} z \)

where \( z \) std. normal r.v.,
\[ \mu = \frac{\mu - \frac{b^2}{2}}{\text{const.}} \]

Then
\[ P(S(t) < B) = P(z < b) \]

where \( B \) and \( b \) are related by
\[ B = S(0) e^{\mu t} + 6\sqrt{\mu} z \]

But \( z \) is a std. normal r.v., so
\[ P(z < b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{x^2}{2}} dx \]

\[ \therefore P(S(t) < B) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{b}{\sqrt{\mu} z}} e^{-\frac{y^2}{2}} dy \]

Change integration variables: Let
\[ y = S(0) e^{\mu t} + 6\sqrt{\mu} x \]
\[ \therefore x = \frac{\ln(y/S(0)) - \mu t}{6\sqrt{\mu}} \]

As \( x \to -\infty \), \( y \to 0 \), and \( (\text{after some work}) \)
\[ P(S(t) < B) = \int_{y=0}^{\frac{B}{\sqrt{2\pi}}} \frac{e^{-\frac{1}{2} \left( \ln \left( \frac{y}{S(0)} - \mu t \right) \right)^2 / 6^2}}{6\sqrt{\mu} y} dy \]

prob. density for a lognormal distribution
Let $h_N = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (\ln\left(\frac{y}{S_0}\right) - \mu T)^2 / \sigma^2 T} \frac{1}{\sqrt{T}} y & y > 0 \\ 0 & y \leq 0. \end{cases}$

Suppose a derivative asset, pays off $\Lambda(S(T))$ at time $T$

Q: what is value at $t=0$, of this derivative asset?
A: use discounted expected value $e^{-rT} E[\Lambda(S(T))]$

discount factor for $0$ to $T$

Q: what should we use for the growth parameter $\nu$?
(Or: $\nu = \rho - \frac{\sigma^2}{2}$; what should we use for $\rho$?)
A: in Black-Scholes "world", use $\nu$ such that $e^{-rT} E[S(T)] = S(0)$. in B-S "world", assets grow at same rate.
Let $\varepsilon$ be std. normal r.v.: then $E[e^{\varepsilon\varepsilon}] = e^{\frac{b^2}{2}}$.

\[ E[e^{\varepsilon\varepsilon}] = \int_{-\infty}^{\infty} e^{\frac{b^2}{2}x^2} e^{-\frac{1}{2}x^2} dx \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\kappa)^2} dx \]

Rule for completing square: $\kappa - 2\kappa x = (x-c)^2 - c^2$

Here, choose $c = b$.

\[ E[e^{\varepsilon\varepsilon}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}((x-b)^2 - b^2)} dx \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-b)^2} dx \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}b^2} \int_{-\infty}^{\infty} e^{-s^2/2} ds \]

Let $s = x - b$,

\[ \text{then: } s \text{ goes from } -\infty \text{ to } \infty \]

\[ = e^{\frac{b^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2/2} ds = e^{\frac{b^2}{2}}. \]
Find $\gamma$ such that $e^{-\gamma T}E[S(T)] = S(0)$,

where $S(T)$ given by lognormal density.

$S(T) = S(0)e^{\gamma T + b\sqrt{T}Z}$, $Z$ std normal,

$E[S(T)] = S(0)E[e^{\gamma T + b\sqrt{T}Z}]$

$= S(0)e^{\gamma T}E[e^{b\sqrt{T}Z}]$

$= S(0)e^{\gamma T}e^{b^2/2}$, where $b = b\sqrt{T}$

$E[S(T)] = S(0)e^{\gamma T}e^{b^2T/2}$

$e^{-\gamma T}E[S(T)] = S(0)e^{(\gamma + \frac{b^2}{2} - \gamma)T} = S(0)$

True if $\gamma = \mu - \frac{\sigma^2}{2}$

But $\gamma = \mu - \frac{\sigma^2}{2}$; true if $\mu = \gamma$. 
Common payoff functions $\Lambda(s)$

- Euro Call

- Cash-or-nothing call

- Asset-or-nothing call

- Euro Put

- Cash-or-nothing put

- Asset-or-nothing put
To work B-S value for a cash-or-nothing call:

\[
\Lambda(y) = \begin{cases} 
1 & \text{if } y > K \\
0 & \text{if } y \leq K
\end{cases}
\]

\[
W(S_0, 0) = \frac{1}{\sqrt{2\pi} \sigma \sqrt{T}} \int_{-\infty}^{\infty} 1 \cdot e^{-\frac{1}{2} \frac{(\ln S_0 - \mu T)^2}{\sigma^2}} \, dy
\]

\[
S_0 = S(0)
\]

Let: \( \frac{\ln S_0 - \mu T}{\sigma \sqrt{T}} = \kappa \), use \( \kappa \) as new integration variable:

\[
\ln y = \sigma \sqrt{T} \kappa + \ln S_0 + \mu T
\]

\[
\frac{d}{d\kappa} \left( \frac{dy}{y} \right) = \sigma \sqrt{T} \cdot 1 + 0 \quad \frac{dy}{y} = \sigma \sqrt{T} \, d\kappa
\]

\[
W(S_0, 0) = \frac{1}{\sqrt{2\pi} \sigma \sqrt{T}} \int_{-\infty}^{\infty} 1 \cdot e^{-\frac{1}{2} \kappa^2} \cdot \frac{\kappa}{\sigma \sqrt{T}} \, d\kappa
\]

\[
\kappa = \frac{\ln K - \ln S_0 - \mu T}{\sigma \sqrt{T}}
\]

But: \( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\kappa} e^{-\frac{x^2}{2}} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-A}^{\kappa} e^{-\frac{x^2}{2}} \, dx = N(-A), \quad N = \text{prob. dist. for standard normal}
\]

\( e^{-\frac{x^2}{2}} \) is even.
\[ W(s, v) = e^{-rT} \frac{\ln(S_0/K) + (r - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \]

Formula for cash-or-nothing call is:

\[ C_{con} = e^{-rT} \frac{\ln(S_0/K) + (r - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \]

But \( t=0 \) is not special:

The value at time \( t, \ 0 < t < T \),
when asset has value \( S = S(t) \),
of cash-or-nothing call, is obtained by replacing \( T \) with \( T-t \) and \( S_0 \) with \( S = S(t) \):

\[ C_{con}^*(S, t) = e^{-r(T-t)} \frac{\ln(S/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = d_2(S, K, T, t, r, \sigma) \]
8.5 Black-Scholes formulas
Black-Scholes value of a derivative asset that pays $\Lambda(S(T))$ is: 

$$e^{-rT}E[\Lambda(S(T))], \quad S(T) = S_0 e^{\mu T + \sigma \sqrt{T} \, \mathcal{Z}}, \quad \mathcal{Z \text{ std. normal}}, \quad \nu = \frac{\sigma^2}{2}.$$ 

Eur. Call which pays $\Lambda(y) = \begin{cases} y - K, & y > K \\ 0, & y \leq K \end{cases}$

Value at $t=0$ of Eur. Call:

$$C_{\text{euro}} = e^{-rT}E[\Lambda(S_0 e^{\mu T + \sigma \sqrt{T} \mathcal{Z}})] = e^{-rT} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \Lambda(S_0 e^{\mu T + \sigma \sqrt{T} x}) \, dx.$$ 

Let $A$ be such that $S_0 e^{\mu T + \sigma \sqrt{T} x} = K$.

For $x \leq A$, $S_0 e^{\mu T + \sigma \sqrt{T} x} \leq K$ so $\Lambda(S_0 e^{\mu T + \sigma \sqrt{T} x}) = 0$.

For $x > A$, $S_0 e^{\mu T + \sigma \sqrt{T} x} > K$ so $\Lambda(S_0 e^{\mu T + \sigma \sqrt{T} x}) = S_0 e^{\mu T + \sigma \sqrt{T} x} - K > 0$.

$$C_{\text{euro}} = e^{-rT} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} (S_0 e^{\mu T + \sigma \sqrt{T} x} - K) \, dx.$$ 

$$= e^{-rT} \int_{-\infty}^{A} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx - K e^{-rT} \int_{A}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx.$$
Complete the square:

\[-\frac{x^2}{2} + 6\sqrt{T}x = -\frac{1}{2} \left( x^2 - 2 \cdot 6\sqrt{T}x \right) = -\frac{1}{2} \left( (x - 6\sqrt{T})^2 - 6^2 \cdot T \right) : \quad x = A \]

\[x' = A - 6\sqrt{T} \]

\[C_{\text{accum}} = e^{-\frac{x'^2}{2}} \mathcal{N}(x') - K_e^{-\frac{\pi T}{2}} \mathcal{N}(A - 6\sqrt{T}) \]

\[= e^{-\left( \frac{\pi T + \alpha^2 + \frac{1}{2}6^2T}{2} \right)} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left( (x')^2 - 6^2 \cdot T \right)} \frac{dx'}{\sqrt{2\pi}} \]

\[= \int_{-\infty}^{\infty} e^{-\frac{(x')^2}{2}} \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}} \frac{dx'}{\sqrt{2\pi}} - Ke^{-\pi T} \int_{-\infty}^{\infty} \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}} \frac{dx}{\sqrt{2\pi}} \]

\[C_{\text{accum}} = \mathcal{N}(A - 6\sqrt{T}) - K_e^{-\pi T} \mathcal{N}(A) \]

\[= \mathcal{N}(d_1) - K_e^{-\pi T} \mathcal{N}(d_2), \text{ where } d_1 = -A + 6\sqrt{T} \]

\[d_2 = -A. \]
Recall payoffs $\Lambda^\text{con}(y) = \begin{cases} 1 & \text{if } y > K \\ 0 & \text{if } y \leq K. \end{cases}$

Cash-or-nothing

$$
\Lambda^\text{con}(y) = \begin{cases} y & \text{if } y > K \\ 0 & \text{if } y \leq K. \end{cases}
$$

Asset-or-nothing

Observe: $\Lambda^\text{euro}(y) = \begin{cases} y - K & \text{if } y > K \\ 0 & \text{if } y \leq K \end{cases}$

euro call

$$
\Lambda^\text{euro}(y) = \Lambda^\text{con}(y) - K \Lambda^\text{con}(y_0).
$$

can interpret

$$
C^\text{euro} = C^\text{aon} - K C^\text{con}.
$$

Also may interpret $N(d_2)$ as prob. that $S(T) > K$. 

Let \( W(S_0, 0) \) and \( W(S, t) \) represent values of a derivative asset:

Interpret: \( W(S_0, 0) \) is value at \( t=0 \), of derivative asset that pays \( \Lambda(S(T)) \) at time \( T \), given that asset at \( t=0 \), is \( S_0 \).

Time to expiration is \( T \):

\[
W(S_0, 0) = e^{-rT} E[\Lambda(S(T))].
\]

Interpret \( W(S, t) \): value at \( t \), of derivative asset that pays \( \Lambda(S(T)) \) at time \( T \), given that asset at \( t \), is \( S \); time to expiration is \( T-t \):

\[
W(S, t) = e^{-r(T-t)} E[\Lambda(S(T))].
\]
"t=0 is not special" means:

can repeat any argument that involved t=0,

but start at t instead:

requires replacing t=0 with t,

T with T-t

time to expiration.

So:

\[ d_1 = \frac{\ln \left( \frac{S_0}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \]

becomes

\[ d_1 = \frac{\ln \left( \frac{S_0}{K} \right) + \left( r + \frac{\sigma^2}{2} \right)(T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = \frac{\ln \left( \frac{S_0}{K} \right) + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \]

becomes

\[ d_2 = \frac{\ln \left( \frac{S_0}{K} \right) + \left( r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma \sqrt{T-t}} \]

Formula for \( C \) becomes

\[ C \text{ is } (S, t, K, r, \sigma, T) = SN(d_1) - Ke^{-r(T-t)} N(d_2), \]

\( d_1, d_2 \) as above.