Math 306 Section T

Midterm Exam 2

April 9, 2015

Name: Solution Key

Problem	Points	Maximum		Problem	Points	Maximum
1		6		4		14
2		10		5		10
3		10		6		10
Subtotal		26		Subtotal		34
				Total		60

Please read the problems carefully and indicate your solutions clearly!
No credit awarded for unclear answers or unclear work.

1. (6 points) Find the general solution to the differential equation:

$$y'' - 2y' + 10y = 0$$

Char eqn: $r^2 - 2r + 10 = 0$

$r = \frac{2 \pm \sqrt{4 - 40}}{2} = 1 \pm \frac{1}{2}\sqrt{36}\, i = 1 \pm 3i$

$$\boxed{y(t) = c_1 e^t \sin(3t) + c_2 e^t \cos(3t)}$$

2. (10 points) Solve the initial value problem:

$$y^{(3)} - 2y'' + y' = 0$$

$$y(0) = 5 \quad y'(0) = 3 \quad y''(0) = 4$$

Char eqn: $r^3 - 2r^2 + r = r(r^2 - 2r + 1) = r(r-1)^2$ $\quad r = 0, 1$

$y(t) = c_1 e^{0t} + (c_2 + c_3 t)e^t$

$= c_1 + c_2 e^t + c_3 t e^t$

ICs:

$y(0) = c_1 + c_2 = 5$

$y'(0) = c_2 + c_3 = 3$

$y''(0) = c_2 + 2c_3 = 4$

$y' = c_2 e^t + c_3 e^t + c_3 t e^t$

$y'' = (c_2 + c_3)e^t + c_3 e^t + c_3 t e^t$

Solve: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$c_3 = 1$

$c_2 = 3 - c_3 = 2$

$c_1 = 5 - c_2 = 3$

$y(t) = 3 + 2e^t + te^t$

3. (10 points) Find the general solution to the differential equation:

$$y''(t) - y(t) = 12e^{2t} - t^2$$

For y_c, char eqn $= r^2 - 1 = (r+1)(r-1) \Longrightarrow y_c(t) = c_1 e^t + c_2 e^{-t}$

$y_p = At^2 + Bt + C + De^{2t}$

$y_p' = 2At + B + 2De^{2t}$

$y_p'' = 2A + 4De^{2t}$

$$y_p'' - y_p = \underbrace{3D}_{=12} e^{2t} + \underbrace{(2A - C)}_{=0} + \underbrace{(-B)}_{=0} t + \underbrace{(-A)}_{=-1} t^2 = 12e^{2t} - t^2$$

$\Longrightarrow$ $D = 4$ $\quad C = 2A = 2$ $\quad B = 0$ $\quad A = 1$

$y_p = t^2 + 2 + 4e^{2t}$

SOLN:

$y(t) = c_1 e^t + c_2 e^{-t} + t^2 + 2 + 4e^{2t}$

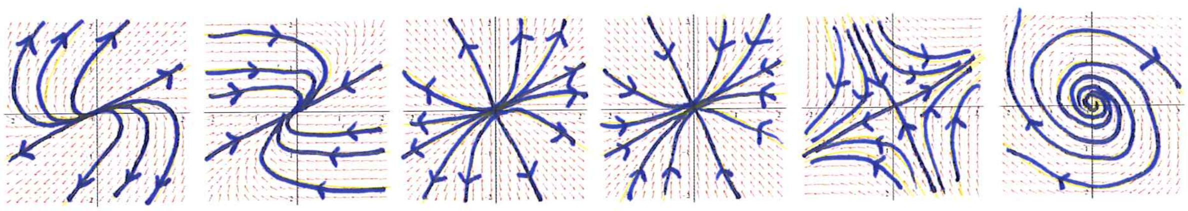
4. (14 points) Consider the system of DEs:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}}_{A} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

(a) Find the general solution.

(b) Find the solution satisfying ICs $x(0) = 2$ and $y(0) = 1$.

(c) Which is the phase portrait for this sytem? Indicate answer here: A



(A) (B) (C) (D) (E) (F)

ⓐ $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ -1 & 5-\lambda \end{vmatrix} = 5 + \lambda^2 - 6\lambda - (-4) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$

$\lambda = 3$ evec: $\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x = 4y \Rightarrow x = 2y$ use $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ evec.

(same equation (bottom row is top/2))

Need chain, use $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (not a multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ so ok)

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = u_2$$

gen soln $y(t) = c_1 \begin{bmatrix} -2 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} -2 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{3t}$

ⓑ ICs $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ falls on eigenvector; use straight-line soln:

$y(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}$ (alternately, solve for $c_1 = 1$, $c_2 = 0$)

ⓒ arrows point outward because $\lambda = 3 > 0$

only one pair of straight trajectories & defect = 1 (improper node)

so not (C), (D), or (E), & not spiraling like (F)

5. (10 points) Consider the system of DEs below and the given solution:

$$\mathbf{v}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\mathbf{v}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

(a) Explain why $\mathbf{v}(t)$ is not the *most general* solution (although it is a solution).

(b) Find the correct general solution.

(a) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ NOT linearly indep.

can rewrite $v(t) = \underbrace{(c_1 + 2c_2)}_{b_1} e^{-2t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = b_1 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

only two lin indep. solutions shown.

(b) $\begin{vmatrix} (-1-\lambda) & 1 & 0 \\ 1 & (-1-\lambda) & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)\left[(-1-\lambda)^2 - 1\right] = (\lambda+2)(-1)\left[\lambda^2 + 2\lambda + 1 - 1\right]$

$= (\lambda+2)(-1)(\lambda^2 + 2\lambda) = (\lambda+2)(-1)(\lambda+2)\lambda \overset{\text{Set}}{=} 0$

eigenvalues $\lambda = -2, 0$

Case $\lambda = -2$ find eigenvec:

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies$ $x = -y$, z = anything

eigenvec form $\begin{bmatrix} s \\ -s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

lin indep solns: $c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-2t}$

Case $\lambda = 0$ find eigen vec:

$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies$ $x = y$, $\frac{x+y}{2} = \frac{2z}{2}$

eigenvec form $\begin{bmatrix} s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ soln $c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

GEN SOLN: $y = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

6. (10 points) Recall that simple harmonic motion is modeled by $mx''(t) + kx(t) = 0$, where $k, m > 0$, and it is convenient to define $\omega = \sqrt{k/m}$.

(a) Rewrite this 2nd-order DE as a system of two 1st-order DEs.

(b) Use the eigenvalue method to find the general solution to your answer for (a).

$x''(t) + \frac{k}{m} x(t) = 0$ or $x'' + \omega x = 0$

let $x = x$, $y = x'$ get $x' = y$, $y' = -\omega x$ (since $y' = x''$)

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{vmatrix} = \lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm \omega i$

Case $\lambda = \omega i$, find eigenvec: $\begin{bmatrix} -\omega i & 1 \\ -\omega^2 & -\omega i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow -\omega i x + y = 0$, $y = \omega i x$

notice bottom row is $-\omega i \times$ top row

eigenvec form $s \begin{bmatrix} 1 \\ \omega i \end{bmatrix}$ let $s = 1$ get $\begin{bmatrix} 1 \\ \omega i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ \omega \end{bmatrix}$

soln: $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ \omega \end{bmatrix} \right) \underbrace{\left(\cos(\omega t) + i \sin \omega(t) \right)}_{= e^{i\omega t}} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(\omega t) - \begin{bmatrix} 0 \\ \omega \end{bmatrix} \sin(\omega t) \right) + i \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(\omega t) + \begin{bmatrix} 0 \\ \omega \end{bmatrix} \cos(\omega t) \right)$

(b) soln to (a) is

$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \left(\begin{bmatrix} \cos \omega t \\ -\omega \sin \omega t \end{bmatrix} \right) + c_2 \left(\begin{bmatrix} \sin \omega t \\ \omega \cos \omega t \end{bmatrix} \right)$