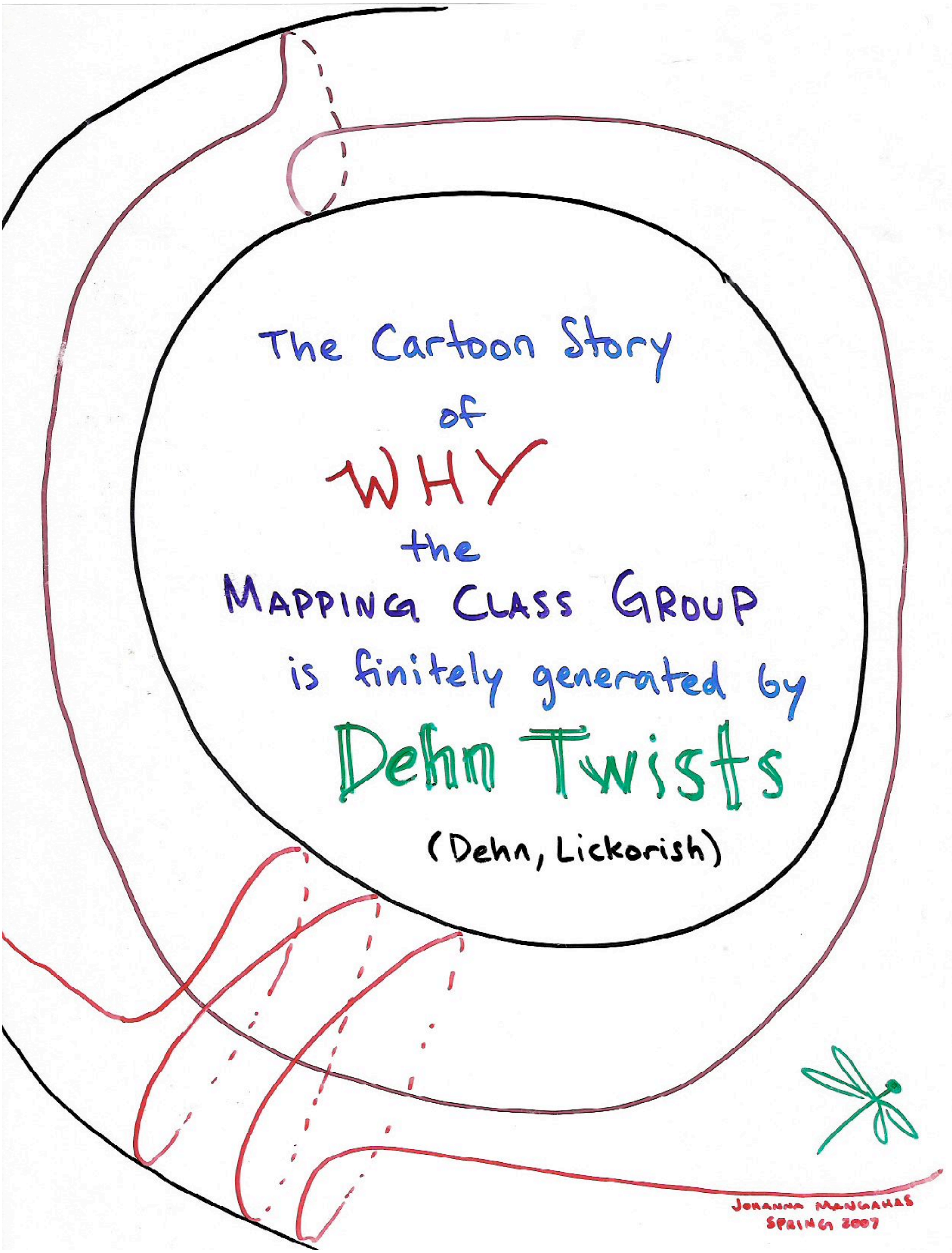
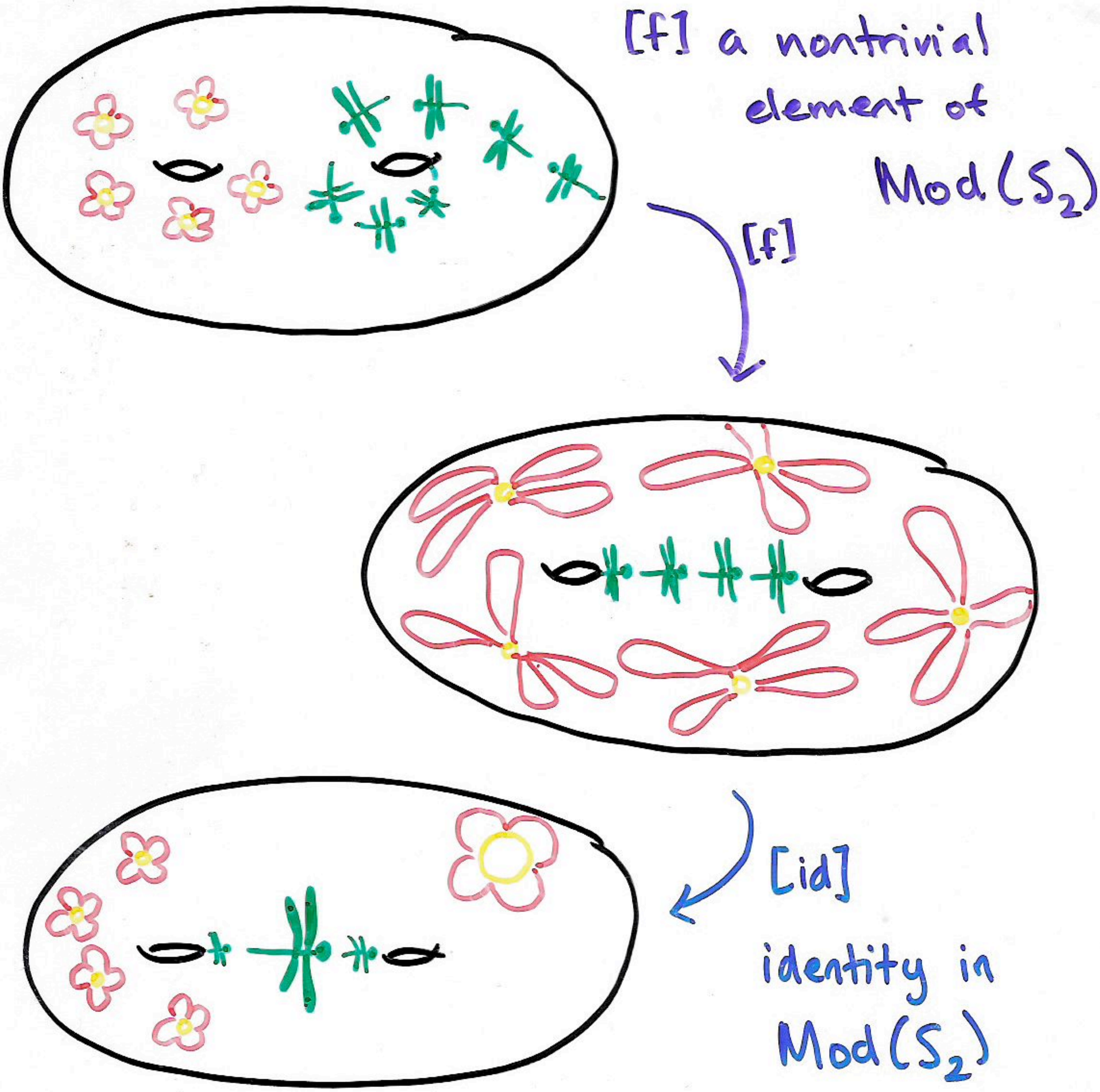


The Cartoon Story of WHY the Mapping Class Group is finitely generated by Dehn Twists

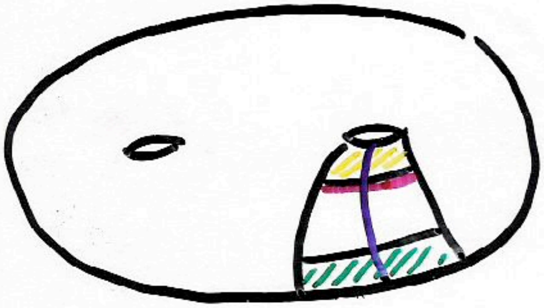
(Dehn, Lickorish)

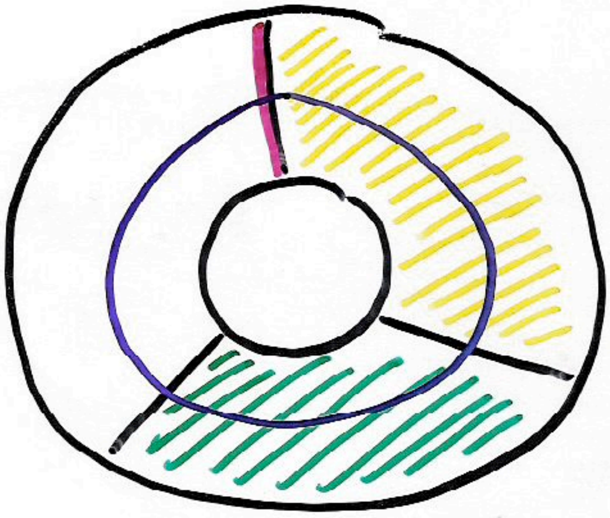
Johanna Mangahas
Spring 2007

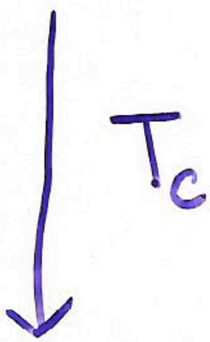
$\mathrm{Mod}(S_g)$: isotopy classes of orient.-preserv. self-homeos of the surface of genus g

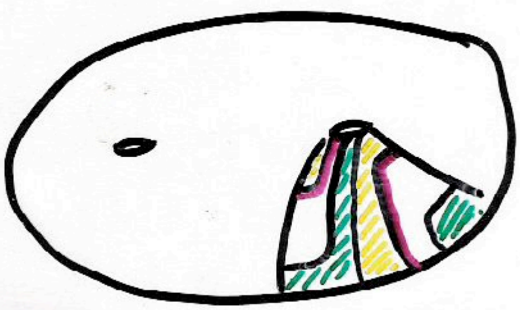


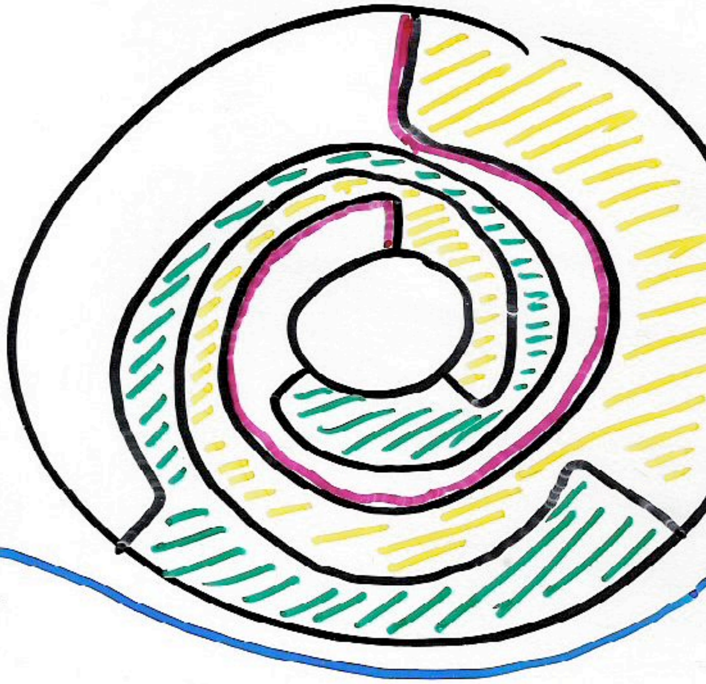
a Dehn twist T_c about curve 





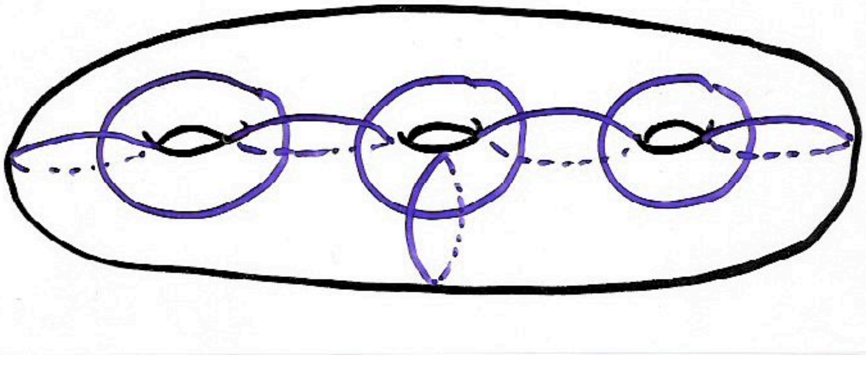




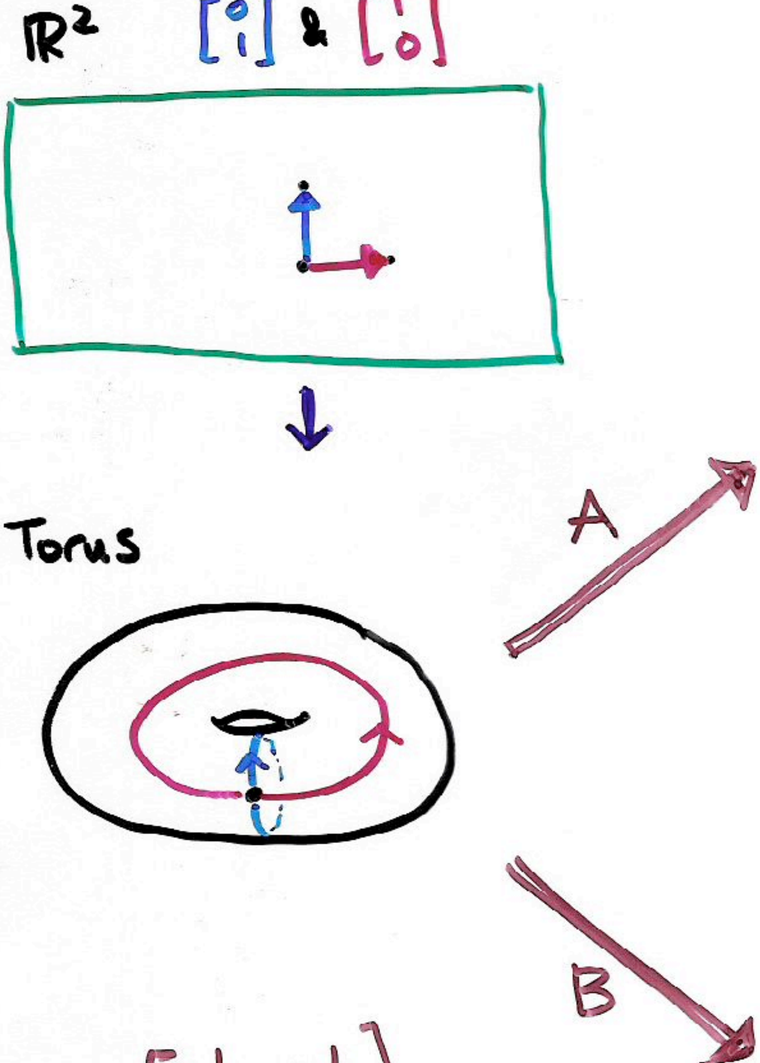


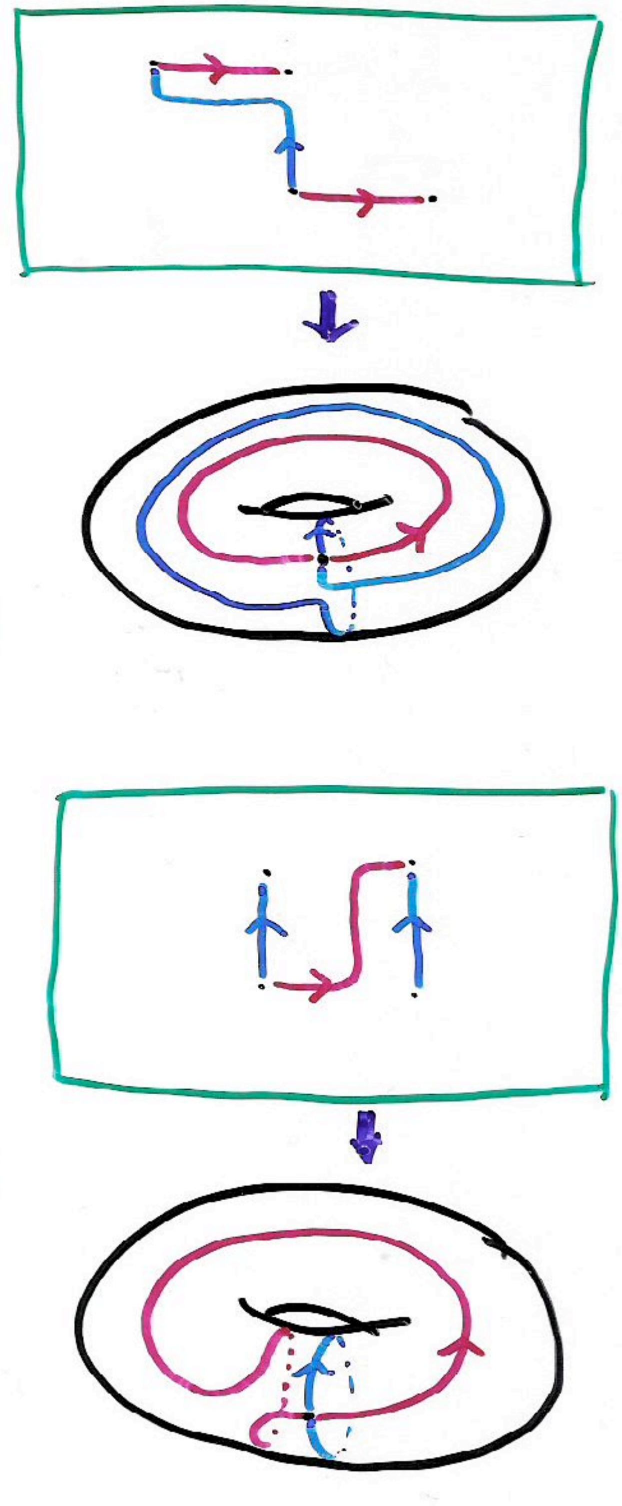
The red arc "turns left" upon meeting c

A generating set for $\mathrm{Mod}(S_3)$:



$\mathrm{Mod}(T) = SL_2(\mathbb{Z})$





$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

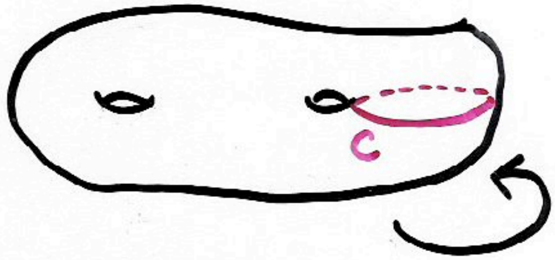
THM: $\mathrm{Mod}(S_g)$ finitely generated by Dehn twists

Proof: induct on genus.

Base case: $\mathrm{Mod}(T)$ generated by two twists

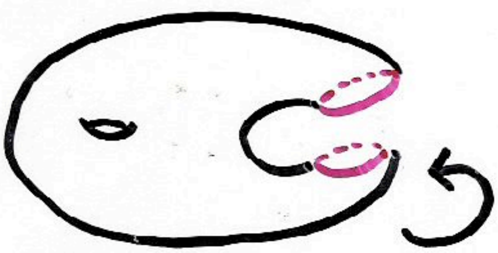
General case:

STEP ① Reduce to f.g.-by-D.t. of $H < \mathrm{Mod}(S_g)$



H := elements that fix a particular nonseparating curve c.

STEP ② Reduce to f.g.-by-D.t. of $\mathrm{PMod}(S_c)$:



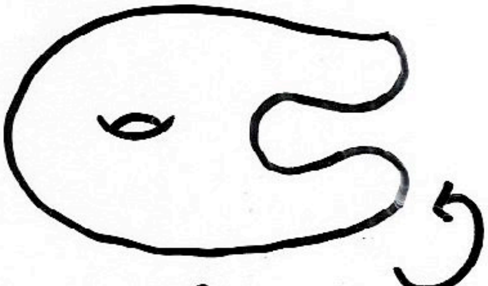
S_c

S_c is S_g cut along c

$\mathrm{PMod}(S_c)$ are isotopy classes of homeos $S_c \circlearrowleft$ that do not permute boundary of S_c.

STEP ③ Reduce to f.g.-by-D.t. of $\mathrm{Mod}(S_{g-1})$

completes induction!



S_{g-1}

Step 1: A group action <u>lemma</u>. Suppose group G acts on connected graph X, D is a subgraph & D-translates cover X.

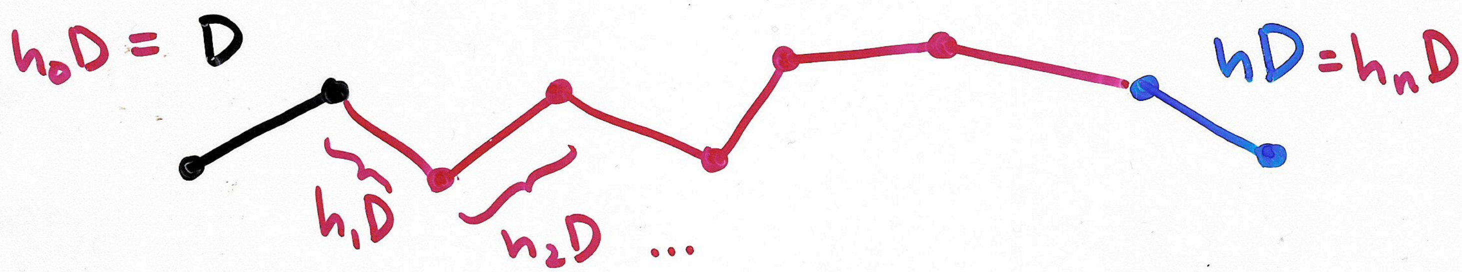
i.e. $G \curvearrowright X$, $D \subset X$, $X = \bigcup_{g \in G} gD$

Then the elements that don't move D off itself generate G, i.e. $G = \langle \mathcal{A} \rangle$, $\mathcal{A} = \{ g \in G \mid gD \cap D \neq \emptyset \}$

<u>proof</u>:

arbitrary $h \in G$. Some path connects D to hD.

Cover by D-translates:



$h_0 = id \in \langle \mathcal{A} \rangle$. $\quad h_{i-1} D \cap h_i D \neq \emptyset$ (overlap of D-translates)

so $h_i^{-1} h_{i-1} D \cap D \neq \emptyset$

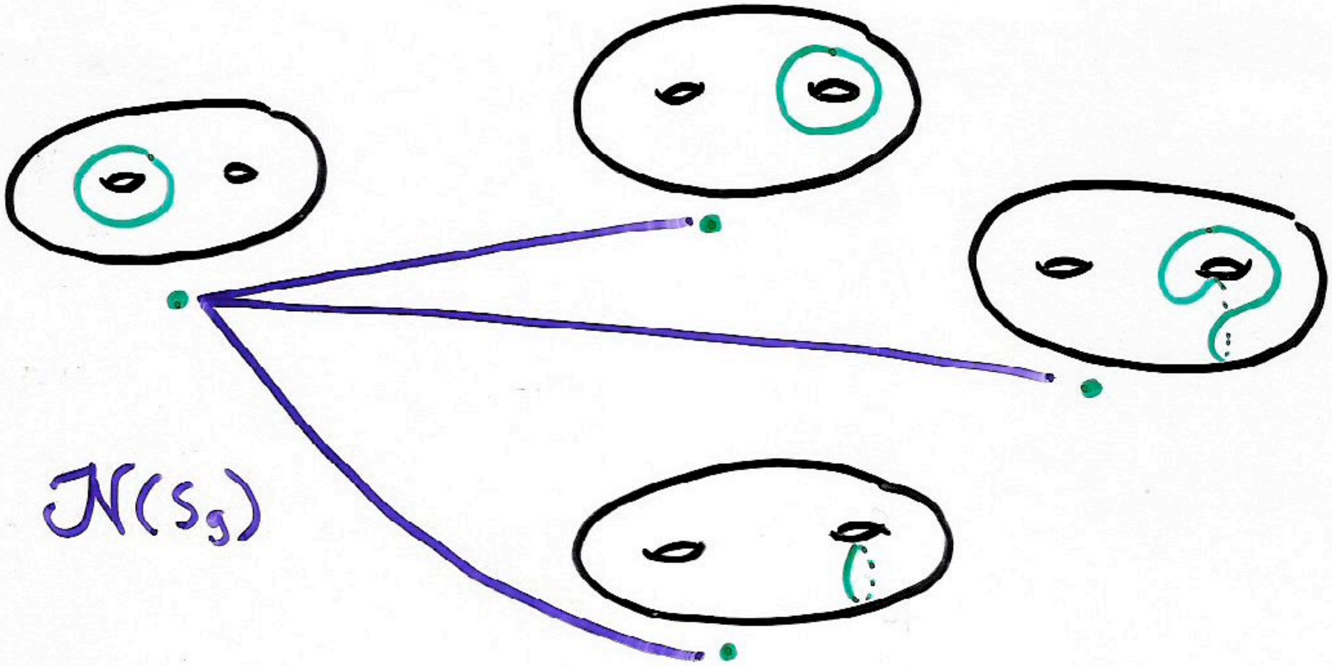
so $h_i^{-1} h_{i-1} \in \langle \mathcal{A} \rangle$

$h_{i-1} \in \langle \mathcal{A} \rangle \Rightarrow h_i \in \langle \mathcal{A} \rangle$

So inductively $h_n = h \in \langle \mathcal{A} \rangle$ ∎

Our graph X is the 1-skeleton of the

Complex of Nonseparating curves on S_g



Vertices are isotopy classes of nonseparating simple closed curves

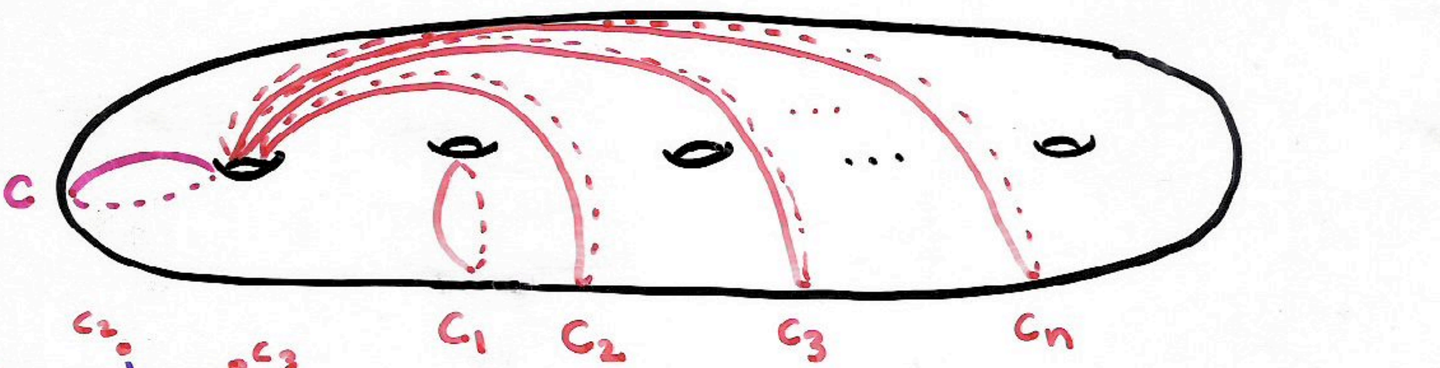
edges connect disjoint curves

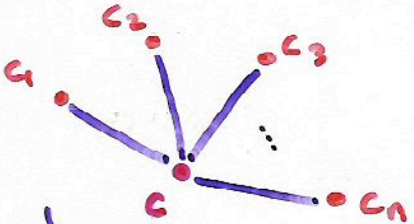
$\text{Mod}(S_g)$ action: $\underset{\substack{\nearrow \\ \text{homeo}}}{[h]} \cdot \underset{\substack{\uparrow \\ \text{curve}}}{[c]} = \underset{\substack{\uparrow \\ \text{image of curve}}}{[h(c)]}$

FACT: $\mathcal{N}(S_g)$ is connected

$\text{Mod}(S_g)$ acts transitively on $\mathcal{N}(S_g)$

Our subgraph D has vertices corresponding to curves shown:





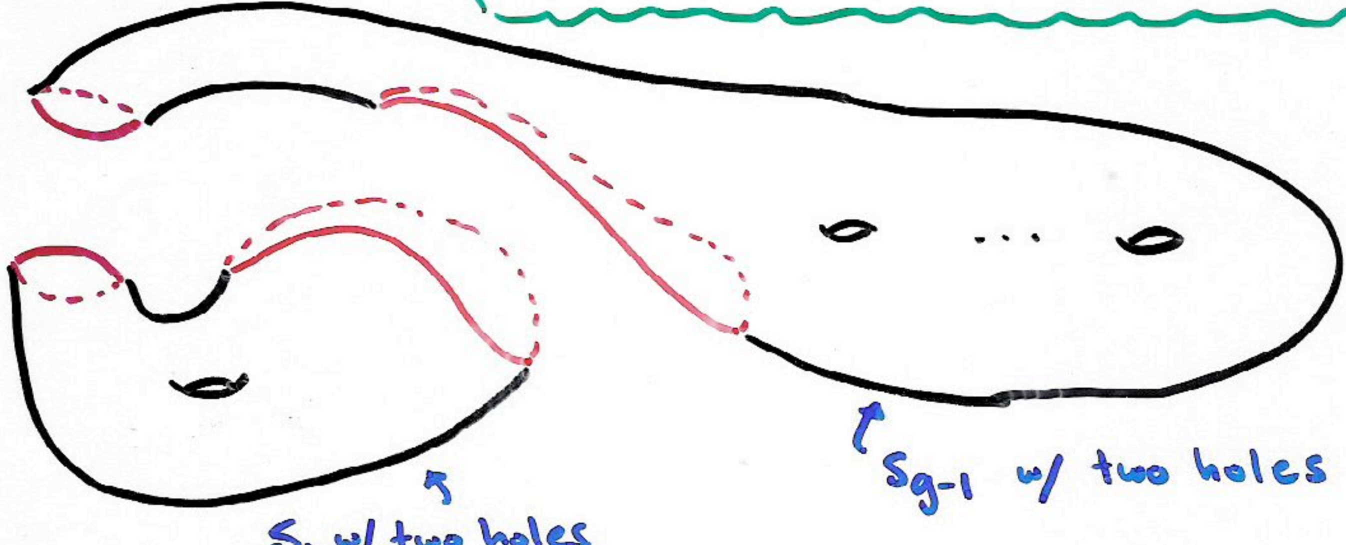
$D \subseteq \mathcal{N}(S_g)$

The classification of surfaces tells us D-translates cover $\mathcal{N}(S_g)$: we have all

edges $\longleftrightarrow$ disjoint pairs of nonsep. scc's

up to $\mathrm{Mod}(S_g)$-action $\longleftrightarrow$ up to homeomorphism

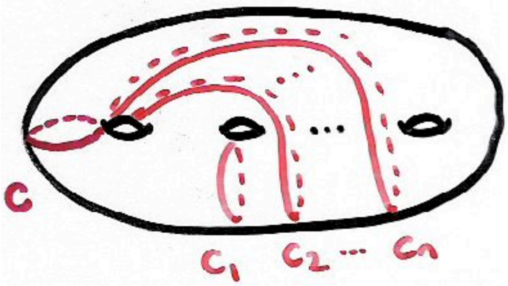
For example: cut along c & c_2



Recall: Step ① was to reduce to f.g.-by-D.t. of H

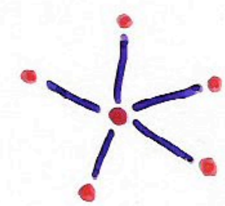
H is the subgroup of $\mathrm{Mod}(S_g)$ fixing curve c.

Lemma $\Longrightarrow$ $\mathrm{Mod}(S_g) = \langle \mathcal{A} \rangle$, $\mathcal{A} = \{ g \mid gD \cap D \neq \emptyset \}$



$\exists\ g_i$ s.t. $g_i(c) = c_i$

$$\boxed{\text{CLAIM: } \langle \mathcal{A} \rangle = \langle g_1, \dots, g_n, H \rangle}$$



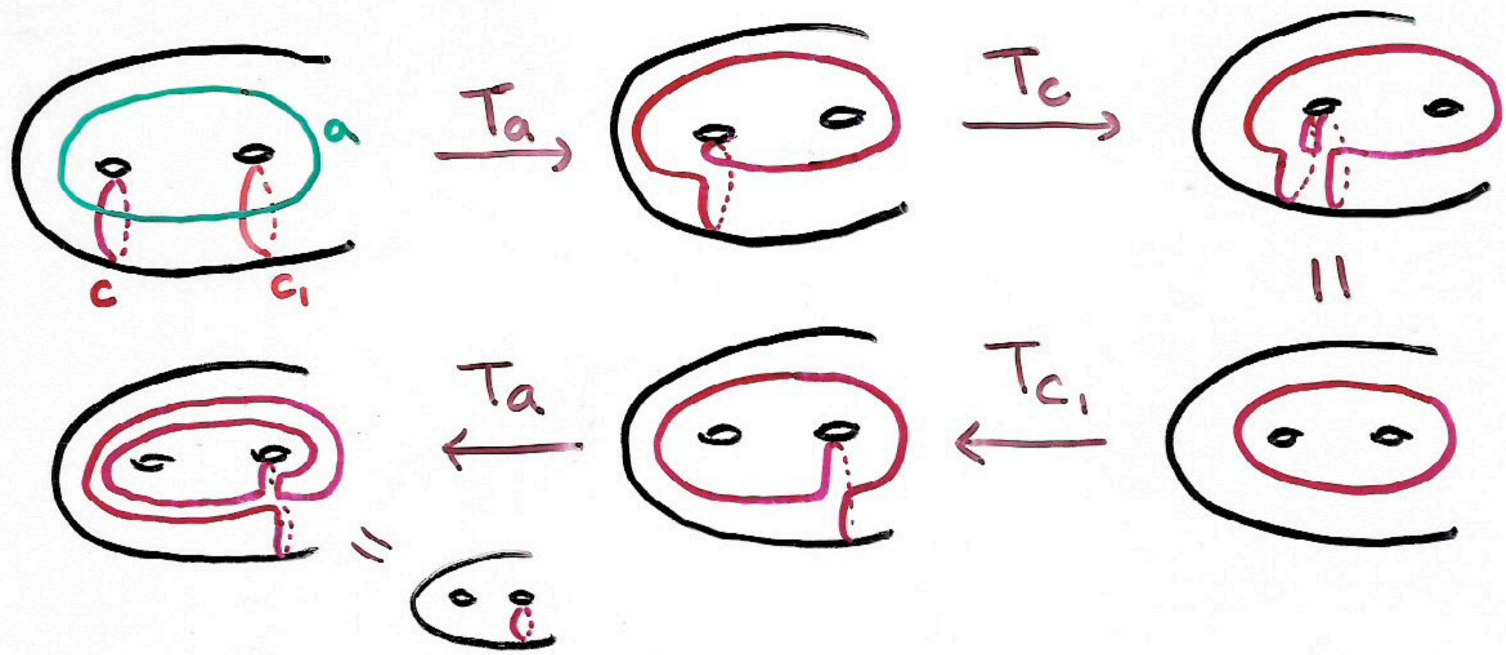
proof: $gD \cap D \neq \emptyset \Rightarrow g(c_j) = c_i$ some i,j

$\Rightarrow g_i^{-1} g g_j(c) = c$

$\Rightarrow g \in g_i H g_j^{-1}$ ■

FACT : Each g_i can be realized by Dehn twists

Example: $g_1(c) = c_1$



Step ②: reduce to f.g.-by-D.t. of $PMod(S_c)$

S_c is S_g cut along c

$PMod(S_c)$ is isotopy classes of self homeos $S_c \circlearrowleft$ that do not permute boundary compnts

Recall H = elts of $Mod(S_g)$ fixing curve c

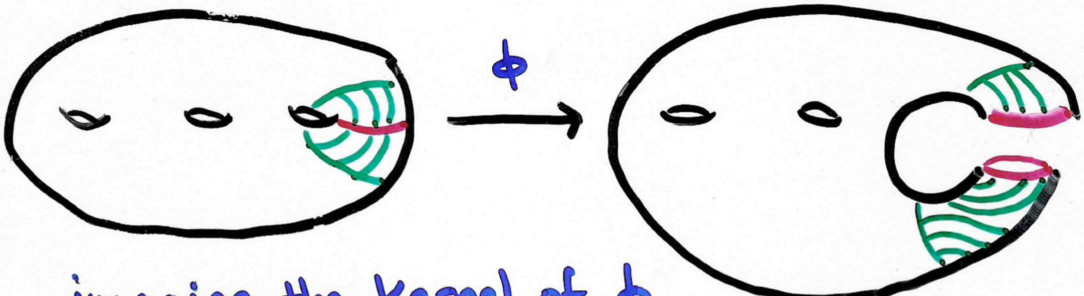
Claim: ① $H = H^+ \cup rH^+$ where H^+ preserves orientation of c

r reverses orientation & is generated by Dehn twists

Claim: ② $1 \longrightarrow \langle T_c \rangle \longrightarrow H^+ \xrightarrow{\phi} PMod(S_c) \longrightarrow 1$

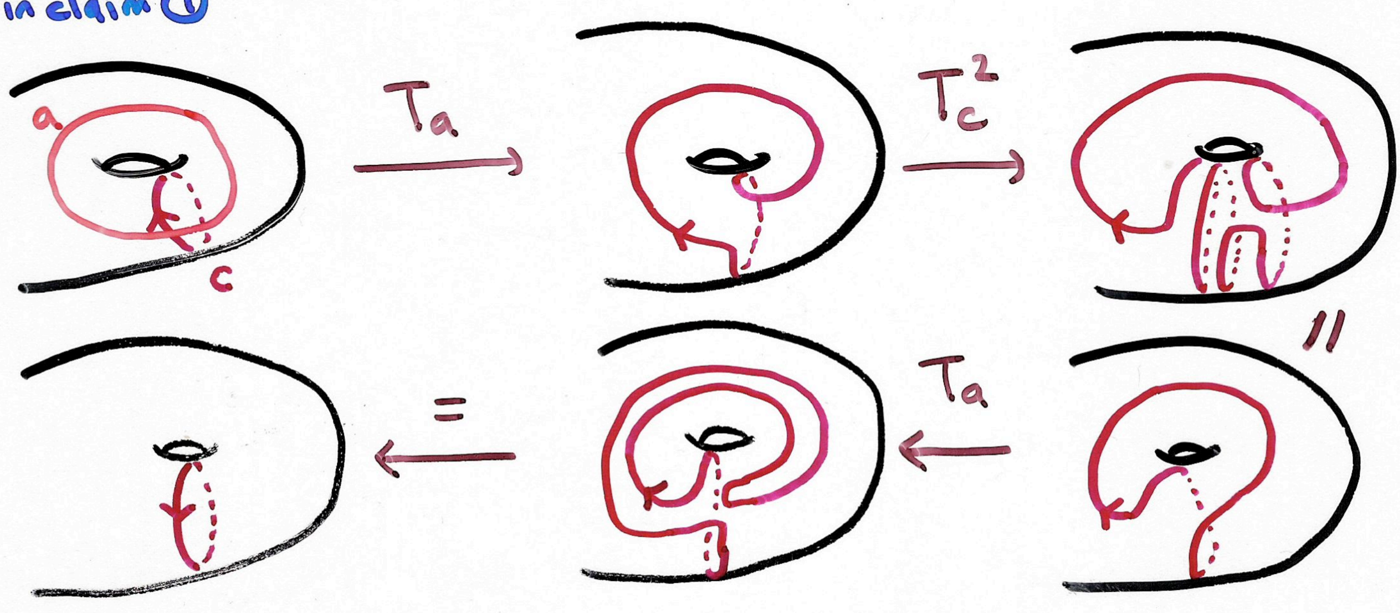
is exact

picture-proof for claim ②:



imagine the kernel of ϕ

The map r in claim ①:



Step ③ Birman Exact Sequence:

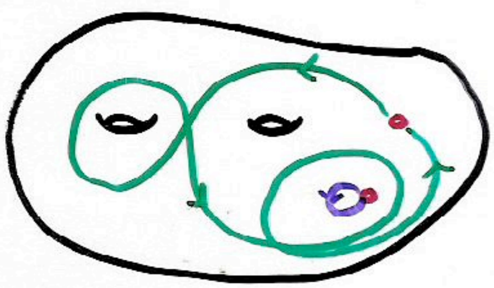
$$1 \longrightarrow PB_2(S_{g-1}) \xrightarrow{p} PMod(S_c) \xrightarrow{f} Mod(S_{g-1}) \longrightarrow 1$$

Braid two punctures around S_{g-1}

(Don't permute boundary)

f: forget the punctures

e.g.



an element of $PB_2(S_{g-1})$ is an

$$\boxed{\{:\} \times I \xrightarrow[\phi]{} S_{g-1}}$$

isotopy of two points.

extends to isotopy of S_{g-1}:

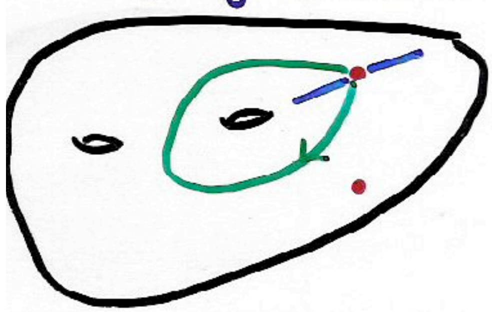
$$S_{g-1} \times I \xrightarrow[\Phi]{} S_{g-1}$$

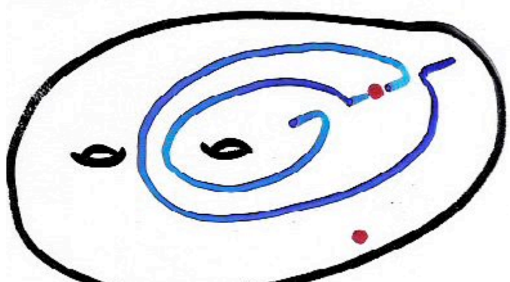
$$\Phi\big|_{S_{g-1} \times \{0\}} = id$$

$$\Phi\big|_{\{:\} \times I} = \phi$$

$$\boxed{p(\phi) := \Phi\big|_{S_{g-1} \times \{1\}}}$$

Im(p) generators:





What happens to blue arc at the end of isotopy? Dehn twists!