Uniform uniform exponential growth of subgroups of the mapping class group

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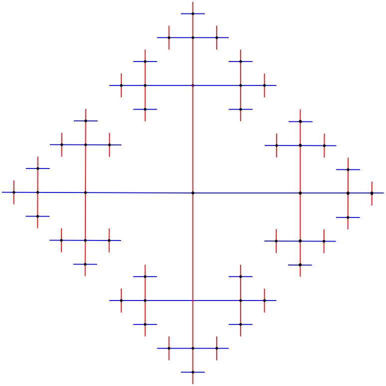

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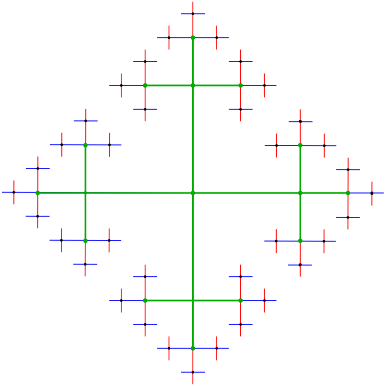

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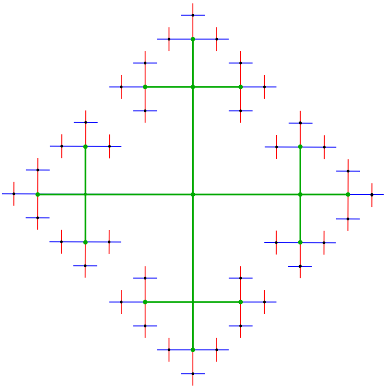
$|B(2; \{a, b\})| = 17$

$|B(n; \{a, b\})| = 2 \cdot 3^n - 1$

$h(F_2; \{a, b\}) = \log 3$



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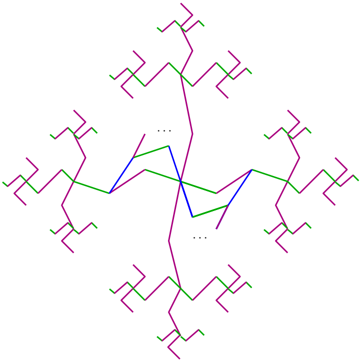

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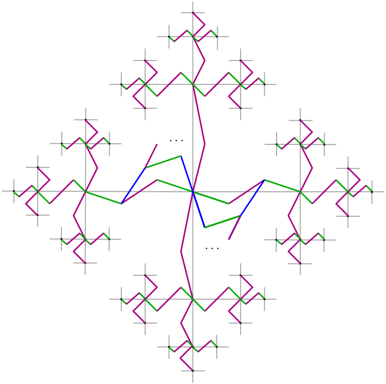
Example: any group G containing F_2. If $u, v \in B(d;S)$ freely generate F_2,

$$|B(nd;S)| > 2 \cdot 3^n - 1$$

$$h(G;S) \geq \log 3/d$$



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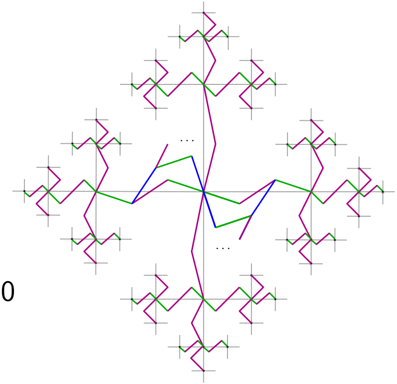
G has *uniform exponential growth* if:

$$h(G) = \inf\{h(G;S) : S \text{ finite set of generators for } G\} > 0$$



Groups with uniform exponential growth

- Non-elementary Gromov-hyperbolic groups (Koubi 1998)
- Finitely generated subgroups of $GL_n(K)$ which are not virtually solvable (Eskin, Mozes, Oh 2002)
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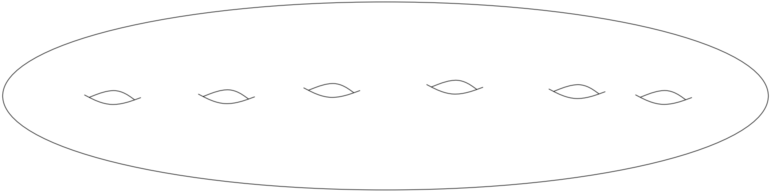
Theorem (Uniform uniform exponential growth)

There exists $d = d(\Sigma)$ such that, if $G < Mod(\Sigma)$ not virtually abelian and finitely generated by S, one has $u, v \in B(d; S)$ freely generating F_2. Hence $h(G) > \log(3)/d(\Sigma) > 0$.

"Pure" elements: $\ker\{\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Aut}(H_1(\Sigma, \mathbb{Z}/3\mathbb{Z}))\}$

f is an automorphism of Σ



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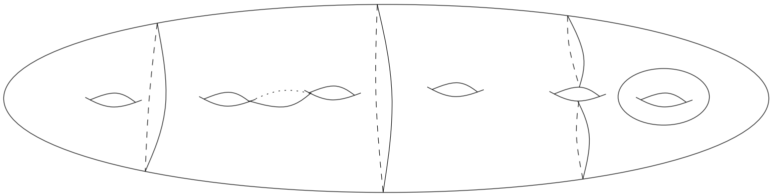

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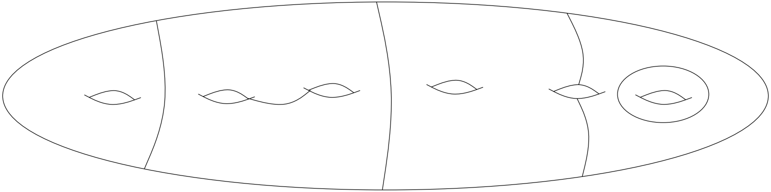

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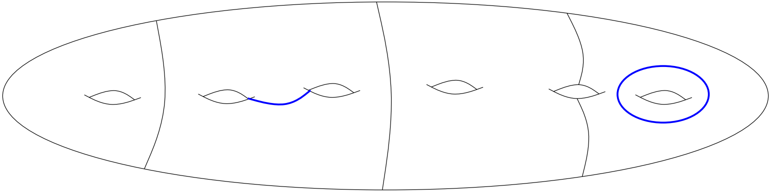

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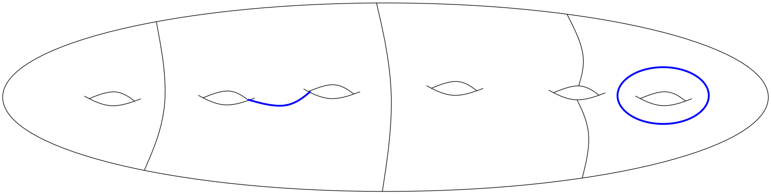

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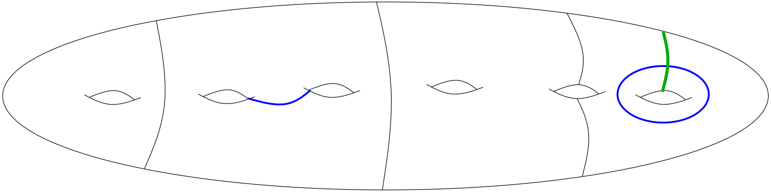

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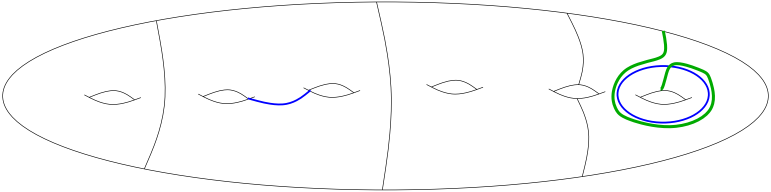

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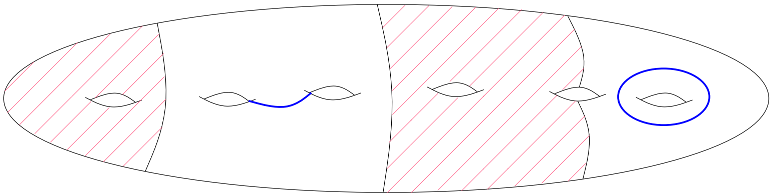

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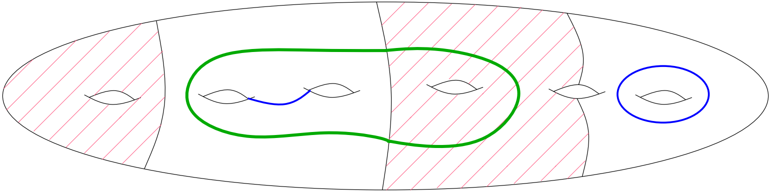

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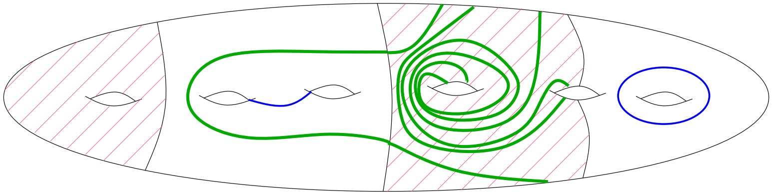

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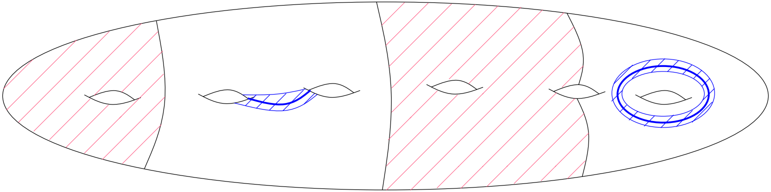

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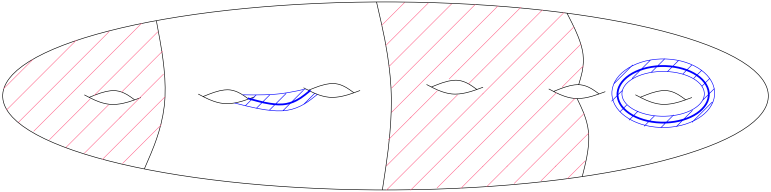

Theorem (Thurston, Ivanov)

A pure mapping class is either:

- *Pseudo-Anosov on the whole surface (pA)*
- *Pseudo-Anosov on some subsurface (rpA)*
- *A composition of Dehn twists about disjoint curves*

Strategy: short words generate F_2

Theorem (McCarthy, Ivanov)

A subgroup of $\mathrm{Mod}(\Sigma)$ *is either virtually abelian or contains* F_2.

Proposition

There exists a power $p = p(S)$ *with the property that, for any pure mapping classes* a, b *such that* $\langle a, b \rangle$ *contains* F_2,

(a) *if* a *is pA*, $\langle a^p, ba^pb^{-1} \rangle \cong F_2$;

(b) *if* a, b *are Dehn twists*, $\langle a^p, b^p \rangle \cong F_2$;

(c) *if* a, b *are rpA with overlapping pA subsurfaces*, $\langle a^p, b^p \rangle \cong F_2$;

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How to find F_2: the Ping-pong Lemma

Lemma

Suppose a and b act on a set X, and suppose there exist nonempty disjoint subsets $X_a, X_b \subset X$ such that $a^k(X_b) \subset X_a$ and $b^k(X_a) \subset X_b$ for all nonzero k. Then $\langle a, b \rangle$ is a rank-2 free group.

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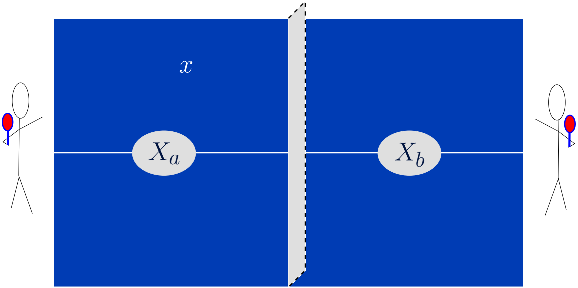

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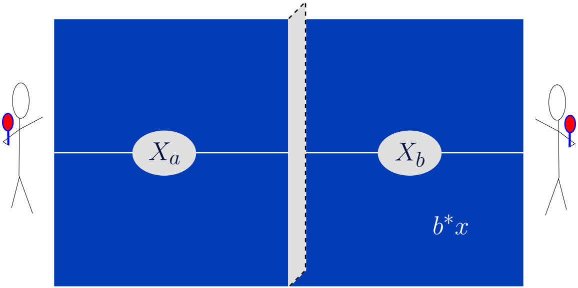

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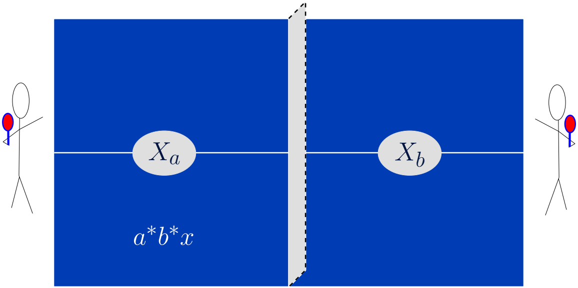

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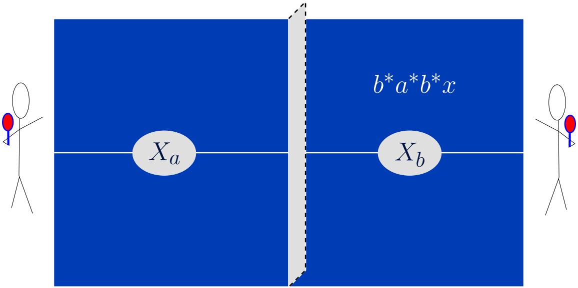

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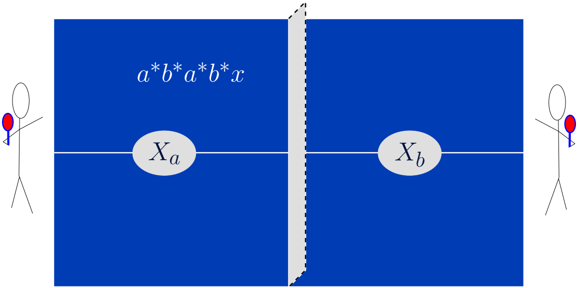

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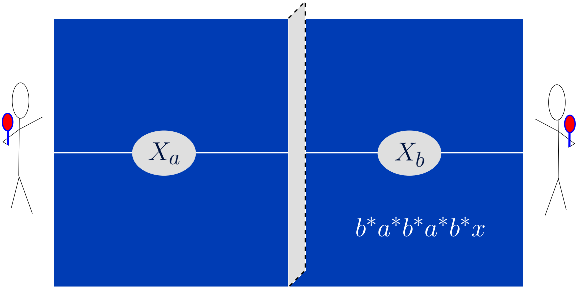

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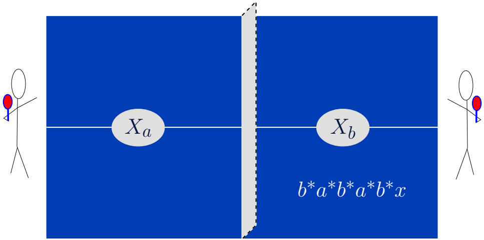
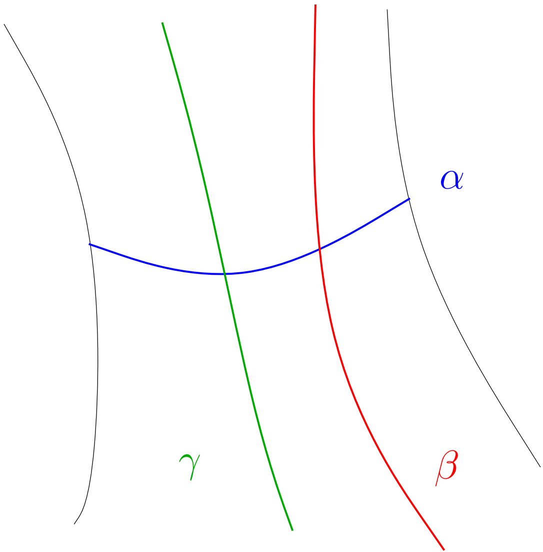


- Any nontrivial word can be conjugated to $b^* a^* b^* \cdots a^* b^*$. $\square$

Dehn twist ping-pong:



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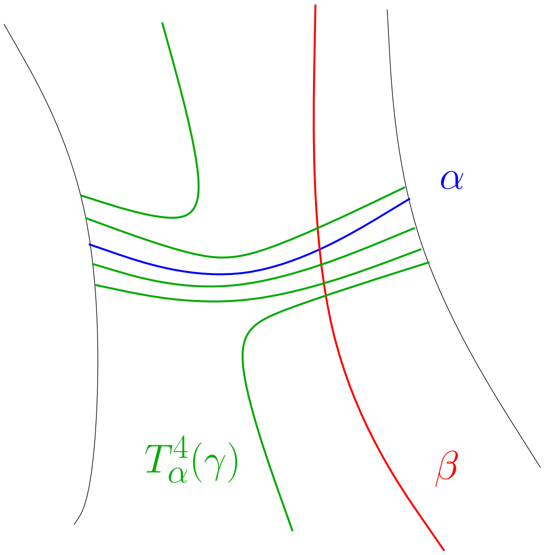
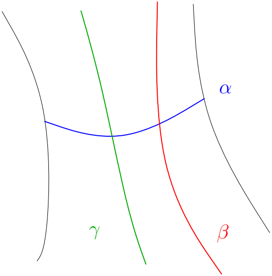
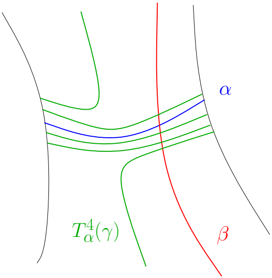




$T_\alpha^4(\{$curves intersecting α more than $\beta\}) \subset$



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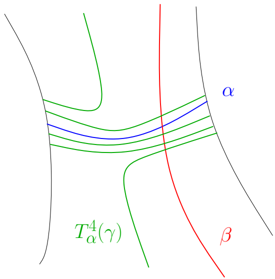
Dehn twist ping-pong: generalize an argument of Hamidi-Tehrani



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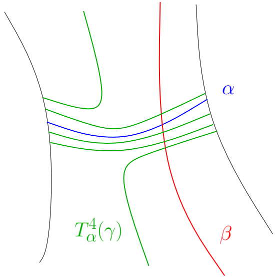

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By ping-pong lemma,
$\langle T_\alpha^4, T_\beta^4 \rangle$ is a rank-2 free group.

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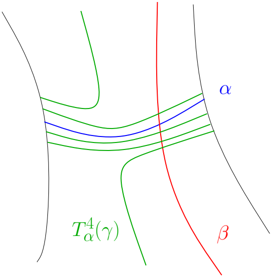

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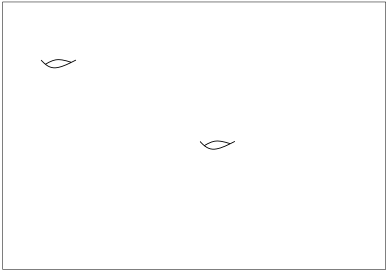
Proposition

Let a and b be compositions of Dehn twists about sets of curves α_i and β_j resp., such that the $\{\alpha_i\}$ are pairwise disjoint, as are $\{\beta_j\}$, but some α_i intersects some β_j. Then for any $k > 4$, $\langle a^k, b^k \rangle$ is a rank-2 free group.

Relative-pseudo-Anosov ping-pong



a and b are rpA mapping classes s.t.
$A = \mathrm{supp}(a)$ and $B = \mathrm{supp}(b)$
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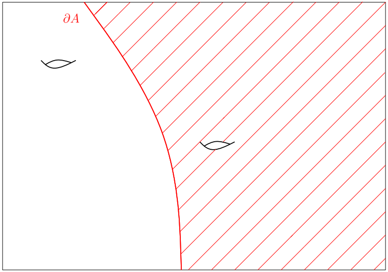

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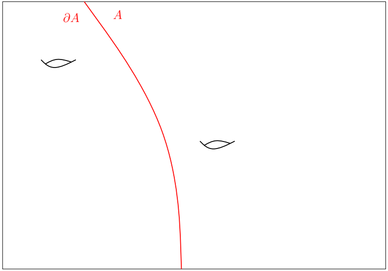

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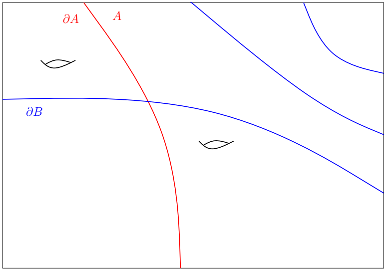

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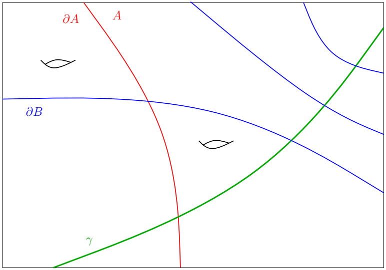
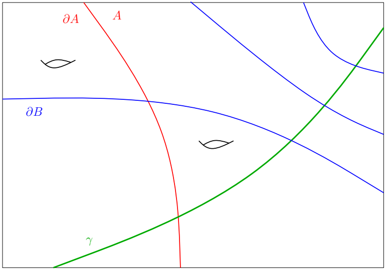
Relative-pseudo-Anosov ping-pong



$A = \mathrm{supp}(a)$ and $B = \mathrm{supp}(b)$
Suppose γ "entangles" ∂B in A.

"Entangles": arcs of $\gamma \cap A$ intersect ∂B many times

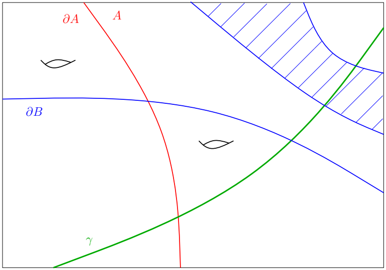
Relative-pseudo-Anosov ping-pong



$A = \mathrm{supp}(a)$ and $B = \mathrm{supp}(b)$
Suppose γ “entangles” ∂B in A.
Then γ “avoids” ∂A in B.

“Avoids”: some arc of $\gamma \cap B$ is disjoint from ∂A

Relative-pseudo-Anosov ping-pong



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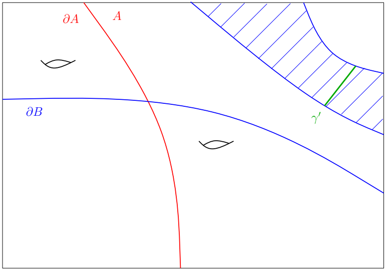

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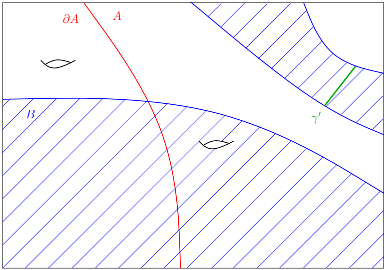
Relative-pseudo-Anosov ping-pong



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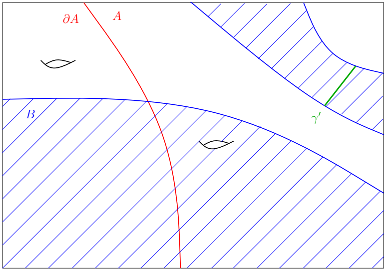
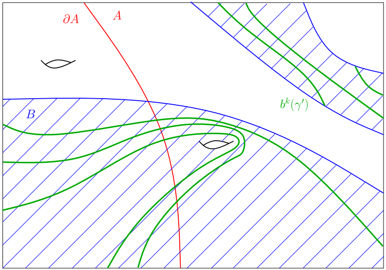
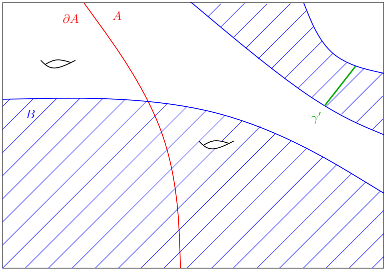
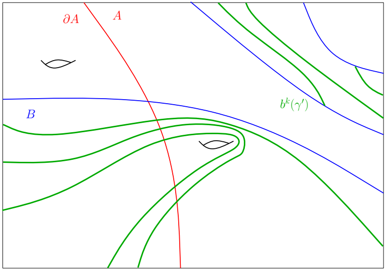


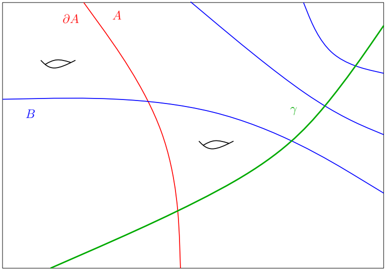


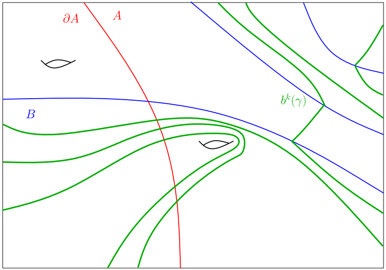
Relative-pseudo-Anosov ping-pong



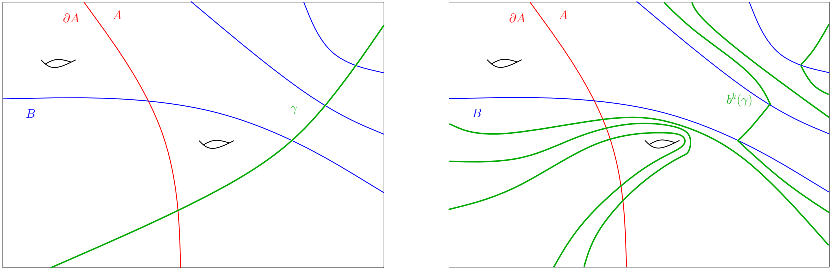


Relative-pseudo-Anosov ping-pong



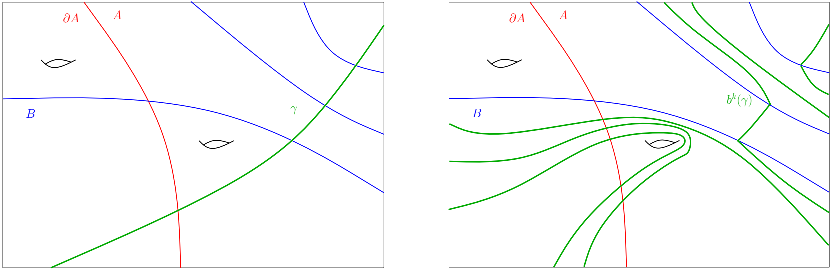


Relative-pseudo-Anosov ping-pong



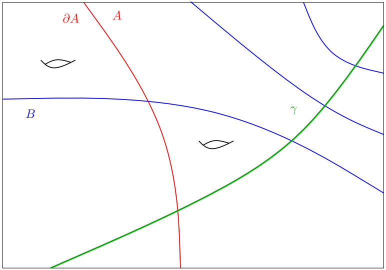
$b^k(\{\text{curves entangling } \partial B \text{ in } A\}) \subset \{\text{curves entangling } \partial A \text{ in } B\}$

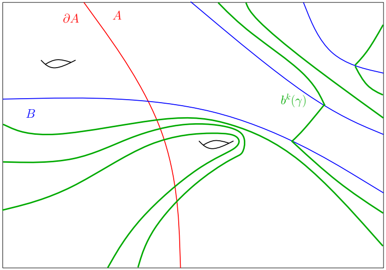
Relative-pseudo-Anosov ping-pong



b^k({curves entangling ∂B in A}) $\subset$ {curves entangling ∂A in B}
a^k({curves entangling ∂A in B}) $\subset$ {curves entangling ∂B in A}

Relative-pseudo-Anosov ping-pong





$b^k(\{\text{curves entangling } \partial B \text{ in } A\}) \subset \{\text{curves entangling } \partial A \text{ in } B\}$
$a^k(\{\text{curves entangling } \partial A \text{ in } B\}) \subset \{\text{curves entangling } \partial B \text{ in } A\}$

Proposition

There exists a power q_{rpA} depending only on S such that, for any rpA mapping classes a and b supported on overlapping connected subsurfaces, and any $k > q_{rpA}$, $\langle a^k, b^k \rangle$ is a rank-2 free group.

All together now

Choose $p > \max\{q_{rpA}, 4, \text{ and } q_{pA} \text{ for } S \text{ or any subsurface of } S\}$.

Proposition

For any pure mapping classes a, b such that $\langle a, b\rangle$ contains F_2, $\langle a^p, b^p\rangle$, $\langle a^p, b^p a^p b^{-p}\rangle$, $\langle b^p, a^p b^p a^{-p}\rangle$, or $\langle a^p, ba^p b^{-1}\rangle \cong F_2$

The lower bound for $h(G)$

- G has finite-index pure subgroup $G' = G \cap \ker$
- For pure subgroup, suffices to consider two-el't generating sets
- $h(G') \geq (\log 3)/3p$.
- $h(G) > (\log 3)/(3p \cdot (2[\mathrm{Mod}(S) : \ker] - 1))$