Convex Cocompactness in Mod(S) via Quasiconvexity in RAAGs

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Convex cocompactness in mapping class groups

Thms (Farb, Mosher; Hamenstädt; Kent, Leininger)

For finitely generated $G < \mathrm{Mod}(S)$, TFAE:

- *G acts cocompactly on its "weak hull", is δ-hyperbolic, ...*
- *Orbits of G are quasiconvex in $\mathrm{Teich}(S)$*
- *Orbit maps of G into $\mathcal{C}(S)$ are quasi-isometric embeddings.*

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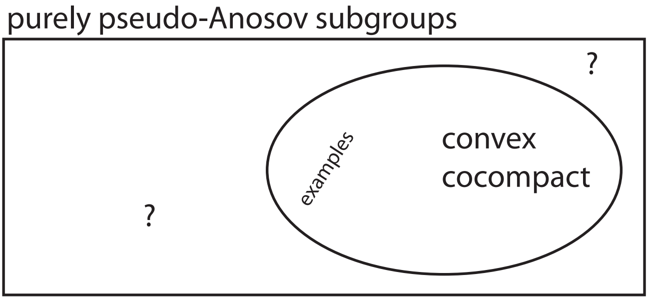
Thms (Farb-Mosher, Hamenstädt)

E_G is word hyperbolic if and only if G is convex cocompact.

Which subgroups of $\mathrm{Mod}(S)$ are convex cocompact?



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Is { **convex cocompact** } *same as* { **f.g. all-pA** } *?*

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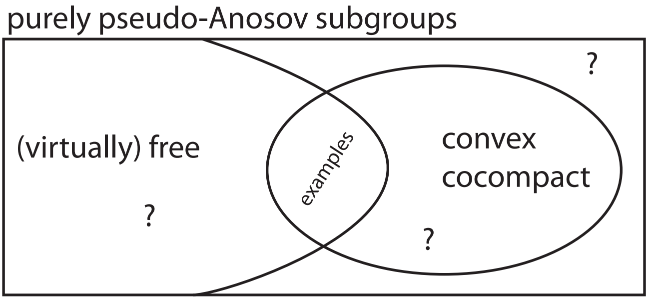

Q:

Is { **convex cocompact** } *same as* { **f.g. all-pA (v. free)** } *?*

RAAGs in mapping class groups

Definition

$A_\Gamma = \langle v_i \text{ vertices of } \Gamma \mid [v_i, v_j] = id \text{ iff } (v_i, v_j) \text{ is an edge of } \Gamma \rangle$

Thms (Koberda, Clay-Leininger-M, Crisp-Paris/-Weiss/-Farb)

Many ways to embed A_Γ in some $\mathrm{Mod}(S)$.

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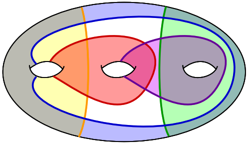
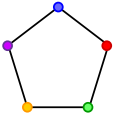
Thm (Clay-Leininger-M)

For partially pA $\{f_1, \ldots, f_n\}$ supported on connected, non-nested X_i with disjointess recorded in the graph Γ, for large enough p_i,

$$A_\Gamma \to \langle f_1^{p_1}, \ldots, f_n^{p_n} \rangle < \mathrm{Mod} S$$

is a quasi-isometric embedding.





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is an admissible embedding.*

*meaning $A_\Gamma \hookrightarrow \mathrm{Mod}(S)$:

(i) Comes with large subsurface curve complex projections, and

(ii) Word partial order matches subsurface partial order

CC in Mod(S) via QC in RAAGs

Thm (M-Taylor)

If $A_\Gamma < \mathrm{Mod}(S)$ is admissible and $G < A_\Gamma < \mathrm{Mod}(S)$ is convex cocompact, then G is (word) quasiconvex in A_Γ.

Thm (M-Taylor)

Suppose $A_\Gamma < \mathrm{Mod}(S)$ is admissible and $G < A_\Gamma$ is fin. gen. and K-quasiconvex. There exists $L = L(K, |\Gamma|)$ such that if $w \in G$ with $0 < |w| < L$ are pseudo-Anosov, then G is convex cocompact (thus all-pseudo-Anosov, thus free).

Corollary

All-pA $G < A_\Gamma < \mathrm{Mod}(S)$ is convex cocompact in $\mathrm{Mod}(S)$ if and only if it is word quasiconvex in A_Γ.

Convex cocompactness in RAAGs

The Cayley graph of A_Γ completes to a CAT(0) cube complex $\widetilde{S_\Gamma}$

Thm (Haglund 2008)

For $G < A_\Gamma$, TFAE:

- *Exists (non-empty) convex subcomplex $C \subset \widetilde{S_\Gamma}$ which is G-invariant and cocompact.*
- *G (word) quasiconvex in A_Γ (vertex orbits $G \cdot v$ are combinatrly. qconvex in $\widetilde{S_\Gamma}$.)*

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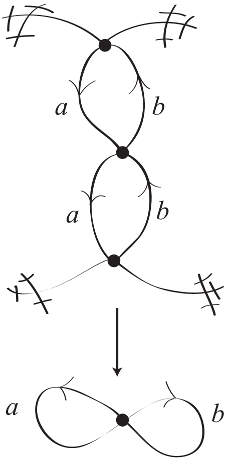
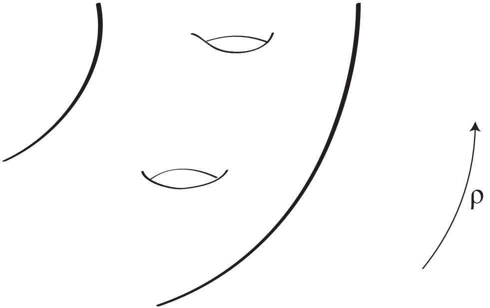
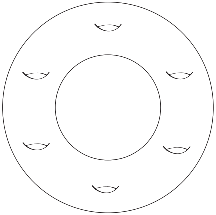


$< ab, a^2b^2 >$

Interesting examples

$n = g - 1 \qquad \rho^n = \mathrm{id}$





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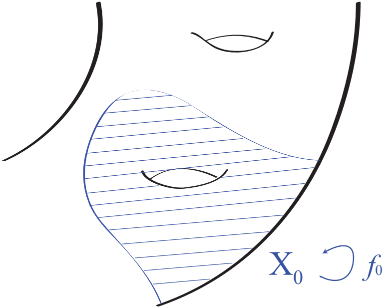

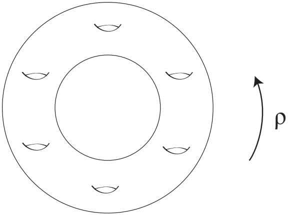

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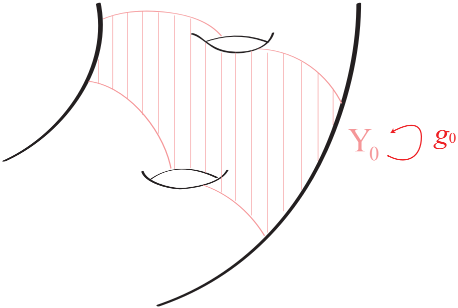

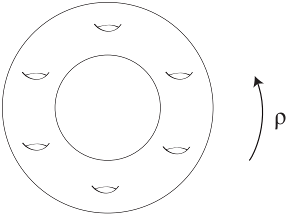

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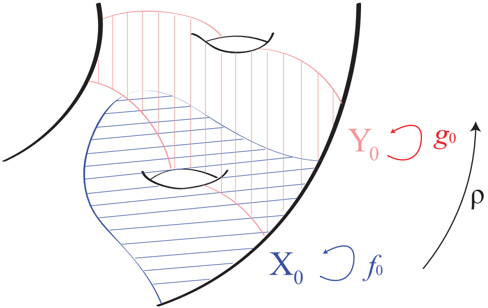

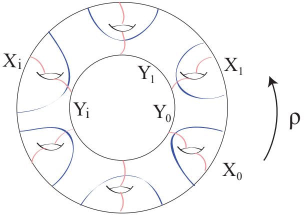

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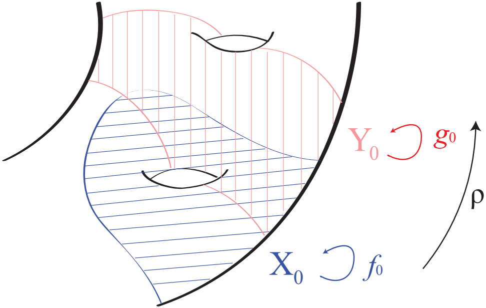

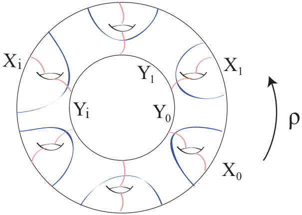

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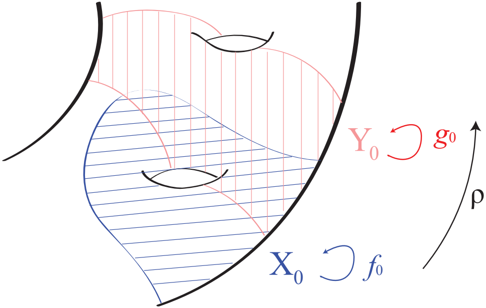

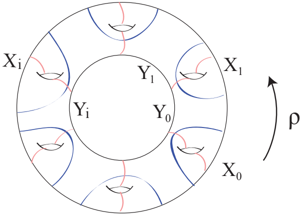

$h = (\rho f_0 g_0)^n = f_1 g_1 f_2 g_2 \cdots f_n g_n \in \langle f_i, g_i \rangle \quad \mathrm{trans}(h) \sim 1/g$

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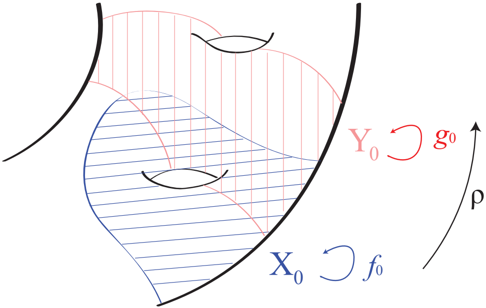

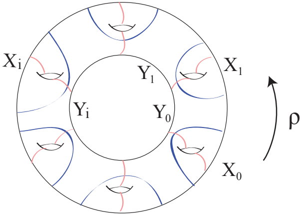

$$h_k = (\rho f_0^k g_0^k)^n = f_1^k g_1^k f_2^k g_2^k \cdots f_n^k g_n^k \in \langle f_i, g_i \rangle \quad \mathrm{trans}(h_k) \sim 1/g$$

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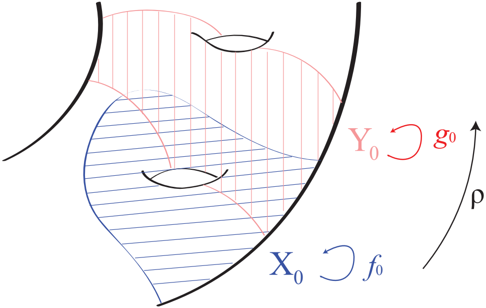

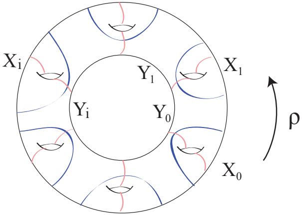

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Thm (M-Taylor)

For any k, $\langle h_1, h_2 \ldots, h_k \rangle \cong F_k$ is convex cocompact

Further questions

Q:

Construct a non-cyclic convex cocompact subgroup containing pseudo-Anosovs with $1/g^2$ translation length in curve complex.

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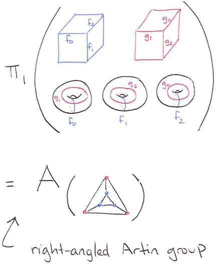
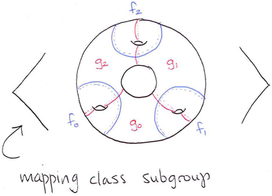
Does G all-pseudo-Anosov imply G convex cocompact in $\mathrm{Mod}(S)$?

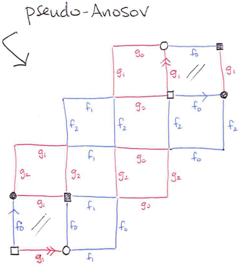
Q:

Does $G < A_\Gamma$ all-loxodromic imply G (word) quasiconvex in A_Γ?

Fun pictures







Consequences of convex cocompactness in Mod(S)

Requirements for word hyperbolicity:

(1) No subgroups $BS(p,q) = \langle a, b | a^{-1} b^p a = b^q \rangle$

(2) Has finite $K(G,1)$ if torsion-free (in general, type FP_∞).

Q: (Gromov, Farb-Mosher)

If G with finite $K(G,1)$ has no BS subgroups, is it hyperbolic?

Example (which might not exist)

If G is all-pA, then E_G has finite $K(G,1)$ and no BS subgroups. Recall if G fails to be convex cocompact, it also fails hyperbolicity.

Q:

Does there exist free, non-quasiconvex $G < A_\Gamma$ and admissible embedding $A_\Gamma < \mathrm{Mod}(S)$ such that G is all-pA?