

A RECIPE FOR SHORT-WORD PSEUDO-ANOSOV

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ABSTRACT. Given any generating set of any subgroup G of the mapping class group of a surface, we find an element f with word length bounded by a constant K depending only on the surface, and with the property that the minimal subsurface supporting a power of f is as large as possible for elements of G . In particular, if G contains a pseudo-Anosov map, we find one of word length at most K . We also find new examples of convex cocompact free subgroups of the mapping class group.

1. INTRODUCTION

Consider $\text{Mod}(S)$, the mapping class group of a surface S , and its action on the isotopy classes of simple closed curves on S . If all elements of a subgroup fix no common family of curves, then the group contains a *pseudo-Anosov* mapping class, that is, a single element which itself fixes no finite family of curves [Iva92]. This paper answers in the affirmative Fujiwara’s question of whether one can always find a “short-word” pseudo-Anosov map (Question 3.4 in [Fuj08]). Where Σ generates the group G , let Σ -length denote the length of an element of G in the word metric induced by Σ . The affirmative statement is:

Theorem 1.1. *There exists a constant $K = K(S)$ with the following property. Suppose $G < \text{Mod}(S)$ is finitely generated by Σ and contains a pseudo-Anosov map. Then G contains a pseudo-Anosov map with Σ -length less than K .*

The proof provides an explicit construction of pseudo-Anosov maps from arbitrary non-pseudo-Anosov elements. In fact, it addresses a broader question. Roughly speaking, a pseudo-Anosov mapping class requires the whole surface for its support. The other elements are called reducible, because they allow one to “reduce” the surface in question. That is, a reducible mapping class, raised to some power, fixes a proper subsurface and thus induces an element of the subsurface’s mapping class group. Mosher has described a unifying approach to both pseudo-Anosov and reducible elements by associating to a mapping class f what he calls its active subsurface $\mathcal{A}(f)$ [Mos]. For the sake of introduction one may think of $\mathcal{A}(f)$ as the smallest subsurface supporting some power of f (noting that these subsurfaces and thus their inclusion are defined only up to isotopy) and observe that f is pseudo-Anosov exactly when $\mathcal{A}(f) = S$. Several foundational mapping class group theorems, including the Tits alternative for $\text{Mod}(S)$ [Iva92, McC85], and subgroup structure results from [BLM83] and [Iva88], elegantly derive from the “Omnibus Subgroup Theorem” of Handel and Mosher [Mos]: given a group $G < \text{Mod}(S)$ there exists $f \in G$ such that for all $g \in G$, $\mathcal{A}(g) \subset \mathcal{A}(f)$. Call such an f *full-support* for G . In this paper we prove the following, which includes Theorem 1.1 as a special case.

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Theorem 3.1. *There exists a constant $K = K(S)$ such that, for any finite subset $\Sigma \subset \text{Mod}(S)$, one may find f full-support for $\langle \Sigma \rangle$ with Σ -length less than K .*

Let us clarify that our proof of the above neither uses nor proves the theorem of Handel and Mosher. Working towards analogous theory for $\text{Out}(F_n)$ (see [HM09]), they formulated the Omnibus Theorem to consolidate the original proofs in [BLM83, Iva92, Iva88, McC85], including results we need to prove Theorem 3.1. While we benefit from their notion of an active subsurface, we define it differently, so that the Omnibus Theorem only rephrases the older results (see Section 2.4).

The proof of Theorem 3.1 spells out short pseudo-Anosov maps explicitly, with the following core construction, concerning special pairs of pure reducible mapping classes we will call *sufficiently different*. These are pure mapping classes a and b with pseudo-Anosov restrictions to proper subsurfaces A and B respectively, such that A and B together fill S , meaning that each curve on S has essential intersection with either A or B . The proposition also identifies subgroups whose action on the curve complex gives a quasi-isometric embedding, so that they are convex cocompact [Ham05, KL08a] in the sense defined by Farb and Mosher [FM] in analogy to Kleinian groups. This last part is proven for interest, and is not necessary for the main theorem.

Proposition 1.2. *There exists a constant $Q = Q(S)$ with the following property. Suppose a and b are sufficiently different pure reducible mapping classes. Then for any $n, m \geq Q$, every nontrivial element of $G = \langle a^n, b^m \rangle$ is pseudo-Anosov except those conjugate to powers of a^n or b^m . Furthermore, G is a rank two free group, and all of its finitely generated all-pseudo-Anosov subgroups are convex cocompact.*

In [Man10], the author considers a more general condition for pairs of pure reducible mapping classes, and finds Q such that G as above is a rank two free group, but need not contain pseudo-Anosov maps (and therefore need not be convex cocompact). A more relevant comparison is Thurston's theorem providing the first concrete examples of pseudo-Anosov diffeomorphisms [Thu88] (see also [Pen88]). He proved that if a and b are Dehn twists about filling curves, then one can find an affine structure on S inducing an embedding of $\langle a, b \rangle$ into $PSL(2, \mathbb{R})$ under which hyperbolic elements of $PSL(2, \mathbb{R})$ correspond to pseudo-Anosov maps in $\langle a, b \rangle$. In particular, every nontrivial element of the free semigroup generated by a and b^{-1} is pseudo-Anosov. More recently, Hamidi-Tehrani [HT02] classified all subgroups generated by a pair of positive Dehn multi-twists. In particular, if α and β are multicurves whose union fills S , and a and b are compositions of positive powers of Dehn twists about components of α and β respectively, then except for finitely many pairs n, m , $\langle a^n, b^m \rangle$ is a rank-two free group whose only reducible elements are those conjugate to powers of a or b (see also [Ish96]). In a different light, one can consider Proposition 1.2 a companion to a theorem of Fujiwara that generates convex cocompact free groups using bounded powers of independent pseudo-Anosov maps; this theorem appears in Section 5.2 as Theorem 5.5.

Plan of the paper. After the background laid out in Section 2, we prove Theorem 3.1 in Section 3, modulo propositions proved in Sections 4 and 5. Section 5 also contains the proof of Proposition 1.2.

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2. PRELIMINARIES

This section consists of five parts. The first presents basic definitions around the mapping class group and curve complex; the second relates the curve complexes of a surface and its subsurfaces. Section 2.3 presents a different view of subsurfaces, which facilitates Section 2.4 on the case for understanding reducible mapping classes as subsurface pseudo-Anosov maps. Finally, Section 2.5 recalls powerful curve complex tools with which we rephrase the classification of elements of $\text{Mod}(S)$.

2.1. Mapping class group and curve complex. Throughout, we consider only compact oriented surfaces with genus g and number of boundary components p . Define the *complexity* $\xi(S)$ of a surface S by $\xi = 3g + p - 3$. We neglect the case where ξ is -2 or zero, which means S is a disk, a closed torus, or a pair of pants, because these are never subsurfaces of interest, as explained in Section 2.2. Annuli, for which $\xi = -1$, feature throughout, but primarily as subsurfaces of higher-complexity surfaces. Let us first assume $\xi \geq 1$, and address the annulus case after. Note that our definitions will not distinguish between boundary components and punctures, except on an annulus. One may find in [FM] a discussion on variant definitions of the mapping class group.

Given a surface S , its *mapping class group* $\text{Mod}(S)$ is the discrete group of isotopy classes of orientation-preserving self-homeomorphism of S , where admissible isotopies fix each component of ∂S setwise. Much (arguably, everything) about $\text{Mod}(S)$ appears in its action on the isotopy classes of those simple closed curves on S that are *essential*: neither homotopically trivial nor boundary-parallel. Let us call these classes *curves* for short. *Pseudo-Anosov* mapping classes are those that fix no finite family of curves; necessarily these have infinite order. Among the rest, we distinguish those that have finite order, and call the remaining *reducible*.

The *intersection* of two curves α and β , written $i(\alpha, \beta)$, is the minimal number of points in $\alpha' \cap \beta'$, where α' ranges over all representatives of the isotopy class denoted by α and likewise β' ranges over representatives of β . Often, we will use the same notation for a curve as an isotopy class and for a representative path on the surface. For specificity and guaranteed minimal intersections, one may represent all curves by closed geodesics with respect to any pre-ordained hyperbolic metric on S . Two curves *fill* a surface if every curve of that surface intersects at least one of them. *Disjoint* curves have zero intersection. All of these definitions extend to *multicurves*, our term for sets of pairwise disjoint curves.

Now let us upgrade the set action of $\text{Mod}(S)$ on curves to a simplicial action on the *curve complex* of S , denoted $\mathcal{C}(S)$. Because $\mathcal{C}(S)$ is a *flag complex*, its data reside entirely in the one-skeleton $\mathcal{C}_1(S)$: higher-dimensional simplices appear whenever the low-dimensional simplices allow it. Thus it suffices to consider only the graph $\mathcal{C}_1(S)$, although we usually refer to the full complex out of habit.

Curves on S comprise the vertex set $\mathcal{C}_0(S)$, and an intersection rule determines the edges of $\mathcal{C}_1(S)$. For a surface with complexity $\xi > 1$, edges join vertices representing disjoint curves. Therefore n -simplices correspond to multicurves with n distinct components, and ξ gives the dimension of $\mathcal{C}(S)$. When $\xi = 1$, S is a punctured torus or four-punctured sphere. Because on these any two distinct curves intersect, we modify the previous definition so that edges join vertices representing

curves which intersect minimally for distinct curves on that surface—that is, once for the punctured torus and twice for the four-punctured sphere. In both cases $\mathcal{C}(S)$ corresponds to the Farey tessellation of the upper half plane (vertices sit on rationals corresponding to slopes of curves on the torus).

Give $\mathcal{C}_1(S)$ the path metric where edges have unit length; this extends simplicially to the full complex. For any two curves α and β , let $d(\alpha, \beta)$ denote their distance in $\mathcal{C}_1(S)$. Immediately one may observe that $\mathcal{C}(S)$ is locally infinite. It is not obvious that $\mathcal{C}(S)$ is connected, but the proof is elementary [Har81]. Furthermore, it has infinite diameter [MM99]. A deep theorem of Masur and Minsky states that $\mathcal{C}(S)$ is δ -hyperbolic, meaning that for some δ , every edge of any geodesic triangle lives in the δ -neighborhood of the other two edges [MM99].

Now suppose S is the annulus $S^1 \times [0, 1]$. We modify the definition of the mapping class group to require that homeomorphisms and isotopy fix ∂S pointwise. Parameterizing S^1 by angle θ and S by (θ, t) , the *Dehn twist* on S maps (θ, t) to $(\theta + 2\pi t, t)$. The cyclic group generated by this twist is the entire mapping class group of the annulus. In this group, let us consider all non-trivial elements pseudo-Anosov, for reasons clarified in Section 2.5. To define $\mathcal{C}(S)$, we contend with the fact that an annulus contains no essential curves. Instead, each vertex of $\mathcal{C}_0(S)$ corresponds to the isotopy class of an arc connecting the two boundary components, again requiring isotopy fix boundary pointwise. Edges connect vertices that represent arcs with disjoint interiors. Although this complex is locally uncountable, it is not difficult to understand: the distance between two distinct vertices equals one plus the minimal number of interior points at which their representative arcs intersect. Again, $\mathcal{C}(S)$ is connected, infinite diameter, and δ -hyperbolic—in fact, it is quasi-isometric to the real line, and $\text{Mod}(S)$ acts on it by translation. Note that our “curve complex” for the annulus is more accurately called an *arc complex*. Generally, we aim to minimize the distinction between annular subsurfaces and subsurfaces with $\xi \geq 1$; for extended treatment of curve complexes and arc complexes, see [MM00].

2.2. Subsurface projection. Let us relate the curve complex of S to that of S' , where $\xi(S) \geq 1$ and S' is an “interesting” subsurface of S . Here, a subsurface is defined only up to isotopy, and assumed *essential*, meaning its boundary curves are either essential in S or shared with ∂S . This rules out the disk. We also disregard pants, which have finite mapping class group. For the remainder of Section 2.2, we assume S' is connected; furthermore, either $\xi(S') \geq 1$ or S' is an annular neighborhood of a curve in S .

For ease of exposition and the convenience of considering $\partial S'$ a multicurve in S , let us make a convention that $\partial S'$ refers only to those boundary curves essential in S . In later sections we consider multiple, possibly nested subsurfaces, but these are always implicitly or explicitly contained in some largest surface S .

Except when S' is an annulus, it is clear one can embed $\mathcal{C}(S')$, or at least its vertex set, in $\mathcal{C}(S)$, but we seek a map in the opposite direction. One can associate curves on the surface to curves on a subsurface via subsurface projection, a notion appearing in [Iva88, Iva92], expanded in [MM00], and recapitulated here. In what follows, we define the projection map $\pi_{S'}$ from $\mathcal{C}_0(S)$ to the powerset $\mathcal{P}(\mathcal{C}_0(S'))$. To start, represent $\gamma \in \mathcal{C}_0(S)$ by a curve minimally intersecting $\partial S'$.

First suppose S' is not an annulus. If $i(\gamma, \partial S') = 0$, then either $\gamma \subset S'$, and we let $\pi_{S'}(\gamma) = \{\gamma\}$, or γ misses S' , and we let $\pi_{S'}(\gamma)$ be the empty set. Otherwise, γ intersects $\partial S'$. For each arc α of $\gamma \cap S'$, take the boundary of a regular neighborhood

of $\alpha \cup \partial S'$ and exclude the component curves which are not essential in S' . Because S' is neither pants nor annulus, some curves remain. Let these comprise $\pi_{S'}(\gamma)$.

In the case that S' is an annulus, when $i(\gamma, \partial S') = 0$ we let $\pi_{S'}(\gamma)$ be the empty set. When γ intersects $\partial S'$, one expects $\pi_{S'}(\gamma)$ to consist of the arcs of γ intersecting S' . However, ambiguity arises because curves such as γ and $\partial S'$ are defined up to ∂S -fixing isotopy, but vertices of $\mathcal{C}(S')$ represent arcs up to $\partial S'$ -fixing isotopy. To remedy this, give S a hyperbolic metric. Consider the cover of S corresponding to the fundamental group of S' embedded in that of S . This cover is a hyperbolic annulus endowed with a canonical “boundary at infinity” coming from the boundary of two-dimensional hyperbolic space (the unit circle, if one uses the Poincaré disk model). Name the closed annulus A , and let $\mathcal{C}(A)$ stand in for $\mathcal{C}(S')$. Each $\gamma \in \mathcal{C}_0(S)$ has a geodesic representative which lifts to A , and if γ intersects S' , some of the lifts connect the boundary components. Let these comprise $\pi'_{S'}(\gamma)$.

We say γ *projects nontrivially to S'* to mean $\pi_{S'}(\gamma)$ is not the empty set. Where X is some collection of curves $\{\gamma_i\}$, let $\pi_{S'}(X) = \bigcup_i \pi_{S'}(\gamma_i)$. If $\pi_{S'}(X)$ is not empty, let $\text{diam}_{S'}(X)$ denote its diameter in $\mathcal{C}(S')$; omit the subscript S' to mean diameter of X in $\mathcal{C}(S)$ itself. We are content with a map from $\mathcal{C}(S)$ to subsets of $\mathcal{C}(S')$, rather than $\mathcal{C}(S')$ directly, precisely because multicurves have bounded-diameter projection:

Lemma 2.1. *Suppose γ is a multicurve in the surface S , and S' is a (connected, essential, non-pants) subsurface. If γ projects nontrivially to S' ,*

$$\text{diam}_{S'}(\pi_{S'}(\gamma)) \leq 2.$$

This fact appears as Lemma 2.3 in [MM00]. Note that complexity ξ in [MM00] differs from our definition by three. Also, there is a minor error in Lemma 2.3, corrected in [Min03] (pg. 28), which we avoid by stating the lemma for multicurves rather than simplices of $\mathcal{C}(S)$ —these do not coincide if $\xi(S) = 1$.

When multicurves α and β both project nontrivially to S' , define their *projection distance* $d_{S'}(\alpha, \beta)$ as $\text{diam}_{S'}(\pi_{S'}(\alpha) \cup \pi_{S'}(\beta))$. This is a “distance” in that it satisfies the triangle inequality and symmetry, but it need not discern curves: Lemma 2.1 implies that disjoint multicurves have a projection distance of at most two, when defined. In the other extreme, one easily finds examples of curves close in $\mathcal{C}(S)$ with large projection distance in $\mathcal{C}(S')$ —a small illustration of the great wealth of structure that opens up when one considers not only $\mathcal{C}(S)$ but curve complexes of all subsurfaces (see, for example, [MM00]).

2.3. Cut-coded subsurfaces and domains. We now present an alternate definition of subsurface that ducks the nuisance of disconnected subsurfaces containing annular components parallel to the boundary of other components. Such subsurfaces are the only ones capable of being mutually nested (via isotopy) yet not topologically equivalent. Our technical antidote may seem tedious, but as an upside, it translates our current notion of a subsurface into an object encoded unambiguously by curves, a recurrent theme of this paper. Moreover, the new viewpoint facilitates the next section’s definition of active subsurfaces, based on Ivanov’s work on $\text{Mod}(S)$ subgroups. The efficient reader is welcome to skim the definition, taking note of Lemmas 2.2 and 2.3, and rely on Figure 1 for intuition.

A *cut-coded subsurface* of S consists of two pieces of information: (1) a multicurve γ and (2) a partition of the non-pants components of $S \setminus \gamma$ into two sets: excluded and included components.

Let us clarify what subsurface S' this data is meant to describe. Label the component curves of γ by $\gamma_1, \dots, \gamma_n$, and the included components A_1, \dots, A_m . The multicurve γ contains each component of ∂A_i , but some γ_j may not be a boundary curve for any A_i . Let N_j be regular neighborhoods of those γ_j belonging to no ∂A_i . Then S' consists of the union of the A_i , seen as subsurfaces, and the implied annuli N_j . Ignoring multiplicity, $\partial S'$ and γ are the same multicurve.

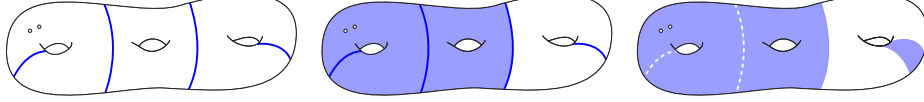


Figure 1. How to define a cut-coded subsurface. From left to right: (1) Set multicurve. (2) Choose included components (shaded). (3) Domains (shaded) correspond to included components and implied annuli.

Call the A_i *component domains* and the N_j *annular domains* of S' . We use *domain* to refer to either kind, or any subsurface that may appear as a domain—in other words, any connected (essential) non-pants subsurface.

Not all subsurfaces are cut-coded, as the latter never include pants, parallel annuli, or annuli parallel to the boundary of a component domain. Cut-coded subsurfaces are exactly those appearing as active subsurfaces of mapping classes, which the next section details.

The cut-coded subsurfaces of S admit a partial order \subset_{cc} detected by subsurface projection. Say a subsurface A *nests* in B if A may be isotoped into B . Now suppose A and B are cut-coded subsurfaces with domains A_i and B_j respectively. Say $A \subset_{cc} B$ if every A_i nests in some B_j . Transitivity and reflexivity of \subset_{cc} are obvious, but antisymmetry is relatively special. If $A \subset_{cc} B$ and $B \subset_{cc} A$, one can check A and B are given by the same data as cut-coded subsurfaces, ultimately because they never contain an annulus parallel to another domain. In contrast, two general disconnected subsurfaces can be mutually nested in each other, but not isotopic (consider, for example, parallel annular components of one subsurface nested in a single annular component of another subsurface).

Call two subsurfaces *disjoint* if they may be isotoped apart, and *overlapping* if they are neither disjoint nor nested; note that overlapping subsurfaces are distinct by definition. Projection determines relations between domains:

Lemma 2.2. *Suppose A and B are domains in S .*

- (i) $\pi_A(\partial B)$ is empty if and only if A nests in B or its complement.
- (ii) A and B overlap if and only if ∂A projects nontrivially to B and ∂B to A .
- (iii) If ∂A projects nontrivially to B but $\pi_A(\partial B)$ is empty, A nests in B .
- (iv) If $\pi_A(\partial B)$ and $\pi_B(\partial A)$ are both empty, A and B are equal or disjoint.

Proof. Statements (ii) – (iv) derive from (i). The forward implication of (i) requires connectedness of A : one may choose curves α_1 and α_2 that fill A , so that A is represented by a regular neighborhood of $\alpha_1 \cup \alpha_2$ with disks and annuli added to fill in homotopically trivial or ∂S -parallel boundary components. If $\pi_A(\partial B)$ is empty, then ∂B is disjoint from $\alpha_1 \cup \alpha_2$. Therefore the two curves, and consequently A itself, can be isotoped either entirely inside B or into its complement. The reverse implication is self-evident. \square

Consider a strictly increasing sequence $A_1 \subsetneq_{cc} A_2 \subsetneq_{cc} \dots \subsetneq_{cc} A_m$ of cut-coded subsurfaces of S . Let α be a maximal multicurve (i.e., a pants decomposition) including ∂A_i for all i . Each step of the sequence corresponds to at least one curve of α appearing for the first time as a component of ∂A_k or an essential curve in A_k , so the maximum length of the sequence is twice the number of components of α . We have just observed:

Lemma 2.3. *If $\xi(S) \geq 1$, a strictly increasing sequence of nonempty cut-coded subsurfaces of S has length at most 2ξ .*

It is easy to construct sequences realizing the upper bound.

2.4. Active subsurfaces and the Omnibus Theorem. Thurston originally classified elements of $\text{Mod}(S)$ by their action on the space of projective measured foliations, a piecewise linear space obtained by completing and projectivizing the space of weighted curves on S . He defined a *pseudo-Anosov diffeomorphism* as one that fixes a pair of transverse projective measured foliations, and proved that any pseudo-Anosov mapping class (i.e., any mapping class fixing no multicurve) has a representative pseudo-Anosov diffeomorphism. In fact he proved more:

Theorem 2.4 (Thurston, [Thu88],[FLP79]). *Every element f of $\text{Mod}(S)$ has a representative diffeomorphism F such that, after cutting S along some 1-dimensional submanifold C , F restricts to a pseudo-Anosov diffeomorphism on the union of some components of $S \setminus C$, and has finite-order on the union of the rest.*

Birman-Lubotzky-McCarthy proved there is a canonical choice for the isotopy class of C , and they gave a simple way to find it [BLM83]. We call this isotopy class the *canonical reduction multicurve* of f . Ivanov generalized to subgroups both the classification and the canonical cutting method [Iva92]. We build the notion of an *active subsurface* according to this last, largest perspective. Let us emphasize that, while we aim to paint a picture that seems perfectly natural, the validity of the definitions in this section depends on multiple lemmas and theorems from [Iva92], which serves as reference for assertions presented without proof. Thus, for everything other than active subsurfaces, we follow the notation in [Iva92].

Ivanov identified a congenial property of many mapping classes. Call a mapping class *pure* if, in the theorem above with the canonical choice for C , the map restricts to either the identity or a pseudo-Anosov map on each component (in particular, no components are permuted). Because one may take the empty set for C in Theorem 2.4, pseudo-Anosov maps are pure. The property of being pure is most useful for reducible mapping classes, because cutting along C “reduces” S to a collection of smaller subsurfaces. The point in what follows is to formalize this procedure.

Call a subgroup pure if it consists entirely of pure mapping classes. Nontrivial pure mapping classes have infinite order, so they let us ignore the complications of torsion. Fortunately, the mapping class group has finite-index pure subgroups (Theorem 3, [Iva92]). One of these is the kernel of the action of the mapping class group on homology with coefficients in $\mathbb{Z}/3\mathbb{Z}$. Name this subgroup $\Gamma_3(S)$.

Let us first define the canonical reduction multicurve $\sigma(G)$ and active subsurface $\mathcal{A}(G)$ of a pure subgroup $G < \text{Mod}(S)$. The multicurve $\sigma(G)$ consists of all curves γ such that (i) G fixes γ , and (ii) if some curve β intersects γ , then G does not fix β . The active subsurface $\mathcal{A}(G)$ is cut-coded with multicurve γ . Included components

correspond to those on which some element of G represents a pseudo-Anosov, rather than finite-order, mapping class, in the following precise sense. For each component Q of $S \setminus \sigma(G)$, we have a homomorphism $\rho_Q : G \rightarrow \text{Mod}(Q)$ such that $\rho_Q(g)$ is the mapping class of $F|_Q$, where F is a homeomorphism representing g . Let G_Q denote the image $\rho_Q(G)$. Because G is pure, for each component Q , the image G_Q either contains a pseudo-Anosov map, in which case Q is included, or is the trivial group, in which case Q is excluded (Theorem 7.16, [Iva92]).

It follows that the annular domains of $\mathcal{A}(G)$ correspond to neighborhoods of those $\gamma \in \sigma(G)$ that only bound components Q for which G_Q is trivial. Because by definition γ is not superfluous, some $g \in G$ restricts, in a neighborhood of γ , to a power of a Dehn twist about γ . Section 2.5 justifies why we consider Dehn twists to be pseudo-Anosov maps for annuli.

If G is an arbitrary subgroup, choose a finite-index pure subgroup H and define $\sigma(G) = \sigma(H)$ and $\mathcal{A}(G) = \mathcal{A}(H)$. Any choice of H gives the same multicurve, and one can always take $H = G \cap \Gamma_3(S)$. G acts on $S \setminus \sigma(G)$, although its elements may permute the components. For each component Q , one can define ρ_Q on the finite-index subgroup of G stabilizing Q . Each image G_Q is finite or contains a pseudo-Anosov map, and $\sigma(G)$ is the minimal multicurve with this property (Theorem 7.16, [Iva92]). It follows that an infinite subgroup G contains a pseudo-Anosov map if and only if $\sigma(G)$ is empty; let us call such a subgroup *irreducible*.

For any mapping class g , let $\sigma(g) = \sigma(\langle g \rangle)$ and $\mathcal{A}(g) = \mathcal{A}(\langle g \rangle)$. In this terminology, we recast Ivanov's Theorem 6.3 [Iva92] as follows:

Theorem 2.5. *For any $G < \text{Mod}(S)$ there exists $f \in G$ such that $\mathcal{A}(f) = \mathcal{A}(G)$.*

After Lemma 2.7 below, we can recognize Theorem 2.5 as Mosher's Omnibus Subgroup Theorem. We do not require this theorem for our proofs, only the validity of the definitions on which it is based. In fact, one can prove Theorem 1.1 with no mention of active subsurfaces for non-pure subgroups. However, we find it conceptually useful to state and prove the more general Theorem 3.1, which, recall, consists of Theorem 2.5 with control on the word length of f .

Facts about active subsurfaces occupy the remainder of this section. Directly from definitions, we derive Lemma 2.6 below. With this we obtain Lemmas 2.7 – 2.9, which enable our proof of Theorem 3.1. Let us refer to domains of $\mathcal{A}(G)$ or $\mathcal{A}(g)$ as domains of G or g respectively. Say G *moves* the curve $\gamma \in \mathcal{C}_0(S)$ if some element of G does not fix γ .

Lemma 2.6. *Suppose H and G are pure subgroups of $\text{Mod}(S)$ and $\gamma \in \mathcal{C}_0(S)$.*

- (a) *H moves γ if and only if γ projects nontrivially to some domain of H .*
- (b) *H and G fix and move the same curves if and only if $\mathcal{A}(H) = \mathcal{A}(G)$.*

Lemma 2.7. *If $H < G$ then $\mathcal{A}(H) \subset_{cc} \mathcal{A}(G)$.*

Proof. We may replace H and G by the pure subgroups obtained by intersecting with $\Gamma_3(S)$. Suppose A is a domain of G and B a domain of H . Because G , and thus H , fixes ∂A , $\pi_B(\partial A)$ is empty. Lemma 2.2 guarantees that B nests in A or its complement. Now suppose B is in the complement of every domain of G . Then any curve essential in B is fixed by G , thus by H . This contradicts that B is a domain of H , unless B has no essential curves. Thus B is an annulus and ∂B consists of (two copies of) a single curve β . Any γ intersecting β is moved by H , hence by G . But because G fixes γ , this means $\gamma \in \sigma(G)$. Because B is the annulus around

γ , B isotopes into $\mathcal{A}(G)$. Thus every domain of H nests in $\mathcal{A}(G)$, which means $\mathcal{A}(H) \subset_{cc} \mathcal{A}(G)$. \square

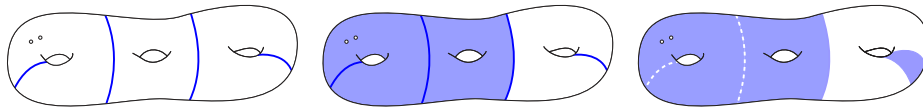


Figure 2. How to define the active subsurface of a group G . (1) Cut along canonical reduction multicurve. (2) Include components (shaded) on which G induces an irreducible subgroup. (3) Every element of G has a power supported on included components and implied annuli, by Lemma 2.7. By Theorem 2.5, some element has exactly this support.

In general, let $\mathcal{A}(X_1, X_2, \dots)$ denote the active subsurface of the group generated by X_1, X_2, \dots , where X_i may be either elements or subgroups of a mapping class group.

Lemma 2.8. *Suppose H and G are pure elements or subgroups of $\text{Mod}(S)$. Then $\mathcal{A}(H) \subset_{cc} \mathcal{A}(G)$ if and only if $\mathcal{A}(G) = \mathcal{A}(G, H)$.*

Proof. Observe that $\mathcal{A}(H) \subset_{cc} \mathcal{A}(G)$ implies that any curve fixed by G is fixed by H , and hence G and $\langle G, H \rangle$ fix the same curves. Apply Lemma 2.6 to $G < \langle G, H \rangle$ for the forward direction, and Lemma 2.7 to $H < \langle G, H \rangle$ for the reverse. \square

Lemma 2.9. *Let G be a pure subgroup of $\text{Mod}(S)$ generated by $\{H, H_1, \dots, H_k, \dots\}$, where H_k are groups or elements. Either $\mathcal{A}(H) = \mathcal{A}(G)$ or $\mathcal{A}(H) \neq \mathcal{A}(H, H_k)$ for some k .*

Proof. Suppose $\mathcal{A}(H) = \mathcal{A}(H, H_k)$ for all k . Let $K_i = \langle H, H_1, \dots, H_i \rangle$. Using induction and Lemma 2.8 one can show $\mathcal{A}(H) = \mathcal{A}(K_i)$ for all i . Because the K_i exhaust G , Lemma 2.7 implies $\mathcal{A}(g) \subset_{cc} \mathcal{A}(H)$ for all $g \in G$. By Lemma 2.6 one knows any curve fixed by H has empty projection to all $\mathcal{A}(g)$, so H and G fix and move the same curves, and consequently $\mathcal{A}(H) = \mathcal{A}(G)$. \square

2.5. Machinations in the curve complex. This section collects several important curve complex results that provide the foundation for our proofs. Together, these results link mapping class behavior with curve complex geometry. Following our theme of interpreting $\text{Mod}(S)$ via $\mathcal{C}(S)$, we note that these results also lead to a $\mathcal{C}(S)$ -centric $\text{Mod}(S)$ classification.

By definition, reducible mapping classes and finite-order mapping classes have bounded orbits in $\mathcal{C}(S)$. That pseudo-Anosov maps have infinite-diameter orbits is corollary to a theorem of Masur and Minsky:

Theorem 2.10 (Minimal translation of pseudo-Anosov maps [MM99]). *There exists $C = C(S) > 0$ such that, for any pseudo-Anosov $g \in \text{Mod}(S)$, vertex $\gamma \in \mathcal{C}_0(S)$, and nonzero integer n ,*

$$d_S(g^n(\gamma), \gamma) \geq C|n|.$$

This gives us a way to recognize pseudo-Anosov maps. It also suggests an alternate classification of elements on $\text{Mod}(S)$, by whether they have finite, finite-diameter, or infinite-diameter orbits in $\mathcal{C}(S)$. If one takes that classification as a

starting point, Dehn twists clearly qualify as pseudo-Anosov for the annulus mapping class group.

For pure mapping classes, one may even refine this classification. Suppose $g \in \text{Mod}(S)$ is pure and has a domain Y . The observation that, for all $\gamma \in \mathcal{C}_0(S)$, $\pi_Y(g(\gamma)) = g(\pi_Y(\gamma))$, yields:

Corollary 2.11 (Minimal translation on subsurfaces). *There exists $c = c(S) > 0$ such that, for any pure element $g \in \text{Mod}(S)$ with domain Y , vertex $\gamma \in \mathcal{C}_0(S)$ such that $\pi_Y(\gamma) \neq \emptyset$, and nonzero integer n ,*

$$d_Y(g^n(\gamma), \gamma) \geq c|n|.$$

Thus every nontrivial pure element g has infinite-diameter orbits in the curve complexes of its domains. Moreover, an orbit of g in the curve complex of one of its domains projects to a bounded diameter subset of the curve complex of any subsurface properly nested in that domain. This is because, if some g -orbit projects to a domain, most of its curves approximate the limiting laminations of g which fill that domain. Projection to the properly nested subsurface fails to distinguish between approximating curves that intersect the subsurface in parallel arcs. Thus the orbit looks roughly constant at large-magnitude powers of g . This is the heuristic behind another theorem of Masur and Minsky, which along with Corollary 2.11 is crucial to our core construction in Section 5:

Theorem 2.12 (Bounded geodesic image [MM00]). *Let Y be a proper domain of S . Let \mathcal{G} be a geodesic in $\mathcal{C}(S)$ whose vertices each project nontrivially to Y . Then there is a constant $M = M(S)$ such that*

$$\text{diam}_Y(\mathcal{G}) \leq M.$$

This theorem enabled Behrstock to obtain Lemma 2.13 below, which our proofs frequently employ. Here we give the elementary proof, with constructive constants, by Chris Leininger. Most of it appeared previously in [Man10], with a numerical error in the proof fixed here; this version also adds the possibility of annular domains.

Lemma 2.13 (Behrstock [Beh06]). *For any pair of overlapping domains Y and Z and any multicurve x projecting nontrivially to both,*

$$d_Y(x, \partial Z) \geq 10 \implies d_Z(x, \partial Y) \leq 4$$

Proof. First we gather the facts that prove the lemma when neither Y nor Z is an annulus. Suppose S' is a subsurface of S and $\xi(S), \xi(S') \geq 1$. Let u_0 and v_0 be curves on S which minimally intersect S' in sets of arcs. Suppose a_u is one these arcs for u_0 , and u a component of the boundary of a neighborhood of $a_u \cup \partial S'$; suppose a_v and v play the same role for v_0 . Then $u \in \pi_{S'}(u_0)$ and $v \in \pi_{S'}(v_0)$. Define intersection number of arcs to be minimal over isotopy of the entire surface. One has:

- (1) If $i(a_u, a_v) = 0$, then $d_{S'}(u_0, v_0) \leq 4$
- (2) If $i(u, v) > 0$, then $i(u, v) \geq 2^{(d_{S'}(u, v) - 2)/2}$
- (3) $i(u, v) \leq 4 + 4 \cdot i(a_u, a_v)$

Statement (1) follows from the proof of Lemma 2.1 (Lemma 2.3 in [MM00]). Straightforward induction proves (2), which Hempel records as Lemma 2.1 in [Hem01]. Fact (3) is the observation that essential curves from the regular neighborhoods of $a_u \cup \partial S'$ and $a_v \cup \partial S'$ intersect at most four times near every intersection

of a_u and a_v , plus at most four more times near $\partial S'$ (this latter four corrects the two appearing in [Man10]).

Now assume $\xi(Y), \xi(Z) \geq 1$. Because $d_Y(x, \partial Z) \geq 10 > 2$, the diameter is realized by curves $u \in \pi_Y(x), v \in \pi_Y(\partial Z)$ such that, by (2), $i(u, v) \geq 2^4 = 16$. By the definition of π_Y , these u and v come from arcs $a_u \subset x \cap Y$ and $a_v \subset \partial Z \cap Y$ respectively. By (3), $i(a_u, a_v) \geq (16 - 4)/4 = 3$. Thus a_u is an arc of x intersected thrice by an arc a_v of ∂Z , within the subsurface Y . Observe that one of the segments of a_u between points of intersection must lie within Z . This segment is an arc a_x of x disjoint from arcs of ∂Y in Z . Fact (1) implies $d_Z(x, \partial Y) \leq 4$.

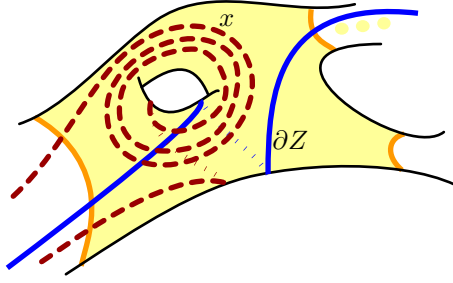


Figure 3. *Key point of Behrstock's lemma.* Because x and ∂Z have large projection distance in Y , one can find an arc of x intersecting an arc of ∂Z three times. Extra-fine dashed lines represent where curves run behind the surface.

The main idea of this proof works for annular domains after a few more relevant facts. Endow S with a hyperbolic metric and let A be an embedded annulus with geodesic core curve α ; let \tilde{A} be the corresponding annular cover of S . Let u and v be geodesic curves in S with lifts \tilde{u} and \tilde{v} traversing the core curve of \tilde{A} . We have already mentioned

$$(4) \quad d_A(u, v) = i(\tilde{u}, \tilde{v}) + 1.$$

Let $\tilde{\alpha}$ be the unique lift of α corresponding to the core curve of \tilde{A} , and let $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be the first lifts of α intersecting \tilde{u} on each side. On the sides of $\tilde{\alpha}_i$ opposite $\tilde{\alpha}$, \tilde{A} is isometric to its pre-image in the universal cover $\tilde{S} = \mathbb{H}^2$. Because geodesics intersect only once in \mathbb{H}^2 ,

- (5) at most two of the intersections of \tilde{u} and \tilde{v} occur outside the open segment of \tilde{u} between the $\tilde{\alpha}_i$.

Now we can retrace the proof above, augmenting it to address the possibility that Y or Z is an annulus. Fixing any hyperbolic metric, we may use geodesic representatives for $x, \partial Y$, and ∂Z . Suppose Y is an annulus with geodesic core curve y and \tilde{Y} is the corresponding annular cover of S . If $d_Y(x, \partial Z) \geq 10$, (4) implies that some lifts \tilde{x} and $\tilde{\partial Z}$ in \tilde{Y} intersect at least nine times. By (5), at least three of these intersections occur on an open segment of \tilde{x} between consecutive lifts of y : in the extreme case, \tilde{x} intersects $\tilde{\partial Z}$ exactly where \tilde{x} intersects the closed lift \tilde{y} and

adjacent lifts \tilde{y}_1 and \tilde{y}_2 on each side, plus three more intersections on the segment of \tilde{x} between \tilde{y}_1 and \tilde{y} , and the other three between \tilde{y} and \tilde{y}_2 . Neighborhoods of either of these segments embed in S . As before, one finds an arc of x intersecting ∂Z at least three times in a neighborhood disjoint from ∂Y . If Z is not an annulus, one repeats the conclusion of the first argument.

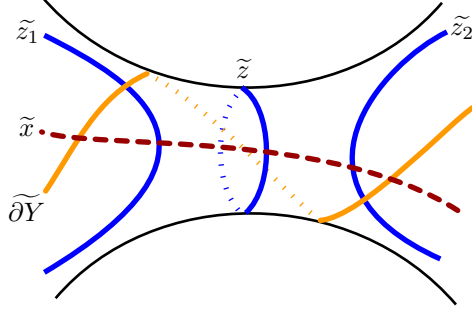


Figure 4. *Behrstock's lemma for the annulus.* If x and z intersect three times with no intersections of ∂Y in between, then in the annular cover corresponding to z , \tilde{x} can intersect $\tilde{\partial Y}$ no more than twice. Extra-fine dashed lines represent parts of curves on the far side of the annulus.

On the other hand, suppose that Z is an annulus with geodesic core curve z , and Y is any domain. The arguments thus far tell us one has an arc of x thrice intersecting z in a neighborhood of that arc disjoint from ∂Y . In the annular cover corresponding to Z , the three intersections correspond to a lift \tilde{x} of x intersecting the closed lift \tilde{z} of z and adjacent lifts \tilde{z}_1, \tilde{z}_2 on each side. Any lift of ∂Y cannot intersect \tilde{x} between these intersections, so (5) implies $i(\tilde{x}, \tilde{\partial Y}) \leq 2$. Fact (4) implies $d_Z(x, \partial y) \leq 3$. \square

Remark. Behrstock's lemma implies that $\mathcal{C}(S)$ -orbits of a mapping class g have bounded projection to subsurfaces that overlap with domains of g . Using stronger results, Theorem 2.12 in particular, one can prove that the g -orbit of any $\gamma \in \mathcal{C}(S)$ projects to an unbounded set in the curve complex of S' if and only if the orbit projects nontrivially to S' and S' is a domain of g . Thus one obtains a refined, curve-complex-based classification of elements of $\text{Mod}(S)$, by letting the phrase “ g is pseudo-Anosov on S' ” mean that the g -orbit of some $\gamma \in \mathcal{C}(S)$ has infinite-diameter projection to $\mathcal{C}(S')$.

3. THE RECIPE

Restated in the terminology introduced in Section 2.4, our main result is:

Theorem 3.1 (Main Theorem). *There exists a constant $K = K(S)$ with the property that, for any nonempty subset $\Sigma \subset \text{Mod}(S)$, there exists $f \in \langle \Sigma \rangle$ with Σ -length less than K , such that $\mathcal{A}(f) = \mathcal{A}(\Sigma)$.*

In Section 3.1, we derive Theorem 3.1 from the same result for a special case:

Theorem 3.2 (Two-generator pure subgroup). *There exists a constant $\ell = \ell(S)$ with the property that, for any reducible mapping classes $a, b \in \text{Mod}(S)$ generating a pure subgroup $\langle a, b \rangle$, there exists $f \in \langle a, b \rangle$ with $\{a, b\}$ -length less than ℓ , such that $\mathcal{A}(f) = \mathcal{A}(a, b)$.*

In Section 3.2, we set up key ingredients for the proof of Theorem 3.2, which we complete in Section 3.3

3.1. Proof of Theorem 3.1 using Theorem 3.2. Recall that $\Gamma_3(S)$ is the kernel of the $\text{Mod}(S)$ action on homology with coefficients in $\mathbb{Z}/3\mathbb{Z}$. By definition, the pure subgroup $H = \langle \Sigma \rangle \cap \Gamma_3(S)$ has the same active subsurface as $\langle \Sigma \rangle$, since H has finite index in $\langle \Sigma \rangle$. Suppose this index is d . Then H is generated by a set Σ' consisting of words less than $2d$ in length with respect to Σ (see, e.g., Lemma 3.4 of [SW92]). Therefore, if we find a full-support mapping class in H whose word length is less than ℓ_1 in Σ' , its word length is less than $2d\ell_1$ in Σ . The cardinality of the automorphism group $\text{Aut}(H_1(S, \mathbb{Z}/3\mathbb{Z}))$ gives an upper bound for d .

Consider the set $\Sigma' = \{h_i\}$ generating H . Renumbering as necessary, Lemmas 2.3 and 2.9 provide a sequence $\mathcal{A}(h_1) \subset_{cc} \mathcal{A}(h_1, h_2) \subset_{cc} \mathcal{A}(h_1, h_2, h_3) \dots$ terminating in at most $2\xi(S)$ steps at $\mathcal{A}(h_1, h_2, \dots, h_k) = \mathcal{A}(H) = \mathcal{A}(\Sigma)$.

Inductively, we define a sequence of mapping classes $\{p_1, \dots, p_k\}$ such that $\mathcal{A}(p_i) = \mathcal{A}(h_1, \dots, h_i)$ and p_i has Σ' -length less than ℓ^i . Let $p_1 = h_1$. Now suppose $i \geq 2$ and we have defined p_{i-1} with the desired properties. By Theorem 3.2 applied to the subgroup $\langle p_{i-1}, h_i \rangle$, we can find p_i with $\{p_{i-1}, h_i\}$ -length less than ℓ , such that $\mathcal{A}(p_i) = \mathcal{A}(p_{i-1}, h_i)$. Since $h_i \in \Sigma'$ and p_{i-1} has Σ' -length less than ℓ^{i-1} , p_i has Σ' -length less than ℓ^i . By Lemma 2.7, $\mathcal{A}(h_i) \subset_{cc} \mathcal{A}(p_i)$. By our inductive assumption that $\mathcal{A}(p_{i-1}) = \mathcal{A}(h_1, \dots, h_{i-1})$, we also know $\mathcal{A}(h_j) \subset_{cc} \mathcal{A}(p_{i-1}) \subset_{cc} \mathcal{A}(p_i)$ for all $j < i$. Lemmas 2.7 – 2.9 tell us $\mathcal{A}(p_i) = \mathcal{A}(h_1, \dots, h_i)$.

In particular, p_k is full-support for H and has Σ' -length less than ℓ^k , where $k \leq 2\xi$. The same p_k is full-support for $\langle \Sigma \rangle$ with Σ -length bounded above by

$$K(S) = 2 \cdot |\text{Aut}(H_1(S, \mathbb{Z}/3\mathbb{Z}))| \cdot \ell^{2\xi}$$

□

3.2. Ingredients for Theorem 3.2. We have reduced our task to generating a short-word full-support mapping class when the generating set has only two pure elements, the premise of Theorem 3.2. Its proof, which we leave to Section 3.3, depends on understanding two ideal situations detailed here.

In the first situation, the two generators a and b are both pseudo-Anosov. Call a pair of pseudo-Anosov maps *independent* if all pairs of nonzero powers fail to commute. In torsion-free groups such as those we consider, either two pseudo-Anosov maps generate a cyclic subgroup, or they are independent (see the proof of Theorem 5.12 in [Iva92]). In the latter case, we have complete understanding of $\langle a^n, b^m \rangle$ for n, m above some uniform bound, thanks to a theorem of Fujiwara:

Theorem 3.3 (Fujiwara [Fuj09]). *There exists a constant $L = L(S)$ with the following property. Suppose $a, b \in \text{Mod}(S)$ are independent pseudo-Anosov maps. Then for any $n, m \geq L$, $\langle a^n, b^m \rangle$ is an all-pseudo-Anosov rank-two free group.*

As always, S above is any compact, oriented surface. Let $T(S)$ be the finite set of topological types of domains in S (i.e., connected non-pants subsurfaces). For the proof of Theorem 3.2, we are interested in $\bar{L}(S) = \max\{L(S') : S' \in T(S)\}$.

In the second situation, a and b are both reducible. Recall that a pair of pure reducible mapping classes $a, b \in \text{Mod}(S)$ are *sufficiently different* if some domains $A \in \mathcal{A}(a)$ and $B \in \mathcal{A}(b)$ together fill S . In Section 5 we prove a version of Fujiwara's theorem for sufficiently different pairs of mapping classes:

Proposition 5.1. *There exists a constant $Q = Q(S)$ with the following property. Suppose $a, b \in \text{Mod}(S)$ are sufficiently different pure reducible mapping classes. Then for any $m, n > Q$, $\langle a^m, b^n \rangle$ is a rank-two free group and its elements are each either pseudo-Anosov or conjugate to a power of a or b .*

In the proof of Theorem 3.2, we make use of $\overline{Q}(S) = \max\{Q(S') : S' \in T(S)\}$.

We also require a recipe for writing a sufficiently different pair of mapping classes, given an arbitrary pair of pure reducible mapping classes generating an irreducible subgroup. Proposition 4.1 of Section 4 fills that need; we state it here for reference. Let $c = c(S)$ be the constant from Corollary 2.11.

Proposition 4.1. *Suppose a and b are pure reducible mapping classes in $\text{Mod}(S)$, S is connected, and $\langle a, b \rangle$ is irreducible. For any $n \geq \xi(S) - 1$ and $k \geq 20/c$,*

$$a_1 = (b^k a^k)^n b^k \cdot a \cdot ((b^k a^k)^n b^k)^{-1} \quad \text{and} \quad b_1 = (a^k b^k)^n a^k \cdot b \cdot ((a^k b^k)^n a^k)^{-1}$$

are sufficiently different pure reducible mapping classes.

3.3. Proof of Theorem 3.2. Let $G = \langle a, b \rangle$, where a and b are from the hypothesis of Theorem 3.2. Finding an element of full-support for G is equivalent to finding a map pseudo-Anosov on $\mathcal{A}(G)$, although we contend with the fact that $\mathcal{A}(G)$ may well be disconnected. On every domain of $\mathcal{A}(G)$, one of the following possibilities occurs:

- (i) a and b are pseudo-Anosov.
- (ii) a and b are both reducible.
- (iii) One of a and b is pseudo-Anosov and the other is reducible.

Our task is to write a word w in a and b that induces a pseudo-Anosov map on every domain of $\mathcal{A}(G)$. Theorem 3.3 is relevant to domains satisfying case (i), while Propositions 4.1 and 5.1 combine to take care of domains in case (ii). We will demonstrate how to convert case (iii) to either case (i) or (ii). Our choice of w must work for all cases simultaneously.

Let $n = \xi(S) - 1$ and $k = 20/c$, where $c = c(S)$ is the constant from Corollary 2.11. Define a_1 and b_1 be as in Proposition 4.1. On each domain of $\mathcal{A}(G)$, the pair a_1, b_1 satisfies one of the cases (i)-(iii)—the same case as a, b , in fact, because a_1 and b_1 are conjugates in G of a and b respectively. Recall that, where L and Q are the constants from Theorem 3.3 and Proposition 5.1 respectively, and $T(S)$ is the finite set of topological types of domains in S , we define $\overline{L} = \max\{L(S') : S' \in T(S)\}$ and $\overline{Q} = \max\{Q(S') : S' \in T(S)\}$. Choose $P \geq \max\{\overline{L}, \overline{Q}, 3/c\}$. Consider the following word:

$$w = b_1^P a_1^P b_1^{-P} a_1^P$$

On domains where possibility (i) holds, either a_1 and b_1 commute and $w = a_1^{2P}$, a pseudo-Anosov map, or a_1 and b_1 are independent and w is pseudo-Anosov by Theorem 3.3. On domains where possibility (ii) holds, w is pseudo-Anosov by Propositions 4.1 and 5.1.

On domains where possibility (iii) holds, and a_1 is the pseudo-Anosov map, we see by re-writing w as $b_1^P a_1^P b_1^{-P} \cdot a_1^P$ that it is the product of powers of pseudo-Anosov

maps, so we may proceed as we did for case (i). Otherwise a_1 is the reducible and b_1 is pseudo-Anosov on the domain, which we will call X . For any γ among the curves of $\sigma(a_1)$ that project nontrivially to X , $d_X(b_1^P(\gamma), \gamma) \geq Pc \geq 3$, where the first inequality employs Corollary 2.11 and the second is by construction. In particular, $\gamma \in \sigma(a_1)$ and $b_1^P(\gamma) \in \sigma(b_1^P a_1 b_1^{-P})$ together fill X , so a_1 and $b_1^P a_1 b_1^{-P}$ are sufficiently different pure reducible mapping classes. Then Proposition 5.1 guarantees w is pseudo-Anosov on X .

We have shown that, on all domains of $\mathcal{A}(G)$, w is pseudo-Anosov. Thus w is full-support for G . In terms of $\{a, b\}$, the word length of w is $4P \cdot (2k(2n+1) + 1)$. Recall $n = \xi - 1$ and $k = 20/c$ where $c = c(S)$ is the constant from Corollary 2.11. For Theorem 3.2, we may use

$$\ell = 4P \cdot (80\xi/c + 1).$$

□

4. SUFFICIENTLY DIFFERENT REDUCIBLES

Recall $c = c(S)$ is the constant from Corollary 2.11. The purpose of this section is to establish the following:

Proposition 4.1. *Suppose a and b are pure reducible elements of $\text{Mod}(S)$, S is connected, and $\langle a, b \rangle$ is irreducible. For any $n \geq \xi(S) - 1$ and $k \geq 20/c$,*

$$a_1 = (b^k a^k)^n b^k \cdot a \cdot ((b^k a^k)^n b^k)^{-1} \quad \text{and} \quad b_1 = (a^k b^k)^n a^k \cdot b \cdot ((a^k b^k)^n a^k)^{-1}$$

are sufficiently different pure reducible mapping classes.

In particular, we will show that any choice of $\alpha \in \sigma(a_1)$ and $\beta \in \sigma(b_1)$ together fill S . Necessarily α and β will bound some domains $X \in \mathcal{A}(a_1)$ and $Y \in \mathcal{A}(b_1)$, and X and Y will together fill S , meaning a_1 and b_1 are sufficiently different.

We prove Proposition 4.1 in two steps. First, via Lemma 4.4, we verify that α projects nontrivially to whichever domains Y of b intersect $\sigma(a)$, and similarly β projects nontrivially to whichever domains X of a intersect $\sigma(b)$. Lemma 4.4 also ensures the projection distances $d_Y(\alpha, \sigma(a))$ and $d_X(\beta, \sigma(b))$ are large enough to apply Lemma 4.5. Lemma 4.5, our second step, gives a condition under which α and β fill S . The proofs of Lemmas 4.4 and 4.5 each occupy their own subsection, followed at the end with the proof of Proposition 4.1.

4.1. The overlap graph and Lemma 4.4. For the purpose of step one, we introduce the *overlap graph* for any pair of pure nontrivial mapping classes a and b in $\text{Mod}(S)$, where S is connected. Let $\{A_i\}_{1 \leq i \leq m}$ and $\{B_j\}_{1 \leq j \leq n}$ be the domains of a and b respectively. The overlap graph $\mathcal{O}(a, b)$ consists of a vertex for every domain of a that overlaps with some domain of b , and for every domain of b that overlaps with some domain of a —recall that two domains overlap if the boundary of each domain projects nontrivially to the other. Edges connect domains that overlap, so that one may color the vertices depending on whether they represent domains of a or b , obtaining a bipartite graph. Assign each edge length one so that $\mathcal{O}(a, b)$ has the usual path metric.

Consider what happens if $\langle a, b \rangle$ is irreducible, or equivalently, $\mathcal{A}(a, b)$ is connected. Then a and b fix no common curve. If a is not pseudo-Anosov, then any domain A_i has essential boundary consisting of curves not fixed by b . This boundary ∂A_i must project nontrivially to some domain B_j of b . If B_j has no essential

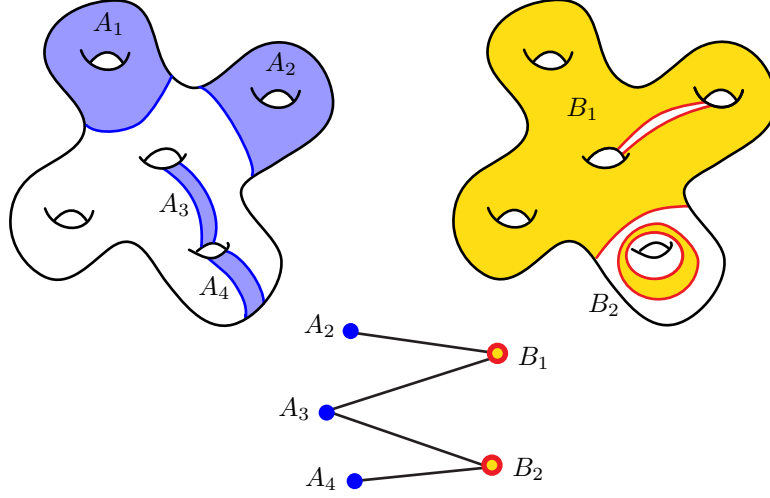


Figure 5. Example of an overlap graph.

boundary, then it is the entire surface, so b is pseudo-Anosov, and it follows from the definitions that $\mathcal{O}(a, b)$ is empty. Otherwise, b is not pseudo-Anosov, and ∂B_j projects nontrivially to some domain of a . If A_i is such a domain, then B_j and A_i overlap. If not, then A_i nests in B_j , and ∂B_j must project nontrivially to some other domain A_k (we are using Lemma 2.2, which details the distinction between nesting and overlapping). In this case, ∂A_k in turn projects nontrivially to B_j , because otherwise B_j would nest in A_k , implying A_i nests in A_k , a contradiction. So B_j and A_k overlap. In either case $\mathcal{O}(a, b)$ has at least two vertices and an edge connecting them.

When $\mathcal{O}(a, b)$ is not empty, we distinguish between the domains represented by vertices—call these domains *overlappers*—and those not represented. From the discussion above, we observe:

Lemma 4.2. *If S is connected and $a, b \in \text{Mod}(S)$ are pure reducible mapping classes generating an irreducible subgroup, then any non-overlapper domain of b nests in an overlapper domain of a .*

Of course, the conclusion also holds with a and b switched. Now we can show:

Lemma 4.3. *If S is connected and $a, b \in \text{Mod}(S)$ are pure reducible mapping classes generating an irreducible subgroup, then $\mathcal{O}(a, b)$ is connected.*

Proof. We find a path between any two vertices of $\mathcal{O}(a, b)$. Realize the domains of a and b as closed submanifolds with minimal pairwise intersection (i.e., ensure that boundary curves intersect essentially and use pairwise disjoint submanifolds to represent domains for the same mapping class). For any two vertices X and Y of $\mathcal{O}(a, b)$, choose points x and y in the corresponding domains, and connect these by a path p . Letting $A = \bigcup_i A_i$ and $B = \bigcup_j B_j$, one sees that each component of $S \setminus (A \cup B)$ is a disk or boundary-parallel annulus with a boundary component consisting of pieces of $\partial A \cup \partial B$. Thus one can isotope p to lie entirely within $A \cup B$, and furthermore transversal to $\partial A \cup \partial B$. The path also lies entirely within

overlappers, by Lemma 4.2. Tracing the path gives a finite sequence of domains, each overlapping with the neighbor before and after, alternating between A_i 's and B_j 's by design. The same sequence appears as a path in $\mathcal{O}(a, b)$ from X to Y . \square

Now we are ready to prove the following:

Lemma 4.4. *Suppose a and b are pure reducible elements of $\text{Mod}(S)$, S is connected, and $\langle a, b \rangle$ is irreducible. Let k, n be integers such that $k \geq 20/c(S)$ and $n \geq \xi(S) - 1$. Let $\alpha = (b^k a^k)^n b^k(\alpha_1)$, where α_1 is any curve in $\sigma(a)$; let Y be any domain of b . If $\sigma(a)$ projects nontrivially to Y , then so does α , and $d_Y(\alpha, \sigma(a)) \geq 14$.*

Proof. We use the overlap graph to track the image of α_1 under alternating applications of b^k and a^k . For ease of exposition, we will not distinguish between overrapper domains and their representative vertices in $\mathcal{O}(a, b)$.

Let Y_0 be the set of vertices of $\mathcal{O}(a, b)$ corresponding to the overlappers of b that intersect α_1 ; note that Y_0 cannot be empty. Let X_0 be the vertices adjacent to Y_0 . For $i \in \mathbb{N}$, let Y_i be the vertices adjacent to X_{i-1} , and let X_i be the vertices adjacent to Y_i . In other words, Y_i consists of the b -domains that overlap with a -domains of X_{i-1} , and X_i consists of the a -domains that overlap with b -domains of Y_i .

Observe that the vertices of Y_{i+1} lie within a radius-two neighborhood of Y_i . Because $\xi(S)$, the maximum number of disjoint curves on S , gives an upper bound on the number of domains for a mapping class in $\text{Mod}(S)$, twice $\xi(S)$ bounds the number of vertices in $\mathcal{O}(a, b)$. By Lemma 4.3, $\mathcal{O}(a, b)$ is connected. It follows that, for $n \geq \xi(S) - 1$, Y_n contains all the b -vertices, that is, all the overrapper domains of b . If a domain Y of b is not an overrapper, then by Lemma 4.2 it nests in some domain of a , and thus cannot intersect $\sigma(a)$. So if $\sigma(a)$ projects nontrivially to Y , Y must be in Y_n . To prove Lemma 4.4 it suffices to establish the following:

(*) For any Y in Y_n , $d_Y((b^k a^k)^n b^k(\alpha_1), \sigma(a)) \geq 14$

We induct on n . We defined Y_0 so that α_1 projects nontrivially to any domain Y in Y_0 . Applying Corollary 2.11,

$$d_Y(b^k(\alpha_1), \alpha_1) \geq c|k| \geq 20 \geq 14$$

Because α_1 is a component of the multicurve $\sigma(a)$, and because projection distances are diameters, we obtain the $n = 0$ case of (*):

$$d_Y(b^k(\alpha_1), \sigma(a)) \geq 14$$

Supposing (*) true for $n = m$, we prove it for $n = m + 1$. Let Y be any domain in Y_m . For any a -domain X , ∂X is a subset of $\sigma(a)$ by definition, and $\text{diam}_Y(\sigma(a)) \leq 2$, by Lemma 2.1. Thus if X overlaps Y , (*) and the triangle inequality give

$$d_Y((b^k a^k)^m b^k(\alpha_1), \partial X) \geq 12$$

In particular, $(b^k a^k)^m b^k(\alpha_1)$ intersects every X that overlaps some Y of Y_m ; these X are exactly the vertices in X_m . That is, for any X in X_m , there exists Y in Y_m such that the above inequality holds. We may apply Behrstock's lemma, giving

$$d_X((b^k a^k)^m b^k(\alpha_1), \partial Y) \leq 4$$

On the other hand, Corollary 2.11 guarantees

$$d_X(a^k(b^k a^k)^m b^k(\alpha_1), (b^k a^k)^m b^k(\alpha_1)) \geq c|k| \geq 20$$

Employing the triangle inequality,

$$d_X(a^k(b^k a^k)^m b^k(\alpha_1), \partial Y) \geq 16$$

Recall the above holds for any X in X_m , with some Y in Y_m depending on X . Now consider any Y' in Y_{m+1} . By construction, Y' overlaps some X in X_m , which in turn overlaps some Y in Y_m . We also know $\text{diam}_X(\partial Y \cup \partial Y') \leq 2$, because $\partial Y'$ and ∂Y are disjoint. Another triangle inequality gives

$$d_X(a^k(b^k a^k)^m b^k(\alpha_1), \partial Y') \geq 14$$

Starting with Behrstock's lemma, we mirror the last four inequalities:

$$\begin{aligned} d_{Y'}(a^k(b^k a^k)^m b^k(\alpha_1), \partial X) &\leq 4 \\ d_{Y'}(b^k a^k(b^k a^k)^m b^k(\alpha_1), a^k(b^k a^k)^m b^k(\alpha_1)) &\geq 20 \\ d_{Y'}(b^k a^k(b^k a^k)^m b^k(\alpha_1), \partial X) &\geq 16 \\ d_{Y'}(b^k a^k(b^k a^k)^m b^k(\alpha_1), \partial X') &\geq 14 \end{aligned}$$

where X' is any domain of X_{m+1} overlapping Y' . Note that $\partial X' \subset \sigma(a)$ as subsets of $\mathcal{C}_0(S)$, and recall Y' was an arbitrary vertex of Y_{m+1} . Thus we may re-write the last inequality to give the $m+1$ step of the induction claim: for any Y in Y_{m+1} ,

$$d_Y((b^k a^k)^{m+1} b^k(\alpha_1), \sigma(a)) \geq 14$$

□

4.2. The filling condition of Lemma 4.5. Our second step to proving Proposition 4.1 is the following lemma, whose proof should feel like *déjà vu*.

Lemma 4.5. *Suppose a and b are pure reducible elements of $\text{Mod}(S)$, S is connected, and $\langle a, b \rangle$ is irreducible. Suppose $\alpha, \beta \in \mathcal{C}_0(S')$ satisfy the following:*

- (i) *If $\sigma(a)$ projects nontrivially to a domain Y of b , then so does α , and $d_Y(\alpha, \sigma(a)) \geq 14$.*
- (ii) *If $\sigma(b)$ projects nontrivially to a domain X of a , then so does β , and $d_X(\beta, \sigma(b)) \geq 14$.*

Then α and β fill S .

Proof. Given an arbitrary curve $\gamma \in \mathcal{C}(S')$, we show it intersects either α or β . Because γ cannot be fixed by both a and b , it projects nontrivially to some domain, and we can choose this domain to be an overlapper, by Lemma 4.2. Without loss of generality we may assume γ projects nontrivially to a domain Y of b , which overlaps with a domain X of a . In the curve complex, ∂X and ∂Y are subsets of $\sigma(a)$ and $\sigma(b)$ respectively. Because multicurves have diameter-two projections (Lemma 2.1), (i) and (ii) imply:

$$d_Y(\alpha, \partial X) \geq 12$$

$$d_X(\beta, \partial Y) \geq 12$$

Assuming γ does not intersect α , we show it must intersect β . Again Lemma 2.1 implies $d_Y(\alpha, \gamma) \leq 2$. The following inequalities employ the triangle inequality, Behrstock's lemma, and another triangle inequality.

$$d_Y(\gamma, \partial X) \geq 10$$

$$d_X(\gamma, \partial Y) \leq 4$$

$$d_X(\gamma, \beta) \geq 8$$

In one more instance of Lemma 2.1, the last inequality shows γ and β intersect. \square

4.3. Proof of Proposition 4.1. Let $\alpha = (b^k a^k)^n b^k(\alpha_1)$, where α_1 is any curve in $\sigma(a)$, and let $\beta = (a^k b^k)^n b^k(\beta_1)$, where β_1 is any curve in $\sigma(b)$. Taken together, Lemma 4.4, the symmetry between a and b , and Lemma 4.5 show that α and β fill S . Since $\alpha \in \sigma(a_1)$ and $\beta \in \sigma(b_1)$, we see a_1 and b_1 are sufficiently different pure reducible mapping classes. \square

5. CONSTRUCTION OF PSEUDO-ANOSOV MAPS AND SUBGROUPS

This section contains the construction at the heart of the proof of our Main Theorem. That is, we prove the following:

Proposition 5.1. *There exists a constant $Q = Q(S)$ with the following property. Suppose $a, b \in \text{Mod}(S)$ are sufficiently different pure reducible mapping classes. Then for any $m, n > Q$, $\langle a^m, b^n \rangle$ is a rank-two free group and its elements are each either pseudo-Anosov or conjugate to a power of a or b .*

When a and b are Dehn twists, Proposition 5.1 restricts to a well-known theorem of Thurston [Thu88]; in the same vein, further Dehn twist investigations appear in [Pen88, Ish96, HT02].

The second part of this section upgrades Proposition 5.1 to Proposition 1.2 given in the introduction, by proving the all-pseudo-Anosov subgroups of $\langle a^m, b^n \rangle$ are convex cocompact.

5.1. From sufficiently different to pseudo-Anosov. Recall that a pseudo-Anosov map is characterized by having infinite-diameter orbits in the curve complex. Thus we will know w is a pseudo-Anosov if distances $d(\gamma, w^n(\gamma))$ grow as n increases, where γ is some vertex in $\mathcal{C}(S)$. Proposition 5.1 is an application of Lemma 5.2 below, which uses the Bounded Geodesic Image Theorem (Theorem 2.12) to glean geometric information about sequences of geodesics. This lemma generalizes Proposition 5.2 in [KL08b], and the author learned its “bootstrapping” strategy from Leininger, who takes a similar tack in [Lei].

Let $M = M(S)$ be the constant from Theorem 2.12. For any vertices w and w' in $\mathcal{C}_0(S)$, we use $[w, w']$ to denote a geodesic connecting w and w' in $\mathcal{C}_1(S)$ (whose geodesics are typically not unique, we should note). Below, properties (ii) and (iii) ensure that the projection in (iv) is well-defined.

Lemma 5.2. *Suppose $\{Y_j\}$ is a sequence of domains in S , and $\{X_j\}$ a sequence of subsets of $\mathcal{C}_0(S)$, for which the following properties hold for all j :*

- (i) $\text{diam}(X_j) \leq 2$
- (ii) For all j , $X_j \cap X_{j+1} = \emptyset$
- (iii) The set of curves that project trivially to Y_j is a subset of X_j
- (iv) $d_{Y_j}(w_{j-1}, w_{j+1}) > 2M$ for any choice of $w_{j-1} \in X_{j-1}$, $w_{j+1} \in X_{j+1}$

Then for any $w_i \in X_i$ and $w_{i+k} \in X_{i+k}$, any geodesic $[w_i, w_{i+k}]$ contains a vertex from X_j for $i \leq j \leq i+k$. Also, the sets X_j are pairwise disjoint. In particular, $[w_i, w_{i+k}]$ has length at least k .

Proof. First let us prove that, for any integer $k \geq 1$, $w_i \in X_i$ and $w_{i+k} \in X_{i+k}$, any geodesic $[w_i, w_{i+k}]$ contains a vertex from X_j for $i \leq j \leq i+k$. Our claim is vacuously true for $k = 1$, and our proof proceeds by induction on k .

The general induction step works even for $k = 2$, but we separate this case to highlight its use of the Bounded Geodesic Image Theorem, Theorem 2.12. By property (iv), $d_{Y_{i+1}}(w_i, w_{i+2}) > 2M > M$, so Theorem 2.12 requires $[w_i, w_{i+2}]$ contain a vertex v disjoint from Y_{i+1} . By property (iii), v is contained in X_{i+1} . Of course, the endpoints of $[w_i, w_{i+2}]$ lie in X_i and X_{i+2} , so the induction claim holds.

Now assume $k > 2$ and for any $w_i \in X_i, w_{i+k-1} \in X_{i+k-1}$, any geodesic $[w_i, w_{i+k-1}]$ intersects X_j for $i \leq j \leq i+k-1$. The lemma will follow easily if we can show that, for any $w_{i+k} \in X_{i+k}$, any geodesic $[w_i, w_{i+k}]$ intersects X_{i+k-1} .

Choosing any $w_{i+k-2} \in X_{i+k-2}$, our first step is to show one can pick a geodesic $[w_i, w_{i+k-2}]$ avoiding X_{i+k-1} . Suppose we are given a geodesic that doesn't, that is, for some $w'_{i+k-1} \in X_{i+k-1}$,

$$[w_i, w_{i+k-2}] = [w_i, w'_{i+k-1}] \cup [w'_{i+k-1}, w_{i+k-2}].$$

Let us require w'_{i+k-1} to be the first vertex along $[w_i, w_{i+k-2}]$ belonging to X_{i+k-1} , so that only the last vertex of $[w_i, w'_{i+k-1}]$ lies in X_{i+k-1} . The induction hypothesis applied to the segment $[w_i, w'_{i+k-1}]$ implies that, for some $w'_{i+k-2} \in X_{i+k-2}$,

$$[w_i, w_{i+k-2}] = [w_i, w'_{i+k-2}] \cup [w'_{i+k-2}, w'_{i+k-1}] \cup [w'_{i+k-1}, w_{i+k-2}].$$

Property (ii) ensures the second and third segments above each have length at least one. Thus their union $[w'_{i+k-2}, w_{i+k-2}]$ has length at least two. Property (i) tells us we can replace it with a length-2 geodesic contained entirely in X_{i+k-2} , giving a new $[w_i, w_{i+k-2}]$ avoiding X_{i+k-1} .

Property (iii) ensures that every vertex of $[w_i, w_{i+k-2}]$ projects nontrivially to Y_{i+k-1} , as does w_{i+k} . Therefore Theorem 2.12, the Bounded Geodesic Image Theorem, applies:

$$d_{Y_{i+k-1}}(w_i, w_{i+k-2}) \leq \text{diam}_{Y_{i+k-1}}([w_i, w_{i+k-2}]) \leq M.$$

Finally, a triangle inequality:

$$\begin{aligned} \text{diam}_{Y_{i+k-1}}([w_i, w_{i+k}]) &\geq d_{Y_{i+k-1}}(w_i, w_{i+k}) \\ &\geq d_{Y_{i+k-1}}(w_{i+k-2}, w_{i+k}) - d_{Y_{i+k-1}}(w_i, w_{i+k-2}) \\ &> 2M - M = M \end{aligned}$$

Again by Theorem 2.12, we know $[w_i, w_{i+k}]$ intersects X_{i+k-1} . The end of the induction is easy: for some $w_{i+k-1} \in X_{i+k-1}$,

$$[w_i, w_{i+k}] = [w_i, w_{i+k-1}] \cup [w_{i+k-1}, w_{i+k}].$$

The induction hypothesis says the first segment on the right intersects each X_j , $i \leq j \leq i+k-1$. So the geodesic on the left intersects X_j , $i \leq j \leq i+k$, as required.

Finally we check that the sets X_j are pairwise disjoint. Suppose $z \in X_i \cap X_{i+k}$ for some nonzero k . By the part of the lemma already proved, the geodesic $[z, z]$ contains vertices in X_j for $i \leq j \leq i+k$: simply put, all those X_j intersect at z . But consecutive X_j do not intersect. \square

Proof of Proposition 5.1. Let $Q = Q(S) = \max\{3, (2M+4)/c\}$ where $M = M(S)$, the constant from Theorem 2.12, and $c = c(S)$, the constant from Corollary 2.11. The hypothesis states that a and b are pure reducible mapping classes, with $A \in \mathcal{A}(a)$ and $B \in \mathcal{A}(b)$, and A and B together fill S . We will show that for any $m, n > Q$, $\langle a^m, b^n \rangle$ is free and nontrivial elements of $\langle a^m, b^n \rangle$ are either pseudo-Anosov or conjugate to powers of a^m or b^n .

If S is a torus or four-punctured sphere, the only nontrivial pure reducible mapping classes are powers of Dehn twists about curves, and any pair of distinct curves fill the surface. In this case the result for $Q = 3$ follows from Theorem 3.5 in the detailed study of [HT02]. For the rest of the proof let us assume $\xi(S) > 1$.

Let $n, m \geq Q$ be arbitrary integers. Choose any components α of ∂A and β of ∂B . Let $C_A \subset \mathcal{C}_0(S)$ be the vertices with empty projection to A , and define C_B analogously. Observe that C_A and C_B sit in 1-neighborhoods of α and β , respectively, so they each have diameter 2 in $\mathcal{C}(S)$. Because any curve intersects either A or B , C_A and C_B contain no common vertices—this is the only place we use the fact that A and B fill S .

Here's the whole point of our choice of Q : for any nonzero integer k ,

$$\begin{aligned} d_A(C_B, a^{mk}(C_B)) &\geq d_A(\beta, a^{mk}(\beta)) \geq |mk|c > Qc \geq 2M + 4 \\ d_B(C_A, b^{nk}(C_A)) &\geq d_B(\alpha, b^{nk}(\alpha)) \geq |nk|c > Qc \geq 2M + 4 \end{aligned}$$

Notice that the domains A, B and sets C_A, C_B play the same roles for a^m and b^n as they do for a and b . Therefore it suffices to prove the following lemma:

Lemma 5.3. *Given pure reducible mapping classes a and b , choose corresponding domains A and B , and define C_A and C_B as above. Suppose C_A and C_B contain no common vertices, and furthermore, for any nonzero k ,*

$$d_A(C_B, a^k(C_B)) > 2M + 4 \quad \text{and} \quad d_B(C_A, b^k(C_A)) > 2M + 4.$$

Then $\langle a, b \rangle$ is free, and every nontrivial element of $\langle a, b \rangle$ is pseudo-Anosov except those conjugate to powers of a or b .

By construction, for any $\gamma, \gamma' \in C_B$ or $\delta, \delta' \in C_A$,

$$(1) \quad d_A(\gamma, a^k(\gamma')) > 2M \quad \text{and} \quad d_B(\delta, b^k(\delta')) > 2M$$

We use this shortly to apply Lemma 5.2.

In the abstract free group on a and b , a word w has reduced form $w = s_1 s_2 \cdots s_R$, where the *syllables* s_i are nontrivial powers of either a or b , and s_i is a power of a if and only if $s_{i\pm 1}$ is a power of b (i.e. powers of a and b alternate). Define *syllable length* $|\cdot|_*$ by $|w|_* = R$.

Claim. For any word w , either $d(w(\alpha), \alpha) \geq |w|_*$ or $d(w(\beta), \beta) \geq |w|_*$.

It immediately follows that $\langle a, b \rangle$ is a rank-two free group. If $|w|_*$ is even, it is easy to check $|w^n|_* = n|w|_*$. The claim implies $d(w^n(\gamma), \gamma) \geq n|w|_*$, where γ is α or β , depending on w . In particular the orbit of w has infinite diameter in the curve complex, so w is a pseudo-Anosov. If $|w|_*$ is odd and neither conjugate to a power of a nor of b , then it is conjugate to v such that $|v|_*$ is even. As v is pseudo-Anosov, so is its conjugate w .

It remains to prove the claim. Towards this, we describe a sequence of domains and $\mathcal{C}_0(S)$ -subsets fulfilling the hypotheses of Lemma 5.2. Let w_0 be the identity, $w_1 = s_1$, $w_2 = s_1 s_2$, and so forth, so that w_i is the word formed by the first i syllables of w . If s_1 is a power of a , let $I(a)$ correspond to the even integers, and $I(b)$ to the odds; if s_1 is a power of b , switch the roles of even and odd. For $r \in \{0, \dots, R\}$, define a sequence of vertices γ_r , domains Y_r , and sets X_r as follows.

$$\begin{array}{llll} \gamma_r = w_r(\alpha) & Y_r = w_r(A) & X_r = w_r(C_A) & \forall r \in I(a) \cap [0, R] \\ \gamma_r = w_r(\beta) & Y_r = w_r(B) & X_r = w_r(C_B) & \forall r \in I(b) \cap [0, R] \end{array}$$

In addition, set $\{\gamma_{-1}, Y_{-1}, X_{-1}\}$ equal to $\{\beta, B, C_B\}$, if s_1 is a power of a , or $\{\alpha, A, C_A\}$, if s_1 is a power of b .

Each X_j is isometric to C_A or C_B , and furthermore pairs X_j, X_{j+1} are isometric to the pair C_A, C_B , disregarding order. Therefore the sequence $\{Y_j, X_j\}$ meets conditions (i) – (iii) of Lemma 5.2. Condition (iv) requires $d_{Y_j}(v_{j-1}, v_{j+1}) > 2M$ for any choice of $v_{j-1} \in X_{j-1}$, $v_{j+1} \in X_{j+1}$. For $j = 0$, this condition simply restates one of the inequalities in (1) above, so let us suppose $j \geq 1$. Without loss of generality, assume $Y_j = w_j(A)$ and $v_{j-1} = w_{j-1}(\gamma_-)$, $v_{j+1} = w_{j+1}(\gamma_+)$ for some $\gamma_-, \gamma_+ \in C_B$. Because subsurface projection commutes naturally with the action of the mapping class group,

$$d_{w_j(A)}(v_{j-1}, v_{j+1}) = d_A(w_j^{-1}(v_{j-1}), w_j^{-1}(v_{j+1})) = d_A(s_j^{-1}(\gamma_-), s_{j+1}(\gamma_+))$$

Exactly one of s_j and s_{j+1} is a power of a , while the other is a power of b . In any case one knows that, for some $\gamma, \gamma' \in C_B$ and nonzero k ,

$$d_A(s_j^{-1}(\gamma_-), s_{j+1}(\gamma_+)) = d_A(a^k(\gamma), \gamma')$$

We used the fact that b fixes C_B setwise. Applying inequality (1) above, we can conclude $d_{w_j(A)}(v_{j-1}, v_{j+1}) > 2M$.

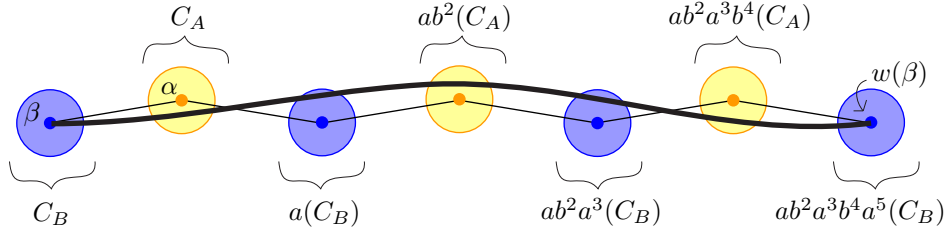


Figure 6. The sequence defined in the proof of Proposition 5.1. Here $w = ab^2a^3b^4a^5$. Vertices correspond to γ_i and shaded circles represent X_i , which each sit in the 1-neighborhood of γ_i . The heavy, smooth line is a geodesic connecting β and $w(\beta)$; it intersects each of the X_i .

By Lemma 5.2, the geodesics $[\gamma_{-1}, \gamma_R]$ and $[\gamma_0, \gamma_R]$ have lengths at least $R + 1$ and R respectively. Depending on w , one of these geodesics is either $[\alpha, w(\alpha)]$ or $[\beta, w(\beta)]$. This proves the claim, thus the lemma, thus the proposition. \square

Remark. One can ask whether Proposition 5.1 above can be proven for n -tuples of pure mapping classes subject to appropriate conditions. For example, if pairwise they fulfill the requirements for a and b in the proposition, do sufficiently high powers generate rank- n free groups? The paper [CLM10] can be seen as expanding the methods here to prove that certain sets of mapping classes have powers that generate right-angled Artin groups, which generalize free groups. However, those powers necessarily depend on the particular set of mapping classes one considers. In fact, without additional conditions, one cannot expect a rank- n version of Proposition 5.1 with Q uniform in S . For example, consider $\langle a, b, a^{n!}b^{n!} \rangle$ where a and b generate a free group. The groups $\langle a^k, b^k, (a^{n!}b^{n!})^k \rangle$ are the rank-three free group for large enough k , but when $k \leq n$ the third generator is superfluous.

5.2. Convex cocompact all-pseudo-Anosov subgroups. Now let us upgrade Proposition 5.1 to Proposition 1.2 of the introduction. The action of a group $G < \text{Mod}(S)$ on the curve complex gives a *quasi-isometric embedding* $G \hookrightarrow \mathcal{C}(S)$ if, for some $\gamma \in \mathcal{C}_0(S)$, $K > 1$, and $C \geq 0$, and for all $w \in G$,

$$(2) \quad K|w| + C \geq d(\gamma, w(\gamma)) \geq |w|/K - C$$

where $|w|$ gives word length with respect to some word metric on G . From (2) one sees that such a group contains no non-trivial reducible elements, by considering the orbit of such an element. Let us call a group *convex cocompact* if it fulfills (2) for some γ , K , and C as above. Hamenstädt and Kent-Leininger proved that this is equivalent to the definition of convex cocompact mapping class subgroups introduced by Farb and Mosher in [FM]. Convex cocompact subgroups which are free of finite rank are called *Schottky*. Farb and Mosher proved these are common in mapping class groups of closed surfaces, and Kent and Leininger include a new proof which also works for non-closed surfaces:

Theorem 5.4 (Abundance of Schottky groups [FM], [KL08a]). *Given a finite set of pseudo-Anosov maps $\{g_1, g_2, \dots, g_k\}$ which are independent (i.e., no pairs of powers commute), there exists l such that for all $m > l$, $\{g_1^m, g_2^m, \dots, g_k^m\}$ is Schottky.*

Fujiwara found a uniform bound for the above theorem in the case of two-generator subgroups; this is the full version of the theorem quoted earlier as Theorem 3.3:

Theorem 5.5 (Fujiwara [Fuj09]). *There exists a constant $L = L(S)$ with the following property. Suppose $a, b \in \text{Mod}(S)$ are independent pseudo-Anosov elements. Then for any $n, m \geq L$, $\langle a^n, b^m \rangle$ is Schottky.*

Proposition 1.2 provides a source of Schottky subgroups with arbitrary rank. It simply adds to Proposition 5.1 the claim that finitely generated all-pseudo-Anosov subgroups of $\langle a, b \rangle$ are Schottky. Recall that a and b are pure mapping classes with essential reduction curves α and β respectively, such that $\alpha \cup \beta$ fills S . Proposition 5.1 tells us that $G = \langle a, b \rangle$ is a free group, and its all-pseudo-Anosov subgroups are exactly those containing no conjugates of powers of a or b . For example, any subgroup of the commutator group $[G, G]$ qualifies.

Proof of Proposition 1.2. Let $G = \langle a, b \rangle$ as in the proposition, and let H be a finitely generated, all-pseudo-Anosov subgroup of G . To show that H is Schottky, we need quasi-isometry constants for the inequalities in (2). For an orbit embedding of a finitely generated group, the upper bound of (2) always comes for free, using any K greater than the largest distance some generator translates γ . So our work is the lower bound. In the proof of Lemma 5.1 we saw that $d(\gamma, w(\gamma)) \geq |w|_*$ where γ is one of α or β , depending on w . It is not hard to check that, in general, $d(\alpha, w(\alpha)) \geq |w|_* - 1$ (the same is true replacing α with β). However, syllable length is not a word metric with respect to any finite generating set for H . In what follows, we define a convenient generating set for H such that, letting $|\cdot|_H$ denote the corresponding word metric, we have $K|w|_* \geq |w|_H$, where K is the size of this generating set. Then we can conclude that $d(\alpha, w(\alpha)) \geq |w|_H/K - 1$, completing the proof.

The existence of this convenient generating set has no relation to our setting of subgroups of mapping class groups, so we isolate this fact as a separate technical lemma. We only need that G is a rank two free group generated by a and b . Suppose

H is a finitely generated subgroup. For $w \in H$, let $|w|$ denote its length in G with respect to the generating set $\{a, b\}$. Let $H_l = \{h \in H : |h| \leq l\}$ and choose L so that H_L generates H . Define $|\cdot|_H$ as word length in H with respect to H_L . Let K be the number of elements in H_L .

Lemma 5.6. *If H contains no element conjugate to a power of a or b , then $K|w|_* \geq |w|_H$.*

Proof. Suppose $w \in H$ has length $|w|_H = n$, and $w = h_1 h_2 \cdots h_n$, where $h_i \in H_L$. One knows that $|h_i h_{i+1}| > |h_i|, |h_{i+1}|$ because otherwise $h_i h_{i+1} \in H_L$, which contradicts that $|w|_H = n$ (one could take a shorter path to w in the Cayley graph of H with respect to H_L , by replacing h_i and h_{i+1} with their product, another generator). In particular this means that strictly less than half of the word h_i , written as a product of a 's and b 's, gets canceled by a piece of the word h_{i+1} in a 's and b 's. Likewise, strictly less than half gets canceled by a piece of the word h_{i-1} . Therefore, at least the middle letter of h_i , if $|h_i|$ is odd (the middle pair of letters if $|h_i|$ is even) gets contributed to the $\{a, b\}$ -spelling of the word w . Incidentally, this is showing that $|w| \geq |w|_H$, implying that finitely generated subgroups of the rank-two free group are quasi-isometrically embedded.

Call the middle letter or pair of letters of each h_i its *core*. Syllable length $|w|_*$ can be shorter than $|w|_H$ only if a string of consecutive h_i 's, say, $h_i h_{i+1} \cdots h_k$, all have a or a^2 at their core, or if they all have b or b^2 at their core, and these cores contribute to the same syllable (power of a or b) in the $\{a, b\}$ -spelling of w . In that case one can write, for $i \leq j \leq k$,

$$h_j = u_j \cdot x^{e(j)} \cdot v_j$$

where x is a or b and $x^{e(j)}$ includes the core of each h_j . Furthermore, $v_j = u_{j+1}^{-1}$, so that

$$h_i h_{i+1} \cdots h_k = u_i x^N v_k$$

where $N = \sum_{i \leq j \leq k} e(j)$. It is possible u_i or v_k are empty words, but N cannot be zero, because otherwise $|h_i h_{i+1} \cdots h_k| \leq |h_i|/2 + |h_k|/2 \leq L$, meaning the entire string can be replaced with a single element of H_L , contradicting the fact that $|w|_H = n$. In this context, suppose $h_i = h_k$. Then $u_k = u_i x^p$ for some p . But because $v_{k-1} = u_k^{-1}$, one has

$$h_i \cdots h_{k-1} = u_i x^{N'} u_i^{-1}$$

for $N' = N - e(k) - p$. However, the stipulation that H contains no elements of G conjugate to powers of a or b , precludes this scenario. Thus if any consecutive string $h_i \cdots h_k$ in the H_L -spelling of w contributes to the same syllable in the $\{a, b\}$ -spelling of w , that string includes at most one instance of each element of H_L .

We have demonstrated a correspondence between letters h_i and syllables of w written with respect to H_L and $\{a, b\}$ respectively: each letter corresponds to at least one syllable (the one in which its core appears), and at most K letters correspond to the same syllable. Therefore $K|w|_* \geq |w|_H$. \square

As described above, Lemma 5.6 completes the proof of Proposition 1.2. \square

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