Convex cocompactness in RAAGs and MCGs

or

The geometry of purely loxodromic subgroups of right-angled Artin groups

or

Some interesting subgroups of MCGs and RAAGs

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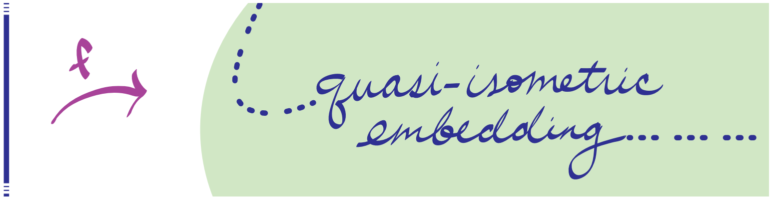
Coarse language



$f : X \to Y$ is an (K, L)–*quasi-isometric embedding* if, $\forall p, q \in X$

$$(1/K) \cdot d_X(p, q) - L \;\leq\; d_Y(f(p), f(q)) \;\leq\; K \cdot d_X(p, q) + L$$

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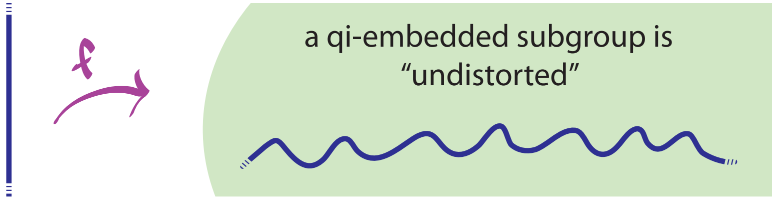

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Whether a subgroup $H < G$ is qi-embedded
does not
depend on generating set

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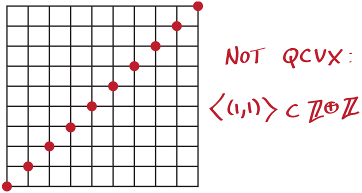

$Z \subset Y$ is a K–*quasiconvex set* if

$$N_K(Z) \text{ contains all geodesics between points in } Z$$

Whether a subgroup $H < G$ is quasiconvex
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Stability (Durham–Taylor)

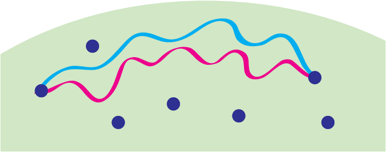
Definition

A f.g. subgroup $H < G$ is *stable* if it is

(1) quasi-isometrically embedded, and

(2) any pair of K–quasigeodesics* between points in H have Hausdorff distance bounded by $M(K)$.

*K–*quasigeodesic*: (K, K)–qi-embedding of an interval.



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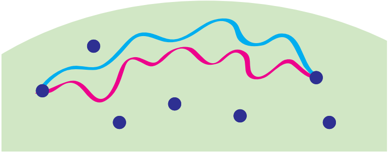

Stable subgroups of G are quasiconvex with respect to **any** word metric on G.

Convex cocompactness in mapping class groups

Definition (Farb–Mosher)

Finitely generated $G < MCG(S)$ is *convex cocompact* if its orbit $G \cdot X \subset \mathrm{Teich}(S)$ is quasiconvex.

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Finitely generated, discrete $G < \mathrm{Isom}(\mathbb{H}^n)$ is convex cocompact iff its orbit $G \cdot p \subset \mathbb{H}^n$ is quasiconvex.

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Well-known theorems (Ivanov, Masur, Masur–Minsky)

$MCG(S) = \mathrm{Isom}(\mathrm{Teich}(S))$ *and* $\mathrm{Teich}(S)$ **is not** *hyperbolic.*

$MCG(S) = \mathrm{Isom}(\mathrm{Curve}(S))$ *and* $\mathrm{Curve}(S)$ **is** *hyperbolic*

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Definition

Pseudo-Anosov mapping classes are elements of $MCG(S)$ with N-S dynamics along translation axis in $\mathrm{Teich}(S)$ (equiv., in $\mathrm{Curve}(S)$)

Goal: motivate theorem on right

Theorems (Kent–Leininger, Hamenstädt, Durham–Taylor)

TFAE for f.g. $G < MCG(S)$

(0) *G is convex cocompact*

(1) *The orbit map $G \hookrightarrow G \cdot v \subset \mathrm{Curve}(S)$ is a q.i.-embedding*

(2) *G is stable in $MCG(S)$.*

Also, these imply G is purely pseudo-Anosov.

Theorem (Koberda–M.–Taylor)

TFAE for f.g. $G < A(\Gamma)$

(1) *The orbit map $G \hookrightarrow G \cdot v \subset \mathrm{Curve}(\Gamma)$ is a q.i.-embedding*

(2) *G is stable in $A(\Gamma)$.*

(3) *G is purely loxodromic.*

MCG convex cocompactness & surface group extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S) & \longrightarrow & E_G & \longrightarrow & G & \longrightarrow & 1 \\ & & || & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(S) & \longrightarrow & \mathrm{Mod}(\mathring{S}) & \longrightarrow & \mathrm{Mod}(S) & \longrightarrow & 1 \end{array}$$

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E_G is word hyperbolic if and only if G is convex cocompact.

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Theorem (Thurston's geometrization of mapping tori)

$$1 \longrightarrow \pi_1(S) \longrightarrow \pi_1(M_\phi) \longrightarrow \langle \phi \rangle \longrightarrow 1$$

M_ϕ is hyperbolizable if and only if ϕ is pseudo-Anosov,
i.e. iff $\langle \phi \rangle$ is convex cocompact.

MCG convex cocompactness & hyperbolic groups

Theorem (Thurston's geometrization, proved by Perelman)

M closed, aspherical 3–mfld admits a hyperbolic metric if and only if $\pi_1(M)$ does not contain $\mathbb{Z} \oplus \mathbb{Z}$.

Question (Gromov)

If H has finite $K(H,1)$ and no subgroups of the form $BS(p,q) = \langle a, b | a^{-1} b^p a = b^q \rangle$, is H hyperbolic?

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Potential counterexample

f.g. purely pA $G \Longrightarrow E_G$ has finite $K(E_G, 1)$ and no BS subgroups.

Question (Farb–Mosher)

Is f.g. purely pA $G < MCG(S)$ necessarily convex cocompact?

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Question (Farb–Mosher)

Is f.g. purely pA $G < MCG(S)$ necessarily convex cocompact?
(No here means no to Gromov, since E_G would not be hyperbolic).

Q: Does f.g. purely pA imply convex cocompact?

Yes answers in special cases

$H < G$ for G from a certain family of $MCG(S)$–subgroups:

- $H < \mathrm{Isom}(\mathbb{H}^2)$ for $\mathbb{H}^2 \subset \mathrm{Teich}(S)$ (Veech groups);
- Leininger–Reid combinations of Veech groups (Leininger)
- $H < \pi_1(M_\phi)$ when M_ϕ is hyperbolic;
  quasiconvex $H < E_G$ when E_G is hyperbolic
  (Dowdall–Kent–Leininger, generalizing Kent–Leininger–Schleimer)
- $H < A(\Gamma) < MCG(S)$ for "admissible" $A(\Gamma)$
  (M.–Taylor, Koberda–M–Taylor)

RAAGs in MCGs

Definition

$A(\Gamma) = \langle v_i \in V(\Gamma) \mid [v_i, v_j] = id \text{ if } (v_i, v_j) \in E(\Gamma) \rangle$

Theorems (Koberda, Clay–Leininger–M, Crisp–Paris/–Weiss/–Farb)

Many ways to embed $A(\Gamma)$ in some $MCG(S)$.

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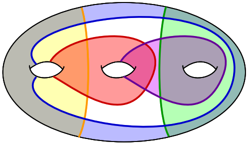
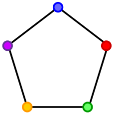
Theorem (Clay–Leininger–M)

For partially pA $\{f_1, \ldots, f_n\}$ supported on connected, non-nested X_i with disjointess recorded in the graph Γ, for large enough p_i,

$$A(\Gamma) \to \langle f_1^{p_1}, \ldots, f_n^{p_n} \rangle < MCG(S)$$

is a quasi-isometric embedding.





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is an admissible embedding.*

*meaning $A(\Gamma) \hookrightarrow MCG(S)$:

(i) Comes with large subsurface curve complex projections, and

(ii) Word partial order matches subsurface partial order

Our special case

Suppose $A(\Gamma) < MCG(S)$ admissible.

Theorem (M–Taylor)

F.g. purely pA $H < A_\Gamma < MCG(S)$ is convex cocompact in $MCG(S)$ if and only if H is combinatorially quasiconvex in $A(\Gamma)$.*

*word metric using standard vertex generators

Easy fact

$\phi \in A(\Gamma) < MCG(S)$ pseudo-Anosov $\Longrightarrow \phi \in A(\Gamma)$ loxodromic.

Corollary (Koberda–M–Taylor)

$H < A(\Gamma) < MCG(S)$ is convex cocompact if and only if H is f.g. purely pA.

Curve(S) and Curve(Γ)

Curve(S):

- Vertices $\longleftrightarrow$ ess. simple closed **curves** on S up to isotopy
- Edge $(\alpha, \beta) \iff \alpha, \beta$ are **disjoint**

Curve(Γ) aka *extension graph* Γ^e of Γ, defined by Kim–Koberda:

- Realize $A(\Gamma) \hookrightarrow MCG(S)$ by
  vertex generators $\longmapsto$ high-powered Dehn twists (Koberda)
- Vertices $\longleftrightarrow$ **base curves** of $A(\Gamma)$–conjugates of vertex gens
- Edge $(\alpha, \beta) \iff \alpha, \beta$ are **disjoint**

Theorem (Kim–Koberda)

Curve(Γ) *is hyperbolic (in fact, it is a quasi-tree).*

Loxodromic elements

Definition

$\phi \in A(\Gamma)$ is loxodromic if $\phi \cdot v \subset \mathrm{Curve}(\Gamma)$ is unbounded.

Note

$\phi \in MCG(S)$ is pseudo-Anosov iff $\phi \cdot v \subset \mathrm{Curve}(S)$ is unbounded.

Theorems (Kim–Koberda, Servatius, Behrstock–Charney)

For $\phi \in A(\Gamma)$, TFAE:

- *ϕ is loxodromic*
- $\phi \notin \mathbb{Z} \oplus \mathbb{Z} < A(\Gamma)$
- *ϕ acts as a rank-1-isometry of $\widetilde{S(\Gamma)}$, the CAT(0) cube complex whose 1–skeleton is $\mathrm{Cayley}(A(\Gamma), V(\Gamma))$*

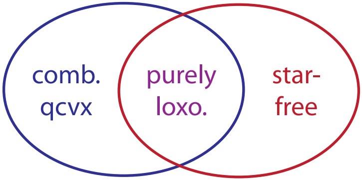
Convex cocompactness in RAAGs

Theorem (Haglund)

For $H < A_\Gamma$, TFAE:

- *Exists (non-empty) convex subcomplex $C \subset \widetilde{S(\Gamma)}$ which is H-invariant and cocompact.*
- *H combinatorially quasiconvex in $A(\Gamma)$,*
  i.e. vertex orbit $H \cdot v$ quasiconvex in $\widetilde{S(\Gamma)}^{(1)}$.

Proposition (K–M–T)



Non-loxodromic elements: join-words and star-words

Theorem (Servatius)

ϕ *not loxodromic* $\Longrightarrow c\phi c^{-1} \in A(J)$ *for a join* $J \subset \Gamma$*.*

Definition

A *join* $J = \Gamma_1 * \Gamma_2$.

join words conj. into $A(J)$

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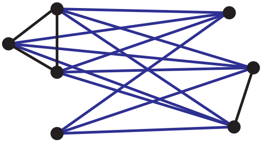

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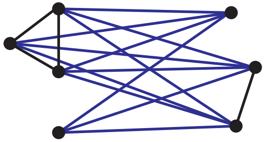
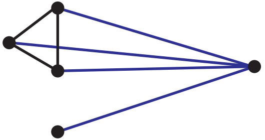


join words conj. into $A(J)$

A *star* $T = \Gamma_1 * v$:



star words conj. into $A(T)$

{purely loxo. (no join words)} $\subsetneq$ {star-free (no star words)}

Star-free RAAG subgroups

Theorem (Koberda–M–Taylor)

If $G < A(\Gamma)$ is finitely generated and star-free, then

(1) *G is a free group,*

(2) *G is quasi-isometrically embedded in $A(\Gamma)$, and*

(3) *$G \cap A(\Lambda)$ is finitely generated, for any subgraph $\Lambda \subset \Gamma$.*