

# Characterizing slopes for torus knots, II

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## Abstract

A slope  $\frac{p}{q}$  is called a characterizing slope for a given knot  $K_0 \subset S^3$  if whenever the  $\frac{p}{q}$ -surgery on a knot  $K \subset S^3$  is homeomorphic to the  $\frac{p}{q}$ -surgery on  $K_0$  via an orientation preserving homeomorphism, then  $K = K_0$ . In a previous paper, we showed that, outside a certain finite set of slopes, only the negative integers could possibly be non-characterizing slopes for the torus knot  $T_{5,2}$ . Applying recent work of Baldwin–Hu–Sivek, we improve our result by showing that a nontrivial slope  $\frac{p}{q}$  is a characterizing slope for  $T_{5,2}$  if  $\frac{p}{q} > -1$  and  $\frac{p}{q} \notin \{0, 1, \pm\frac{1}{2}, \pm\frac{1}{3}\}$ . In particular, every nontrivial L-space slope of  $T_{5,2}$  is characterizing for  $T_{5,2}$ . As a consequence, if a nontrivial  $\frac{p}{q}$ -surgery on a non-torus knot in  $S^3$  yields a manifold of finite fundamental group, then  $|p| > 9$ .

## 1 Introduction

For a knot  $K \subset S^3$  and a slope  $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ , let  $S^3_{p/q}(K)$  be the manifold obtained by the  $\frac{p}{q}$ -surgery on  $K$ . A slope  $\frac{p}{q}$  is said to be *characterizing* for a given knot  $K_0 \subset S^3$  if whenever  $S^3_{p/q}(K) \cong S^3_{p/q}(K_0)$  for a knot  $K \subset S^3$ , then  $K = K_0$ . Here, “ $\cong$ ” stands for an orientation preserving homeomorphism. Obviously the trivial slope  $\frac{1}{0}$  is never characterizing for any knot.

A long standing conjecture due to Gordon states that all nontrivial slopes are characterizing for the unknot. This conjecture was originally proved using Monopole Floer homology [18], and there were also proofs via Heegaard Floer homology [30, 32]. Based on work of Ghiggini [13], Ozsváth and Szabó [33] proved the same result for the trefoil knot and the figure–8 knot. To date, the unknot, the trefoil knot and the figure–8 knot are the only knots for which it is known that all but finitely many slopes are characterizing.

The next simplest knot is the torus knot  $T_{5,2}$ . It is reasonable to expect that all nontrivial slopes are characterizing for  $T_{5,2}$ . In [28], it is proved that a nontrivial slope  $\frac{p}{q}$  is characterizing for  $T_{5,2}$  unless  $\frac{p}{q}$  is a negative integer or  $-47 \leq p \leq 32$  and  $1 \leq q \leq 8$ . It is also known that the slopes

$$8, 9, 10, 11, 12, \frac{17}{2}, \frac{19}{2}, \frac{21}{2}, \frac{23}{2}, \frac{28}{3}, \frac{29}{3}, \frac{31}{3}, \frac{32}{3}$$

are characterizing [1, 5, 16, 29]. Characterizing slopes for general torus knots have been studied in [19, 20, 27, 28].

In this paper, we further narrow down the range of possible non-characterizing slopes for  $T_{5,2}$ . Our main result is the following theorem.

**Theorem 1.1.** *A nontrivial slope  $\frac{p}{q}$  is characterizing for  $T_{5,2}$  if  $\frac{p}{q} > -1$  and  $\frac{p}{q} \notin \{0, 1, \pm\frac{1}{2}, \pm\frac{1}{3}\}$ .*

Recall that a rational homology 3-sphere  $Y$  is an  $L$ -space if the rank of  $\widehat{HF}(Y)$  is equal to the order of  $H_1(Y; \mathbb{Z})$ . For  $T_{5,2}$ , the surgery slopes which yield L-spaces are the ones greater than or equal to  $3 = 2g(T_{5,2}) - 1$ . Thus Theorem 1.1 implies that all nontrivial L-space slopes of  $T_{5,2}$  are characterizing for  $T_{5,2}$ .

**Corollary 1.2.** *If a nontrivial  $p/q$ -surgery, with  $|p| \leq 9$ , on a nontrivial knot  $K$  in  $S^3$  produces a manifold of finite fundamental group, then  $K$  is one of the torus knots  $T_{3,\pm 2}$  and  $T_{5,\pm 2}$ .*

*Proof.* By changing  $K$  to its mirror image, we may also assume that  $p/q$  is positive.

When  $S^3_{p/q}(K)$  has cyclic fundamental group, i.e. when it is a lens space, we may further assume that  $p/q = p$  is an integer for otherwise  $K$  is a torus knot [8] and  $S^3_{p/q}(K)$  is never a lens space when  $p \leq 9$  and  $q \geq 2$  [24]. Now [15, Theorem 1.4] implies that when  $p \leq 9$ , the knot  $K$  is a fibered knot of genus at most 2, and  $S^3_p(K) \cong S^3_p(T_{3,2})$  or  $S^3_p(T_{5,2})$ . Now we use [13] or Theorem 1.1 to get that  $K = T_{3,2}$  or  $T_{5,2}$ .

When  $S^3_{p/q}(K)$  has finite but non-cyclic fundamental group, it is shown in [9, Theorems 2, 3 and Table 1] that either  $K$  is  $T_{3,2}$  or  $T_{5,2}$ , or  $p/q = 7$  or  $8$ ,  $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ . We may now apply Theorem 1.1 to conclude that  $K = T_{5,2}$ .  $\diamond$

The bound 9 in Corollary 1.2 appears to be the best one could get for hyperbolic knots in  $S^3$  with the current techniques. The 10-surgery on  $T_{4,3}$  is a spherical space form with non-cyclic fundamental group, and the current techniques could not rule out the possibility that the same surgery on a hyperbolic knot with the same knot Floer homology as  $T_{4,3}$  yields the same manifold. Conjecturally on a hyperbolic knot  $K$  in  $S^3$  if a nontrivial  $\frac{p}{q}$ -surgery yields a manifold of finite fundamental group, then  $|p| \geq 17$ , a bound which can be realized on the  $(-2, 3, 7)$ -pretzel knot.

Our proof of Theorem 1.1, given in the next section, is mainly based on our earlier work [28] and a recent paper of Baldwin–Hu–Sivek [2].

**Acknowledgements.** The first author was partially supported by NSF grant number DMS-1811900.

## 2 Proof of Theorem 1.1

For a knot  $K$  in  $S^3$ ,  $\Delta_K(T)$  denotes the symmetric Alexander polynomial of  $K$ . The following theorem and remark are [28, Theorem 4.1] and the remark after the proof.

**Theorem 2.1.** *Suppose that  $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$  for a knot  $K \subset S^3$  and a nontrivial slope  $\frac{p}{q}$ . Then one of the following two cases happens:*

1)  $K$  is a genus  $(n+1)$  fibered knot for some  $n \geq 1$  with

$$\Delta_K(T) = (T^{n+1} + T^{-(n+1)}) - 2(T^n + T^{-n}) + (T^{n-1} + T^{1-n}) + (T + T^{-1}) - 1. \quad (1)$$

2)  $K$  is a genus 1 knot with  $\Delta_K(T) = 3T - 5 + 3T^{-1}$ .

Moreover, if

$$\frac{p}{q} \in \left\{ \frac{p}{q} > 1 \right\} \cup \left\{ \frac{p}{q} < -6, |q| \geq 2 \right\},$$

then the number  $n$  in the first case must be 1.

**Remark 2.2.** We have the following addendum to Theorem 2.1:

- (a) If  $p$  is even, then case 2) of Theorem 2.1 cannot happen and in case 1) of Theorem 2.1, the number  $n$  must be odd.
- (b) If  $p$  is divisible by 3, then case 2) cannot happen and in case 1), the number  $n$  is not divisible by 3.

The following result is implicitly contained in [28, Subsection 4.1]. Background information about Heegaard Floer homology can be found in [28, Section 3].

**Proposition 2.3.** *Suppose that  $S_{p/q}^3(K) \cong S_{p/q}^3(T_{5,2})$  for a knot  $K \subset S^3$  and a nontrivial slope  $\frac{p}{q} > 1$ . Then*

$$\widehat{HFK}(S^3, K) \cong \widehat{HFK}(S^3, T_{5,2}) \quad (2)$$

as a bigraded group.

*Proof.* In [28, Subsection 4.1], it is proved that  $\Delta_K = \Delta_{T_{5,2}}$  and  $V_0(K) = V_1(K) = 1$ . By [28, Proposition 4.2],

$$t_s(K) = V_s(K) + \text{rank} H_{\text{red}}(A_s^+).$$

Since  $t_0(K) = t_1(K) = 1$  and  $t_s(K) = 0$  when  $s > 1$ , we have

$$HF_{\text{red}}(A_s^+) = 0 \quad \text{whenever } s \geq 0.$$

So  $K$  is an L-space knot. Since  $\Delta_K = \Delta_{T_{5,2}}$ , we get (2) by [31].  $\diamond$

**Remark 2.4.** In [28, Proposition 4.7], it is proved that if  $\frac{p}{q} < -6$  and  $|q| \geq 2$ , then  $\Delta_K = \Delta_{T_{5,2}}$ . However, we cannot conclude that (2) holds. Algebraically, it is possible that  $\widehat{HFK}(S^3, K, 1)$  has rank 3, while  $HF^+(S_{p/q}^3(K)) \cong HF^+(S_{p/q}^3(T_{5,2}))$ .

For any hyperbolic knot  $K$  in  $S^3$ , a result of Gabai and Mosher [25] states that the complement of  $K$  contains an essential lamination  $\lambda$  which has an associated degeneracy locus  $d(\lambda)$  in the form of  $d(\lambda) = \frac{m}{n}$ ,  $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ , such that if

$$\Delta\left(\frac{p}{q}, d(\lambda)\right) := |pn - qm| \geq 2,$$

then  $\lambda$  remains an essential lamination in  $S_{p/q}^3(K)$ . Since  $S_{1/0}^3(K) = S^3$  does not contain any essential lamination, it follows that  $d(\lambda) = m/0$  or  $m/1$ . Furthermore we have

**Fact (i)** If  $\Delta\left(\frac{p}{q}, d(\lambda)\right) \geq 3$ , then  $S_{p/q}^3(K)$  is an irreducible, atoroidal and non-Seifert fibered manifold [36, Theorem 2.5].

**Fact (ii)** If  $K$  is fibered and  $d(\lambda) = m/1$  where  $\lambda$  is the stable lamination of  $K$ , then  $|m| \geq 2$  [12, Theorem 8.8]. (Another proof of this was given in [34].)

**Proposition 2.5.** *Suppose that  $S_{p/q}^3(K) \cong S_{p/q}^3(T_{5,2})$ ,  $|\frac{p}{q}| < 1$  and  $\frac{p}{q} \notin \{0, \pm\frac{1}{2}, \pm\frac{1}{3}\}$ . Then  $K = T_{5,2}$ .*

*Proof.* When  $K$  is a torus knot, Proposition 2.5 holds by [19, Lemma 4.2].

Next suppose that  $K$  is a satellite knot. Let  $K'$  be a companion knot of  $K$ . Since  $S_{p/q}^3(K) \cong S_{p/q}^3(T_{5,2})$  is atoroidal and irreducible (since  $p/q \neq 10$ ) [24], the work of Gabai [10] implies that  $K$  is a 0-bridge or 1-bridge braid in a tubular neighborhood  $V$  of  $K'$ . If  $K$  is a 0-bridge braid, then  $K$  is a  $(r, s)$ -cable of  $K'$  and the surgery slope  $p/q$  must be  $(qrs \pm 1)/q$  [14]. If  $K$  is a 1-bridge braid, then it follows from [11, Lemma 3.2] that  $\frac{p}{q} \in \mathbb{Z}$ . In both cases we get a contradiction with the assumption that  $|p/q| < 1$ .

So  $K$  is hyperbolic. By Theorem 2.1, either  $K$  is fibered or  $g(K) = 1$ . If  $K$  is fibered and  $d(\lambda) = \frac{m}{0}$ , since  $|q| \geq 3$ ,  $\Delta(d(\lambda), \frac{p}{q}) \geq 3$ . Hence  $S_{p/q}^3(K)$  is not Seifert fibered by Fact (i) above, which contradicts the assumption  $S_{p/q}^3(K) \cong S_{p/q}^3(T_{5,2})$ . So we may assume that when  $K$  is fibered,  $d(\lambda) = m/1$  for the stable lamination  $\lambda$  of  $K$ , and therefore  $|m| \geq 2$  by Fact (ii) above. Since  $|\frac{p}{q}| < 1$  and  $|m| \geq 2$ ,  $\Delta(d(\lambda), \frac{p}{q}) \geq 3$ . Again by Fact (i) we get a contradiction with the assumption  $S_{p/q}^3(K) \cong S_{p/q}^3(T_{5,2})$ .

So  $K$  is hyperbolic and  $g(K) = 1$ . By [4, Theorem 1.5]  $0 < |p| \leq 3$  since  $S_{p/q}^3(K) \cong S_{p/q}^3(T_{5,2})$  is a Seifert fibered space. By Remark 2.2,  $|p| = 1$ . Since  $\frac{p}{q} \notin \{0, \pm\frac{1}{2}, \pm\frac{1}{3}\}$ , we again have  $\Delta(d(\lambda), \frac{p}{q}) \geq 3$  (whether  $d(\lambda) = \frac{m}{0}$  or  $\frac{m}{1}$ ), which leads to a contradiction as in the preceding paragraph.  $\diamond$

**Proposition 2.6.** *Suppose that  $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$  for a nontrivial slope  $\frac{p}{q}$  with  $\frac{p}{q} > 1$ . Then  $K = T_{5,2}$ .*

*Proof.* To get a contradiction we assume that  $K \neq T_{5,2}$ .

By Proposition 2.3, we have  $\widehat{HFK}(S^3, K) \cong \widehat{HFK}(S^3, T_{5,2})$ . In [2, Section 3], Baldwin–Hu–Sivek proved that if  $\widehat{HFK}(S^3, K) \cong \widehat{HFK}(S^3, T_{5,2})$  and  $K \neq T_{5,2}$ , then  $K$  is a hyperbolic, doubly-periodic, genus two fibered knot, the degeneracy locus of the stable lamination of  $K$  is 4, and moreover there exists a pseudo-Anosov 5-braid  $\beta$  whose closure  $B = \hat{\beta}$  is an unknot with braid axis  $A$ , such that  $K$  is the lift of  $A$  in the branched double cover  $\Sigma(S^3, B) \cong S^3$ . Let  $V$  be the exterior of  $A$  in  $S^3$  and  $M_K$  the exterior of  $K$  in  $S^3$ . Then  $V$  is a solid torus and  $M_K$  is a double branched cover of  $V$  with  $B$  as the branched set in  $V$ . Let  $\tau$  be the corresponding covering involution on  $M_K$  and  $U$  the branched set in  $M_K$ . Then  $U$  is the fixed point set of  $\tau$  which is a knot disjoint from  $\partial M_K$ , and  $(M_K, U)/\tau = (V, B)$ . The restriction of  $\tau$  on  $\partial M_K$  is a free action—an order two rotation along the longitude factor of  $\partial M_K$ .

We also use  $M_K(p/q)$  to denote the surgery manifold  $S^3_{p/q}(K)$  and similarly  $V(p/q)$  for  $S^3_{p/q}(A)$ . Note that the involution  $\tau$  on  $M_K$  extends to an involution  $\tau_{p/q}$  on  $M_K(p/q)$ . In fact if we let  $N_{p/q}$  denote the filling solid torus in forming  $M_K(p/q) = M_K \cup N_{p/q}$  and let  $K_{p/q}$  be the center circle of  $N_{p/q}$ , then the fixed point set of  $\tau_{p/q}$  is

$$Fix(\tau_{p/q}) = \begin{cases} U, & \text{if } p \text{ is odd} \\ U \cup K_{p/q}, & \text{if } p \text{ is even} \end{cases} \quad (3)$$

and

$$M_K(p/q)/\tau_{p/q} = \begin{cases} V(p/2q), & \text{if } p \text{ is odd} \\ V((p/2)/q), & \text{if } p \text{ is even.} \end{cases} \quad (4)$$

Let  $Y$  be the exterior of  $U$  in  $M_K$  and  $W$  the exterior of  $B$  in  $V$ . Then  $Y$  is a free double cover of  $W$ . Since  $B$  is the closure of a pseudo-Anosov braid in  $V$ ,  $W$  is hyperbolic. Hence  $Y$  is also hyperbolic.

Note that  $M_K(p/q) \cong M_{T_{5,2}}(p/q)$  is a Seifert fibered space whose base orbifold is  $S^2(2, 5, d)$  with  $d = |p - 10q|$  if  $p/q \neq 10$ , and is a connected sum of two lens spaces of orders 2 and 5 if  $p/q = 10$  [24]. Since  $K$  is a doubly periodic knot, by [17]  $M_K(p/q)$  is irreducible and thus  $p/q \neq 10$ , by [35]  $M_K(p/q)$  is not a lens space, and by [22]  $M_K(p/q)$  is not a prism manifold. Thus  $M_K(p/q)$  is a Seifert fibered space whose base orbifold is  $S^2(2, 5, d)$  with  $d = |p - 10q| > 2$ . So there is a unique Seifert structure on  $M_K(p/q)$ . By Thurston’s Orbifold Theorem [3, 7], we may assume that the unique Seifert structure on  $M_K(p/q)$  is  $\tau_{p/q}$ -invariant, i.e.  $\tau_{p/q}$  sends every Seifert fiber to a Seifert fiber preserving the order of singularity.

Since the base orbifold of the Seifert fibered space  $M_K(p/q)$  is orientable, the Seifert fibers of  $M_K(p/q)$  can be coherently oriented. If  $\tau_{p/q}$  preserves the orientations of the Seifert fibers of  $M_K(p/q)$ , then  $Fix(\tau_{p/q})$  consists of Seifert fibers (see [4, Lemma 4.3]). Since  $K$  is hyperbolic,  $K_{p/q}$  cannot be a component of  $Fix(\tau_{p/q})$ . Thus by Formula (3),  $p$  is odd and  $Fix(\tau_{p/q}) = U$ , and by Formula (4),  $M_K(p/q)/\tau_{p/q} = V(p/2q)$ . Moreover, if we let  $Y(\partial M_K, p/q)$  denote the Dehn filling of  $Y$  along the component  $\partial M_K$  of  $\partial Y$  with the slope  $p/q$ , and similarly define  $W(\partial V, p/2q)$ , then  $Y(\partial M_K, p/q)$  is Seifert fibered and is a free double cover of  $W(\partial V, p/2q)$ . So the latter manifold  $W(\partial V, p/2q)$  is also Seifert fibered. But  $W(\partial V, 1/0)$  is a solid torus (it is the exterior of the unknot  $B$  in  $S^3$ ). We get a contradiction with [21, Corollary 15].

Hence  $\tau_{p/q}$  reverses the orientations of the Seifert fibers of  $M_K(p/q)$ . Since  $p/q > 1$ , we may assume that both  $p$  and  $q$  are positive. Let  $\mu$  be the meridian of  $K$ , then  $[\mu]$  generates  $H_1(M_K) \cong \mathbb{Z}$  and  $H_1(M_K(p/q)) \cong \mathbb{Z}/p\mathbb{Z}$ . Clearly  $\tau_*[\mu] = [\mu]$ , so

$$\tau_* = \text{id on } H_1(M_K), \quad (\tau_{p/q})_* = \text{id on } H_1(M_K(p/q)). \quad (5)$$

To apply a homological argument, we describe the Seifert fibered structure of  $M_{T_{5,2}}(p/q)$  explicitly as follows. Let  $V_0 \cup V_1$  be a standard genus one Heegaard splitting of  $S^3$ . We may assume that  $T_{5,2}$  is embedded in  $\partial V_0$  and is homologous to  $5\mathcal{L} + 2\mathcal{M}$ , where  $\mathcal{L}$  is the canonical longitude of  $V_0$ , and  $\mathcal{M}$  the meridian of  $V_0$ . Let  $\mu_0 \subset \partial M_{T_{5,2}}$  be the meridian of  $T_{5,2}$ , let  $\lambda_0$  be the canonical longitude of  $T_{5,2}$  and

let  $\mathcal{C}_i$  be the core of  $V_i$ ,  $i = 1, 2$ . Then  $M_{T_{5,2}}$  is Seifert fibered with  $C_0$  and  $C_1$  as two singular fibers of order 5 and 2 respectively. A regular fiber  $\mathcal{F}$  of  $M_{T_{5,2}}$  in  $\partial M_{T_{5,2}}$  has slope  $10\mu_0 + \lambda_0$ . If  $p/q \neq 10$ , the Seifert structure of  $M_{T_{5,2}}$  extends to one on  $M_{T_{5,2}}(p/q)$  such that the core  $\mathcal{C}'$  of the filling solid torus is an order  $d = |p - 10q|$  singular fiber if  $d > 1$ . In  $H_1(M_{T_{5,2}}(p/q))$ , we have

$$[\mathcal{F}] = 10[\mu_0], \quad [C_0] = 2[\mu_0], \quad [C_1] = 5[\mu_0], \quad [C'] = \pm q'[\mu_0], \quad (6)$$

where  $q' \in \mathbb{Z}$  satisfies that  $qq' \equiv 1 \pmod{p}$ .

Since  $\tau_{p/q}$  sends a regular fiber to a regular fiber reversing its orientation, we see, using (5) and (6), that  $10[\mu_0] = -10[\mu_0]$  in  $\mathbb{Z}/p\mathbb{Z}$ . So  $p|20$ . We claim that  $p = 1$  or  $2$ . Suppose otherwise that  $p > 2$ . We already know that  $M_K(p/q)$  has three singular fibers of orders 2, 5,  $d = |p - 10q| > 2$ . If  $d \neq 5$ , then  $\tau_{p/q}$  sends the order  $d$  singular fiber  $\mathcal{C}'$  to itself with opposite orientation. Hence  $q'[\mu_0] = -q'[\mu_0]$  in  $\mathbb{Z}/p\mathbb{Z}$ . Since  $\gcd(p, q') = 1$ , we get  $p|2$ , which is not possible. So we must have  $d = 5$ , which, together with the condition  $p|20$ , implies that  $p/q = 5$ . By (6) the two order 5 singular fibers  $C_0, C'$  are homologous to  $2[\mu_0]$  and  $\pm[\mu_0]$  respectively. By (5),  $\tau_{p/q}$  must send  $C_0$  to itself, and  $C'$  to itself. So  $2[\mu_0] = -2[\mu_0]$  in  $\mathbb{Z}/5\mathbb{Z}$ , which is not possible.

Recall that  $K$  has a degeneracy locus 4. Since  $\frac{p}{q} \leq 2$ , we have  $\Delta(\frac{p}{q}, 4) \geq 3$  unless  $\frac{p}{q} = 2$ . By Fact (i), we only need to consider the case  $\frac{p}{q} = 2$ .

By Formula (3) the fixed point set of  $\tau_2$  is  $U \cup K_2$  and by Formula (4),  $M_K(2)/\tau_2 = V(1)$  which is  $S^3$ . Hence the branched set  $B \cup K_2^*$  in  $M_K(2)/\tau_2 = V(1) = S^3$  is a Montesinos link of two components [23], where  $K_2^*$  is the image of  $K_2$  under the map  $M_K(2) \rightarrow V(1)$ , which is also the core of the filling solid torus of  $V(1)$ . Note that  $K_2^*$  is an unknot in  $S^3$  while  $B$  is the closure of a 5-braid in the exterior of  $K_2^*$  which is a solid torus. Hence the linking number between  $B$  and  $K_2^*$  is 5.

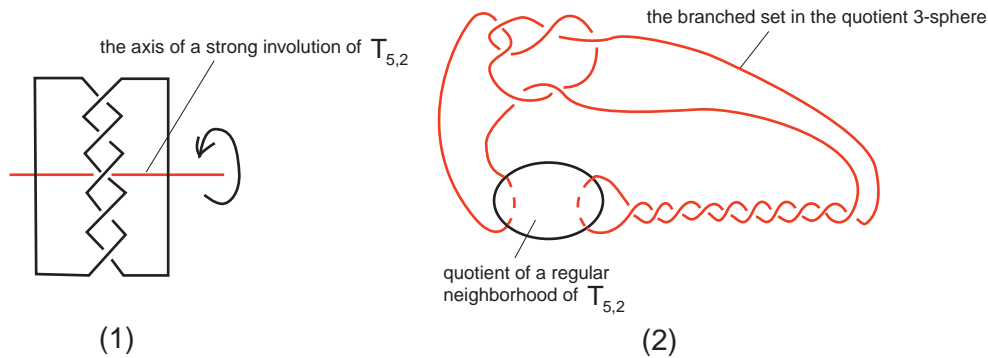


Figure 1: The strong involution on  $(S^3, T_{5,2})$  and the quotient

On the other hand  $M_{T_{5,2}}(2)$  is a double branched cover of  $S^3$  whose branched set in  $S^3$  can be explicitly constructed as follows. The knot  $T_{5,2}$  is strongly invertible and the quotient spaces under the strong involution of  $S^3$ , a regular neighborhood of  $T_{5,2}$  and the exterior, as well as the branched set are as shown in Figure 1. Furthermore  $M_{T_{5,2}}(2)$  is a double branched cover of  $S^3$  whose branch set can be obtained by replacing the rational  $1/0$ -tangle in Figure 1 (2) with the rational  $2$ -tangle. The resulting branched set is the two-component Montesinos link shown in Figure 2.

Note that  $M_K(2) = M_{T_{5,2}}(2)$  has base orbifold  $S^2(2, 5, 8)$  and so it has a unique Seifert fibration structure. Since the number of singular fibers is 3, the classification of Montesinos links [6] implies that  $M_{T_{5,2}}(2)$  is the double branched cover of  $S^3$  over a unique Montesinos link with 3 rational tangles. Therefore the link shown in Figure 2 should be the link  $B \cup K_2^*$ . However the linking number between the two components of the link in Figure 2 is 3, yielding a final contradiction with the early conclusion that the linking number between  $B$  and  $K_2^*$  is 5.  $\diamond$

**Remark 2.7.** Proposition 2.6 can be proved without using the degeneracy locus condition.

Now the combination of Propositions 2.5 and 2.6 gives Theorem 1.1.

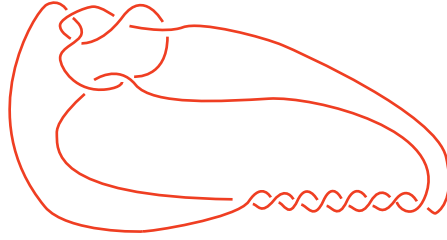


Figure 2: The branched sets for the 2-surgery on  $T_{5,2}$

## References

- [1] Kenneth L. Baker, *Small genus knots in lens spaces have small bridge number*, *Algebr. Geom. Topol.* **6** (2006), 1519–1621.
- [2] John A. Baldwin, Ying Hu, and Steven Sivek, *Khovanov homology and the cinquefoil* (2021), preprint, available at <https://arxiv.org/abs/2105.12102>.
- [3] Michel Boileau and Joan Porti, *Geometrization of 3-orbifolds of cyclic type*, *Astérisque* **272** (2001), 208 (English, with English and French summaries). Appendix A by Michael Heusener and Porti.
- [4] Steven Boyer, Cameron McA. Gordon, and Xingru Zhang, *Dehn fillings of knot manifolds containing essential once-punctured tori*, *Trans. Amer. Math. Soc.* **366** (2014), no. 1, 341–393.
- [5] S. Boyer and X. Zhang, *Finite Dehn surgery on knots*, *J. Amer. Math. Soc.* **9** (1996), no. 4, 1005–1050.
- [6] Gerhard Burde and Heiner Zieschang, *Knots*, 2nd ed., *De Gruyter Studies in Mathematics*, vol. 5, Walter de Gruyter & Co., Berlin, 2003.
- [7] Daryl Cooper, Craig D. Hodgson, and Steven P. Kerckhoff, *Three-dimensional orbifolds and cone-manifolds*, *MSJ Memoirs*, vol. 5, Mathematical Society of Japan, Tokyo, 2000. With a postface by Sadayoshi Kojima.
- [8] Marc Culler, C. McA. Gordon, J. Luecke, and Peter B. Shalen, *Dehn surgery on knots*, *Ann. of Math. (2)* **125** (1987), no. 2, 237–300.
- [9] Margaret I. Doig, *Finite knot surgeries and Heegaard Floer homology*, *Algebr. Geom. Topol.* **15** (2015), no. 2, 667–690.
- [10] David Gabai, *Surgery on knots in solid tori*, *Topology* **28** (1989), no. 1, 1–6.
- [11] ———, *1-bridge braids in solid tori*, *Topology Appl.* **37** (1990), no. 3, 221–235.
- [12] ———, *Problems in foliations and laminations*, *Geometric topology* (Athens, GA, 1993), *AMS/IP Stud. Adv. Math.*, vol. 2, Amer. Math. Soc., Providence, RI, 1997, pp. 1–33.
- [13] Paolo Ghiggini, *Knot Floer homology detects genus-one fibred knots*, *Amer. J. Math.* **130** (2008), no. 5, 1151–1169.
- [14] C. McA. Gordon, *Dehn surgery and satellite knots*, *Trans. Amer. Math. Soc.* **275** (1983), no. 2, 687–708.
- [15] Joshua Evan Greene, *The lens space realization problem*, *Ann. of Math. (2)* **177** (2013), no. 2, 449–511.
- [16] ———, *L-space surgeries, genus bounds, and the cabling conjecture*, *J. Differential Geom.* **100** (2015), no. 3, 491–506.
- [17] Chuichiro Hayashi and Koya Shimokawa, *Symmetric knots satisfy the cabling conjecture*, *Math. Proc. Cambridge Philos. Soc.* **123** (1998), no. 3, 501–529.
- [18] P. Kronheimer, T. Mrowka, P. Ozsváth, and Z. Szabó, *Monopoles and lens space surgeries*, *Ann. of Math. (2)* **165** (2007), no. 2, 457–546.
- [19] Duncan McCoy, *Non-integer characterizing slopes for torus knots*, *Comm. Anal. Geom.* **28** (2020), no. 7, 1647–1682.
- [20] ———, *Surgeries, sharp 4-manifolds and the Alexander polynomial* (2014), preprint, available at <https://arxiv.org/abs/1412.0572>.
- [21] W. Menasco and X. Zhang, *Notes on tangles, 2-handle additions and exceptional Dehn fillings*, *Pacific J. Math.* **198** (2001), no. 1, 149–174.
- [22] Katura Miyazaki and Kimihiko Motegi, *Seifert fibering surgery on periodic knots*, *Proceedings of the First Joint Japan-Mexico Meeting in Topology* (Morelia, 1999), 2002, pp. 275–285.
- [23] José M. Montesinos, *Seifert manifolds that are ramified two-sheeted cyclic coverings*, *Bol. Soc. Mat. Mexicana (2)* **18** (1973), 1–32 (Spanish).
- [24] Louise Moser, *Elementary surgery along a torus knot*, *Pacific J. Math.* **38** (1971), 737–745.
- [25] Lee Mosher, *Laminations and flows transverse to finite depth foliations. Part I: branched surfaces and dynamics*, preprint.
- [26] Yi Ni, *Knot Floer homology detects fibred knots*, *Invent. Math.* **170** (2007), no. 3, 577–608.

- [27] ———, *Seifert fibered and reducible surgeries on hyperbolic fibered knots* (2020), preprint, available at <https://arxiv.org/abs/2007.11774>.
- [28] Yi Ni and Xingru Zhang, *Characterizing slopes for torus knots*, *Algebr. Geom. Topol.* **14** (2014), no. 3, 1249–1274.
- [29] ———, *Finite Dehn surgeries on knots in  $S^3$* , *Algebr. Geom. Topol.* **18** (2018), no. 1, 441–492.
- [30] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and genus bounds*, *Geom. Topol.* **8** (2004), 311–334.
- [31] ———, *On knot Floer homology and lens space surgeries*, *Topology* **44** (2005), no. 6, 1281–1300.
- [32] ———, *Knot Floer homology and rational surgeries*, *Algebr. Geom. Topol.* **11** (2011), no. 1, 1–68.
- [33] ———, *The Dehn surgery characterization of the trefoil and the figure eight knot*, *J. Symplectic Geom.* **17** (2019), no. 1, 251–265.
- [34] Rachel Roberts, *Taut foliations in punctured surface bundles. II*, *Proc. London Math. Soc. (3)* **83** (2001), no. 2, 443–471.
- [35] Shi Cheng Wang and Qing Zhou, *Symmetry of knots and cyclic surgery*, *Trans. Amer. Math. Soc.* **330** (1992), no. 2, 665–676.
- [36] Ying-Qing Wu, *Sutured manifold hierarchies, essential laminations, and Dehn surgery*, *J. Differential Geom.* **48** (1998), no. 3, 407–437.