Remarks on \( SU(2) \)-simple knots and \( SU(2) \)-cyclic 3-manifolds

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\textit{Dedicated to Steve Boyer on the occasion of his 65th birthday}

Abstract. We give some remarks on two closely related issues as stated in the title. In particular we show that a Montesinos knot is \( SU(2) \)-simple if and only if it is a 2-bridge knot, extending a result of Zentner for 3-tangle summand pretzel knots. We conjecture with some evidence that an \( SU(2) \)-cyclic rational homology 3-sphere is an \( L \)-space.

For a knot \( K \) in \( S^3 \), \( M_K \) will be its exterior and \( \mu \) a meridian slope of \( K \). Up to a choice of an orientation for \( \mu \) and a choice of the base point for \( \pi_1(M_K) \), we may also consider \( \mu \) as an element of \( \pi_1(M_K) \). A representation \( \rho : \pi_1(M_K) \to SU(2) \) is called trace free if the trace of \( \rho(\mu) \) is zero (which is obviously well defined). An \( SU(2) \)-representation of \( \pi_1(M_K) \) is called binary dihedral if its image is isomorphic to a binary dihedral group. Note that every binary dihedral representation of \( \pi_1(M_K) \) is trace free [K Proof of Theorem 10]. A knot \( K \) is called \( SU(2) \)-simple if every irreducible trace free \( SU(2) \)-representation of \( \pi_1(M_K) \) is binary dihedral.

A Montesinos link is usually denoted by \( K(e; q_1/p_1, q_2/p_2, \ldots, q_n/p_n) \) where \( q_i/p_i \) represents a rational tangle, \( |p_i| > 1 \) and \( (q_i, p_i) = 1 \) for all \( i \) (see Figure 1). By combining the \( e \) twists in the figure with one of the tangles, we may assume that \( e = 0 \), and we will simply write a Montesinos link as \( K(q_1/p_1, \ldots, q_n/p_n) \) and sometimes we refer it as a cyclic tangle sum of \( n \) rational tangles. When \( q_i = 1, i = 1, \ldots, n \), we get a pretzel link. In [Z] it was shown that every pretzel

\begin{figure}
\centering
\includegraphics[width=\textwidth]{montesinos-link.png}
\caption{A Montesinos link \( K(e; q_1/p_1, q_2/p_2, \ldots, q_n/p_n) \).}
\label{fig:montesinos-link}
\end{figure}

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Every Montesinos knot $K(q_1/p_1, \cdots, q_n/p_n)$, where $n \geq 3$ and $|p_i| > 1$ for all $i = 1, \ldots, n$, is not $SU(2)$-simple.

Theorem 1. For any $SU(2)$-representation $\rho : \pi_1(M_K) \to SU(2)$, let $\bar{\rho} : \pi_1(M_K) \to PSU(2)$ be the induced $PSU(2)$-representation. If $\rho : \pi_1(M_K) \to SU(2)$ is a trace free representation, then $\rho(\mu)$ is an order 4 matrix and $\rho(\mu^2) = -I$, where $I$ is the identity matrix of $SU(2)$. So $\bar{\rho} : \pi_1(M_K) \to PSU(2)$ factors through the quotient group $\pi_1(M_K)/\langle \mu^2 \rangle$, where $\langle \mu^2 \rangle$ denotes the normal subgroup of $\pi_1(M_K)$ generated by $\mu^2$. Let $\Sigma_2(K)$ denote the double branched cover of $(S^3, K)$, then $\pi_1(\Sigma_2(K))$ is an index two subgroup of $\pi_1(M_K)/\langle \mu^2 \rangle$. It is known that a trace free irreducible representation $\rho : \pi_1(M_K) \to SU(2)$ is binary dihedral if and only if the restriction of $\bar{\rho}$ on $\pi_1(\Sigma_2(K))$ has nontrivial cyclic image [K] Section I.E]. For any 2-bridge knot $K$, $\Sigma_2(K)$ is a lens space and so every irreducible trace free $SU(2)$-representation of $\pi_1(M_K)$ is binary dihedral, that is, every 2-bridge knot is $SU(2)$-simple. As a Montesinos knot is a 2-bridge knot if and only if it has less than three rational tangle summands, we have

Corollary 2. A Montesinos knot is $SU(2)$-simple if and only if it is a 2-bridge knot.

By [KM] Corollary 7.17, every nontrivial knot in $S^3$ has an irreducible trace free $SU(2)$-representation. It follows that if the double branched cover of a nontrivial knot $K$ is a homology 3-sphere, i.e. if the knot determinant $|\Delta_K(-1)| = 1$ where $\Delta_K(t)$ is the Alexander polynomial of $K$, then $K$ is not $SU(2)$-simple.

A 3-manifold $Y$ is called $SU(2)$-cyclic (resp. $PSU(2)$-cyclic) if every $SU(2)$-representation (resp. $PSU(2)$-representation) of $\pi_1(Y)$ has cyclic image. In general $PSU(2)$-cyclic is a stronger condition than $SU(2)$-cyclic, that is, $PSU(2)$-cyclic implies $SU(2)$-cyclic but not the other way around. Since $\Sigma_2(K)$ is an $\mathbb{Z}_2$-homology 3-sphere [R] Cor 3 of 8D), every $PSU(2)$-representation of $\pi_1(\Sigma_2(K))$ lifts to an $SU(2)$-representation [BZ] Page 752 and thus $\Sigma_2(K)$ is $SU(2)$-cyclic if and only if it is $PSU(2)$-cyclic. So if $\Sigma_2(K)$ is $SU(2)$-cyclic, then $K$ is an $SU(2)$-simple knot. The following question concerns the converse.

Question 3. Is there an $SU(2)$-simple knot $K$ in $S^3$ whose double branched cover $\Sigma_2(K)$ is not $SU(2)$-cyclic (that is, the double branched cover $\Sigma_2(K)$ has irreducible $PSU(2)$-representations but none of them extend to $M_K$)?

One may consider an $SU(2)$-cyclic 3-manifold as an $SU(2)$-representation $L$-space. The following conjecture suggests that for a rational homology 3-sphere being $SU(2)$-cyclic is more restrictive than being a usual $L$-space in the Heegaard Floer homology sense.

Conjecture 4. If a rational homology 3-sphere is $SU(2)$-cyclic, then it is an $L$-space.

Certainly the converse of Conjecture 4 does not hold; there are many $L$-spaces which are not $SU(2)$-cyclic. For instance, the double branched covers of all alternating Montesinos knots of at least three tangle summands are not $SU(2)$-cyclic but are $L$-spaces.
Here are some evidences for the conjecture. Let \( K_1 = T(p_1, q_1) \) and \( K_2 = T(p_2, q_2) \) be two torus knots in \( S^3 \), and let \( M_1 \) and \( M_2 \) be their exteriors. Let \( Y(T(p_1, q_1), T(p_2, q_2)) \) be the graph manifold obtained by gluing \( M_1 \) and \( M_2 \) along their boundary tori by an orientation reversing homeomorphism \( h : \partial M_1 \to \partial M_2 \) which identifies the meridian slope in \( \partial M_1 \) to the Seifert fiber slope in \( \partial M_2 \) and identifies the Seifert fiber slope in \( \partial M_1 \) with the meridian slope in \( \partial M_2 \). By [Mot] Proposition 5 \( Y(T(p_1, q_2), T(p_2, q_2)) \) has only cyclic \( \text{PSL}_2(\mathbb{C}) \)-representations. (Although it was assumed in [Mot] that all \( p_1, q_1, p_2, q_2 \) are positive, the same argument with obvious modification works without this assumption). Therefore \( Y(T(p_1, q_1), T(p_2, q_2)) \) is \( SU(2) \)-cyclic.

PROPOSITION 5. \( Y(T(p_1, q_1), T(p_2, q_2)) \) is an \( L \)-space.

PROOF. We prove this assertion by applying [HW] Theorem 1.6]. By that theorem, we just need to verify that \( h(L^0_{M_1}) \cup L^0_{M_2} \cong \mathbb{Q} \cup \{0\} \), where \( L^0_{M_1} \) is the interior of the set of \( L \)-space filling slopes of \( M_1, i = 1, 2 \). Note that a general torus knot can be expressed as \( T(p, q) \) with \( (p, q) = 1 \) and \( |p|, q \geq 2 \). By [OS] Corollary 1.4

(1) \( L^0(M_i) = \{ \) slopes in the open interval \( (p_i q_i - p_i - q_i, \infty), \) if \( p_i > 0 \),

slopes in the open interval \( (-\infty, p_i q_i - p_i + q_i), \) if \( p_i < 0 \).

Let \( \mu_i, \lambda_i \) be the meridian and longitude of \( K_i \). Note that \( p_i q_i \) is the Seifert fiber slope in \( \partial M_i \). We have \( h(\mu_1) = \mu_2^{p_2 q_2} \lambda_2 \) and \( h(\mu_1^{p_1 q_1} \lambda_1) = \mu_2 \). Hence for a general slope \( m/n \) in \( \partial M_1 \), where \( m, n \) are relative prime,

\[
h(\mu_1^{m \lambda_1}) = h(\mu_1^{m-p_1 q_1 n} (\mu_1^{p_2 q_2} \lambda_2)^n) = (\mu_2^{p_2 q_2} \lambda_2)^{m-p_1 q_1 n} \mu_2^n
\]

Now suppose \( a/b \) is a slope in \( \partial M_2 \), where \( a, b \) are relatively prime. Choose \( n = a - p_2 q_2 b \) and \( m = p_1 q_1 (a - p_2 q_2 b) + b \), then \( m, n \) are relatively prime, \( h(m/n) = a/b \), and

(2) \( \frac{m}{n} = p_1 q_1 + \frac{b}{a - p_2 q_2 b} = p_1 q_1 + \frac{1}{\frac{a}{b} - p_2 q_2} \).

Case 1. \( p_1 > 0 \) and \( p_2 > 0 \).

For any \( a/b \notin L^0(M_2) \), i.e. either \( a/b = 1/0 \) or \( a/b \) is finite and \( a/b \leq p_2 q_2 - p_2 - q_2 \) by \( \textbf{1} \), choose correspondingly in \( \partial M_1 \) the slope \( m/n = p_1 q_1 \) or as in \( \textbf{2} \) which yields \( m/n \geq p_1 q_1 + \frac{1}{p_2 - q_2} > p_1 q_1 - 1 \). So in either case \( m/n \in L^0(M_1) \) by \( \textbf{1} \) and \( h(m/n) = a/b \), which means \( h(L^0_{M_1}) \cup L^0_{M_2} \cong \mathbb{Q} \cup \{1/0\} \) in this case.

Case 2. \( p_1 > 0 \) and \( p_2 < 0 \).

For any \( a/b \notin L^0(M_2) \), we may assume that \( a/b \) is finite and so \( a/b \geq p_2 q_2 - p_2 + q_2 \) by \( \textbf{1} \). So \( \frac{a}{b} - p_2 q_2 \) is positive. Choose the slope \( m/n \) in \( \partial M_1 \) as in \( \textbf{2} \) which yields \( m/n > p_1 q_1 \) in this case. So \( m/n \in L^0(M_1) \) by \( \textbf{1} \) and \( h(m/n) = a/b \). Thus \( h(L^0_{M_1}) \cup L^0_{M_2} \cong \mathbb{Q} \cup \{1/0\} \) holds in this case.

Case 3. \( p_1 < 0 \) and \( p_2 > 0 \).

This case is really Case 2 if we switch \( K_1 \) and \( K_2 \).

Case 4. \( p_1 < 0 \) and \( p_2 < 0 \).

For any \( a/b \notin L^0(M_2) \), again we may assume \( a/b \) is finite and so \( a/b \geq p_2 q_2 - p_2 + q_2 \) by \( \textbf{1} \). Choose the slope \( m/n \) in \( \partial M_1 \) as in \( \textbf{2} \) which yields \( m/n \leq p_1 q_1 + \frac{1}{p_2 - q_2} < p_1 q_1 + 1 \). So \( m/n \in L^0(M_1) \), \( h(m/n) = a/b \) and we have \( h(L^0_{M_1}) \cup L^0_{M_2} \cong \mathbb{Q} \cup \{1/0\} \).
The proof of Proposition 5 is now completed. □

It was shown in [ZZ] that if \( p_1 q_1 p_2 q_2 - 1 \) is odd, then \( Y(T(p_1, q_1), T(p_2, q_2)) \) is the double branched cover of an alternating knot in \( S^3 \), so \( Y(T(p_1, q_1), T(p_2, q_2)) \) is an \( L \)-space and the knot in \( S^3 \) is an \( SU(2) \)-simple knot (and is an arborescent knot) but is not a 2-bridge knot.

**Remark 6.** It is pointed out by Steven Sivek and the referee that Proposition 5 actually follows from [ZZ], the proof of the result of [ZZ] cited above generalizes immediately to give the conclusion that \( Y(T(p_1, q_1), T(p_2, q_2)) \) is the double branched cover of an alternating link in \( S^3 \) and thus is an \( L \)-space. Proving Proposition 5 this way gives a little more information for free, because branched double covers of alternating links are known to be \( L \)-spaces in pretty much every version of Floer homology including monopole Floer homology and (framed) instanton homology. By contrast, the proof using [HW] does not apply in instanton homology.

There are also examples of hyperbolic rational homology 3-spheres which are \( SU(2) \)-cyclic [C]. These examples are also double branched covers of alternating knots in \( S^3 \) and thus are \( L \)-spaces. These alternating knots are thus \( SU(2) \)-simple but are not arborescent.

Conjecture [H] can be equivalently stated as: if an rational homology 3-sphere is not an \( L \)-space, then it has an irreducible \( SU(2) \)-representation. There are evidences supporting the conjecture from this point of view. For instances Dehn surgery on any nontrivial knot in \( S^3 \) with any slope in the interval \((-1, 1)\) yields a manifold which is not an \( L \)-space [OS] and is not \( SU(2) \)-cyclic either [KM].

Steven Sivek and the referee provide the following remark with further evidence for Conjecture 4.

**Remark 7.** A rational homology 3-sphere \( Y \) which is \( SU(2) \)-cyclic is conjecturally an instanton homology \( L \)-space (meaning \( I^\#(Y) \) has rank \(|H_1(Y)|\)). On the other hand \( I^\#(Y) \) is conjecturally isomorphic to \( \hat{HF}(Y) \). It was shown in [BS] Theorem 4.6) that if \( Y \) is a \( SU(2) \)-cyclic rational homology 3-sphere whose fundamental group is cyclically finite, then \( Y \) is an instanton homology \( L \)-space. Here the notion of cyclically finite was first defined in [BN] meaning that as \( \rho \) ranges over reducible representations of \( \pi_1(Y) \to SU(2) \), all of the finite cyclic covers of \( Y \) corresponding to subgroups \( ker(ad(\rho)) < \pi_1(Y) \) are rational homology 3-spheres.

**Proof of Theorem 1.** Let \( K = K(q_1/p_1, \ldots, q_n/p_n) \) be a Montesinos knot with \( n \geq 3 \). We need to show that \( \pi_1(M_K) \) has an irreducible trace free \( SU(2) \)-representation which is not binary dihedral. Here is an outline of how the proof goes. We show that the double branched cover \( \Sigma_2(K) \) has an irreducible \( PSU(2) \)-representation \( \bar{\rho}_0 \) which can be extended to an \( PSU(2) \)-representation \( \bar{\rho} \) of \( \pi_1(M_K) \) up to conjugation. This \( PSU(2) \)-representation \( \bar{\rho} \) lifts to an \( SU(2) \)-representation \( \rho \) of \( \pi_1(M_K) \) which is automatically trace free. Since \( \bar{\rho}_0 \) is an irreducible representation, \( \rho \) is not binary dihedral. The existence of \( \bar{\rho}_0 \) is provided by [H]. We first apply some ideas from [Mat] to show that \( \bar{\rho}_0 \) extends to a unique \( PSL_2(\C) \)-representation \( \bar{\rho} \) of \( \pi_1(M_K) \). Then we further show that this \( \bar{\rho} \) is conjugate to an \( SU(2) \)-representation by applying some results from [HP] [CD].

Now we give the details of the proof. For a finitely generated group \( \Gamma \), \( \bar{R}(\Gamma) = Hom(\Gamma, PSL_2(\C)) \) denotes the \( PSL_2(\C) \) representation variety of \( \Gamma \) and \( \hat{X}(\Gamma) \) the
$PSL_2(\mathbb{C})$ character variety of $\Gamma$. Let $t : \tilde{R}(\Gamma) \to \tilde{X}(\Gamma)$ be the map which sends a representation $\tilde{\rho}$ to its character $\chi_{\tilde{\rho}}$. We shall write an element in $PSL_2(\mathbb{C})$ as $A$ which is the image of an element $\bar{A}$ in $SL_2(\mathbb{C})$ under the quotient map $SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ and for convenience we sometimes call elements in $PSL_2(\mathbb{C})$ as matrices. For any $A \in PSL_2(\mathbb{C})$ define $tr^2(A) = (\text{trace}(A))^2$ which is obviously well defined. Recall that the character $\chi_{\tilde{\rho}}$ of an $PSL_2(\mathbb{C})$-representation $\tilde{\rho}$ is the function $\chi_{\tilde{\rho}} : \Gamma \to \mathbb{C}$ defined by $\chi_{\tilde{\rho}}(\gamma) = tr^2(\tilde{\rho}(\gamma))$.

A character $\chi_{\tilde{\rho}}$ is real if $\chi_{\tilde{\rho}}(\gamma) \in \mathbb{R}$ for all $\gamma \in \Gamma$. If we consider $\tilde{X}(\Gamma)$ as an algebraic subset in $\mathbb{C}^n$ (for some $n$), then real characters of $\tilde{X}(\Gamma)$ correspond to real points of $\bar{X}(\Gamma)$, i.e. points of $\bar{X}(\Gamma) \cap \mathbb{R}^n$. If $\sigma : \mathbb{C}^n \to \mathbb{C}^n$ (for each $n \geq 1$) denotes the operation of coordinatewise taking complex conjugation, then any complex affine algebraic set $Y$ in $\mathbb{C}^n$ defined over $\mathbb{Q}$ is invariant under $\sigma$ and the set of real points of $Y$ is precisely the fixed point set of $\sigma$ in $Y$. Note that $\tilde{R}(\Gamma)$ and $\tilde{X}(\Gamma)$ are both algebraic sets defined over $\mathbb{Q}$ and that the map $t : \tilde{R}(\Gamma) \to \tilde{X}(\Gamma)$ is an algebraic map defined over $\mathbb{Q}$, we thus have the following commutative diagram of maps:

$$
\begin{array}{ccc}
R(\Gamma) & \xrightarrow{\sigma} & R(\Gamma) \\
\downarrow t & & \downarrow t \\
\tilde{X}(\Gamma) & \xrightarrow{\sigma} & \tilde{X}(\Gamma).
\end{array}
$$

It follows that $\sigma(\chi_{\tilde{\rho}}) = \chi_{\sigma(\tilde{\rho})}$.

Recall that a representation $\tilde{\rho} \in \tilde{R}(\Gamma)$ is called irreducible if the image of $\tilde{\rho}$ cannot be conjugated into the set $\{ \bar{A} ; \bar{A} \text{ upper triangular} \}$ (BZ, Definition on page 752). Two irreducible representations in $\tilde{R}(\Gamma)$ are conjugate if and only if they have the same character (This property is proved in BZ, the second paragraph on page 753).

If $W$ is a compact manifold, $R(W)$ and $\tilde{X}(W)$ denote $R(\pi_1 W)$ and $\tilde{X}(\pi_1 W)$ respectively.

Let $K = K(q_1/p_1, \ldots, q_n/p_n)$ and $M = M_K$. We may assume that all $p_i$ are positive (by changing the sign of $q_i$ if necessary). Let $p : \tilde{M} \to M$ be the 2-fold cyclic covering and let $\tilde{\mu} = p^{-1}(\mu)$ which is a connected simple closed essential curve in $\partial \tilde{M}$ which doubles covers $\mu$. Then $p_* : \pi_1(\tilde{M}) \to \pi_1(M)$ is an injection and we may consider $\pi_1(M)$ as an index two normal subgroup of $\pi_1(\tilde{M})$, in which $\tilde{\mu} = \mu^2$. Dehn filling $\tilde{M}(\tilde{\mu})$ of $\tilde{M}$ with the slope $\tilde{\mu}$ is the double branched cover $\Sigma_2(K)$ of $(S^3, K)$. The covering involution $\tau$ on $\tilde{M}$ extends to one on $\tilde{M}(\tilde{\mu})$ which we still denote by $\tau$. Montesinos proved in [Mont1, Mont2] that $\tilde{M}(\tilde{\mu})$ admits a Seifert fibering invariant under the covering involution $\tau$, the base orbifold of the Seifert fibred space is $S^2(p_1, \ldots, p_n)$ which is the 2-sphere with $n$ cone points of orders $p_1, \ldots, p_n$, and $\tau$ descends down to an involution $\tau$ on $S^2(p_1, \ldots, p_n)$ which is a reflection in a circle passing through all the cone points (see Figure 2).

We denote the orbifold fundamental group of $S^2(p_1, \ldots, p_n)$ by $\Delta(p_1, \ldots, p_n)$ which has the following presentation:

$$
\Delta(p_1, \ldots, p_n) = \langle a_1, \ldots, a_n ; a_i^{-p_i} = 1, i = 1, \ldots, n, a_1 a_2 \cdots a_n = 1 \rangle.
$$

Geometrically the element $a_i$ is represented by the loop $b_{i-1}^{-1} b_i$ shown in Figure 2 ($b_0$ is the trivial loop). Note that there is a quotient homomorphism from $\pi_1(\tilde{M}(\tilde{\mu}))$ onto $\Delta(p_1, \ldots, p_n)$.

It was shown in [Mat] Section 3.3 that when $n = 3$ any irreducible $PSL_2(\mathbb{C})$-representation of $\pi_1(M)$ which factors through $\pi_1(\tilde{M}(\tilde{\mu}))$ has a unique extension to
\( \pi_1(M) \). Note that this extended representation can be lifted to a trace free \( SL_2(\mathbb{C}) \)-representation of \( \pi_1(M) \). We shall slightly extend this result to the following

\[
\tau_1^{2^{n-1}} b b_{n-1} b_3 b_2 b_1 x_0 \_ c \_ c \_ c_n \_ c_{n-1} \]

Figure 2. The orbifold \( S^2(p_1, \ldots, p_n) \), its involution \( \tau \) and the generating set \( b_1, ..., b_{n-1} \) for \( \Delta(p_1, ..., p_n) \), where \( x_0 \) is the base point and \( c_1, ..., c_n \) are cone points of orders \( p_1, ..., p_n \) respectively.

**Proposition 8.** Let \( \delta \) be the composition of the three quotient homomorphisms

\[
\pi_1(M) \rightarrow \pi_1(M(\check{\mu})) \rightarrow \Delta(p_1, p_2, ..., p_n) \rightarrow \Delta(p_1, p_2, p_3).
\]

Let \( \phi : \Delta(p_1, p_2, p_3) \rightarrow PSL_2(\mathbb{C}) \) be any irreducible representation. Then \( \bar{\rho}_0 = \phi \circ \delta \) has a unique extension to \( \pi_1(M_K) \).

**Proof.** The proof for uniqueness is verbatim as that given in [Mat] on page 38-39. We need to note that as \( K(q_1/p_1, \ldots, q_n/p_n) \) is a knot at most one of \( p_i \)'s is even. So [Mat] Lemma 2.4.9 still applies to our current case, i.e. the center of the image group of \( \bar{\rho}_0 \) is the trivial group.

**Claim 9.** There will be an extension \( \bar{\rho} \) if and only if there is \( \check{A} \in PSL_2(\mathbb{C}) \) such that \( \check{A}^2 = \check{I} \) (where \( \check{I} \) is the identity matrix of \( SL_2(\mathbb{C}) \)) and \( \check{A}\bar{\rho}_0(\beta)\check{A}^{-1} = \bar{\rho}_0(\mu \beta \mu^{-1}) \) for all \( \beta \in \pi_1(M) \).

Again this claim can be proved verbatim as that of [Mat] Claim 3.3.2].

So to finish the proof of Proposition 8 we just need to find an \( \check{A} \in PSL_2(\mathbb{C}) \) with the properties stated in Claim 9 which is what we are going to do in the rest of the proof of Proposition 8. Recall that \( \check{M}(\check{\mu}) \) is the Dehn filling of \( \check{M} \) with a solid torus \( N \) whose meridian slope is identified with the slope \( \check{\mu} \). The core circle of \( N \) is the fixed point set of \( \tau \) in \( \check{M}(\check{\mu}) \). Let \( D \) be a meridian disk of \( N \) such that the fixed point of \( \tau \) in \( D \) (the center point of \( D \)) is disjoint from the singular fibers of the Seifert fibred space \( \check{M}(\check{\mu}) \). Choose a point \( \check{x} \) in \( \partial D \) and let \( \check{x}_0 \) be the center.
point of $D$. Then arguing as on [Mat] Page 40 we have the following commutative diagram:

$$\pi_1(\tilde{M}, \tilde{x}) \rightarrow \pi_1(\tilde{M}(\mu), \tilde{x}) \rightarrow \pi_1(\tilde{M}(\mu), \tilde{x}_0) \rightarrow \Delta(p_1, \ldots, p_n)$$

where $(\cdot)\mu : \pi_1(\tilde{M}, \tilde{x}) \rightarrow \pi_1(\tilde{M}, \tilde{x})$ corresponds to the conjugation action by $\mu$, i.e. $(\beta)\mu = \mu \beta \mu^{-1}$ and $\Delta(p_1, \ldots, p_n)$ is the orbifold fundamental group of $S^2(p_1, \ldots, p_n)$ whose base point is the image $x_0$ of the point $\tilde{x}_0$ under the quotient map $\tilde{M}(\mu) \rightarrow S^2(p_1, \ldots, p_n)$.

Figure 2 shows the generating set $b_1, \ldots, b_{n-1}$ of the orbifold fundamental group $\Delta(p_1, \ldots, p_n)$ of $S^2(p_1, \ldots, p_n)$. In fact we have

$$a_1 = b_1, a_2 = b_1^{-1}b_2, a_3 = b_2^{-1}b_3, \ldots, a_{n-1} = b_{n-2}^{-1}b_{n-1}, a_n = b_n^{-1}$$

and conversely

$$b_1 = a_1, b_2 = a_1a_2, b_3 = a_1a_2a_3, \ldots, b_{n-1} = a_1a_2 \cdots a_{n-1}, b_n = a_n^{-1}.$$

Obviously from Figure 2 the induced isomorphism $\tilde{\tau}_* : \Delta(p_1, \ldots, p_n) \rightarrow \Delta(p_1, \ldots, p_n)$ sends $b_i$ to $b_i^{-1}, i = 1, \ldots, n-1$. So we have

$$\tilde{\tau}_*(a_1) = a_1^{-1}, \tilde{\tau}_*(a_2) = b_1b_2^{-1} = a_1a_2^{-1}a_1^{-1}, \tilde{\tau}_*(a_3) = b_2b_3^{-1} = a_1a_2a_3^{-1}a_2^{-1}a_1^{-1},$$

$$\ldots, \tilde{\tau}_*(a_{n-1}) = b_{n-2}b_{n-1}^{-1} = a_1a_2 \cdots a_{n-1}a_{n-2}^{-1}a_{n-1}^{-1}, \tilde{\tau}_*(a_n) = \tilde{\tau}_*(b_n^{-1}) = b_{n-1}^{-1} = a_n^{-1}.$$

Since the quotient homomorphism

$$\Delta(p_1, \ldots, p_n) = \langle a_1, \ldots, a_n ; a_i^p = 1, i = 1, \ldots, n, a_1a_2 \cdots a_n = 1 \rangle \rightarrow \Delta(p_1, p_2, p_3) = \langle a_1, a_2, a_3 ; a_1^p = 1, i = 1, 2, 3, a_1a_2a_3 = 1 \rangle$$

sends $a_i$ to $a_i$, $i = 1, 2, 3$, and sends $a_i$ to $1$, $i = 4, \ldots, n$ we see that $\tilde{\tau}_*$ descents to an isomorphism $\tilde{\tau}_\# : \Delta(p_1, p_2, p_3) \rightarrow \Delta(p_1, p_2, p_3)$ such that $\tilde{\tau}_\#(a_1) = a_1^{-1}, \tilde{\tau}_\#(a_2) = a_1a_2^{-1}a_1^{-1}, \tilde{\tau}_\#(a_3) = a_1a_2a_3^{-1}a_2^{-1}a_1^{-1}$ and we have the following commutative diagram:

$$\pi_1(\tilde{M}) \rightarrow \pi_1(\tilde{M}(\mu)) \rightarrow \Delta(p_1, \ldots, p_n) \rightarrow \Delta(p_1, p_2, p_3) \rightarrow PSL_2(\mathbb{C})$$

So $\tilde{\tau}_\#(a_1a_2) = a_2^{-1}a_1^{-1} = (a_1a_2)^{-1}$. Since $\Delta(p_1, p_2, p_3)$ is generated by $a_1, a_2$, we see by applying [BZ: Lemma 3.1] that $\phi$ and $\phi \circ \tilde{\tau}_\#$ have the same $PSL_2(\mathbb{C})$ character. (In fact if $\phi(a_1) = A_1$ and $\phi(a_2) = A_2$, then $\phi(a_1a_2) = A_1A_2 = A_1^{-1}A_2^{-1}$ and $\phi(a_1a_2a_3) = (A_2^{-1}A_1^{-1})^{-1} = (A_2^{-1}A_1^{-1})(A_1A_2^{-1})^{-1} = (A_2^{-1}A_1^{-1})^{-1}$. Now let $F_2$ be the free group on two generators $\xi_1$ and $\xi_2$. Let $p_1$ and $p_2$ be the $SL_2(\mathbb{C})$ representations of $F_2$ defined by $p_1(\xi_1) = A_1, i = 1, 2,$ and $p_2(\xi_1) = A_1^{-1}, p_2(\xi_2) = A_1A_2^{-1}$. Then one can easily verify that $\text{tr}(p_1(\xi_1)) = \text{tr}(p_2(\xi_1)), \text{tr}(p_1(\xi_2)) = \text{tr}(p_2(\xi_2))$ and $\text{tr}(p_1(\xi_1\xi_2)) = \text{tr}(p_2(\xi_1\xi_2))$. So [BZ: Lemma 3.1] applies.) So $\phi$ and $\phi \circ \tilde{\tau}_\#$ are conjugate $PSL_2(\mathbb{C})$ representations, that is, there is $\tilde{A} \in PSL_2(\mathbb{C})$ with $\tilde{A}\phi\tilde{A}^{-1} = \phi \circ \tilde{\tau}_\#$.
can be conjugated into an

tation of the total geodesic plane mentioned above, i.e. the image of \( \tilde{H} \) group completes the proof of Theorem 1. □

Now by [B], every triangle group \( \Delta(p_1,p_2,p_3) \) has an irreducible \( SO(3) \cong PSU(2) \)-representation. Therefore there is an irreducible representation \( \tilde{\rho}_0 \) as given in Proposition \( \S \) with its image contained in \( PSU(2) \). So the character \( \chi_{\tilde{\rho}_0} \) of \( \tilde{\rho}_0 \) is real valued. Let \( \tilde{\rho} \) be the unique extension of \( \tilde{\rho}_0 \) to \( \pi_1(M) \) as guaranteed by Proposition \( \S \). The rest of proof is to show that \( \tilde{\rho} \) is also a \( PSU(2) \)-representation.

Claim 10. The character \( \chi_{\tilde{\rho}} \) of \( \tilde{\rho} \) is real valued.

Suppose otherwise. Recall that \( \sigma : \tilde{X}(M) \rightarrow \tilde{X}(M) \) is the operation of taking complex conjugation and a character is real valued if and only if it is a fixed point of \( \sigma \). So \( \chi_{\tilde{\rho}} \neq \sigma(\chi_{\tilde{\rho}}) = \chi_{\sigma(\tilde{\rho})} \) are two different characters of irreducible representations and thus \( \tilde{\rho} \) and \( \sigma(\tilde{\rho}) \) are non-conjugate representations. But \( \chi_{\tilde{\rho}_0} = \sigma(\chi_{\tilde{\rho}_0}) = \chi_{\sigma(\tilde{\rho}_0)} \) and \( \tilde{\rho}_0 \) is irreducible. Hence \( \tilde{\rho}_0 \) and \( \sigma(\tilde{\rho}_0) \) are conjugate representations, that is, there is \( B \in PSL_2(\mathbb{C}) \) such that \( \tilde{\rho}_0 = B\sigma(\tilde{\rho}_0)B^{-1} \). Hence \( \tilde{\rho} \) and \( B\sigma(\tilde{\rho})B^{-1} \) are non-conjugate representations which have the same restriction on \( \pi_1(M) \). We get a contradiction with Proposition \( \S \).

By [HP] Lemma 10.1] an \( PSL_2(\mathbb{C}) \)-character \( \chi_{\tilde{\rho}} \) of a finitely generated group is real valued if and only if the image of \( \tilde{\rho} \) can be conjugated into \( PSU(2) \) or \( PGL_2(\mathbb{R}) \). So our current representation \( \tilde{\rho} \) can be conjugated into \( PSU(2) \) or \( PGL_2(\mathbb{R}) \). If it can be conjugated into \( PSU(2) \), then we are done because this conjugated representation lifts to a trace free \( SU(2) \)-representation which is not binary dihedral. So suppose that \( \tilde{\rho} \) is conjugate to an \( PGL_2(\mathbb{R}) \)-representation \( \tilde{\rho}' \). As noted in [HP] right after Lemma 10.1, \( PGL_2(\mathbb{R}) \subset PGL_2(\mathbb{C}) \cong PSL_2(\mathbb{C}) \) has two components, the identity component is \( PSL_2(\mathbb{R}) \) and the other component consists of matrices of determinant \( -1 \) (which in \( PSL_2(\mathbb{C}) \) are represented by matrices with entries in \( \mathbb{C} \) with zero real part). Considering the action of \( PSL_2(\mathbb{C}) \) on hyperbolic space \( \mathbb{H}^3 \) by orientation preserving isometries, the group \( PGL_2(\mathbb{R}) \) is the stabilizer of a total geodesic plane in \( \mathbb{H}^3 \), and an element of \( PGL_2(\mathbb{R}) \) is contained in \( PSL_2(\mathbb{R}) \) if and only if it preserves the orientation of the plane (cf [JS] Section 2.8). Since \( \pi_1(M) \) is the unique index two normal subgroup of \( \pi_1(M) \), \( \pi_1(M) \) is generated by elements \( \gamma^2, \gamma \in \pi_1(M) \). (To see this assertion holds, it is easy to verify that the subgroup \( G \) of \( \pi_1(M) \) generated by squares is a normal subgroup and the quotient \( \pi_1(M)/G \) is an abelian group in which every element has order two. Since \( H_1(M) = \mathbb{Z}, \pi_1(M)/G \) has to be the cyclic group of order two.) Therefore the image of the restriction \( \tilde{\rho}_0' \) of \( \tilde{\rho}' \) on \( \pi_1(M) \) consists of elements preserving the orientation of the total geodesic plane mentioned above, i.e. the image of \( \tilde{\rho}_0' \) is contained in \( PSL_2(\mathbb{R}) \). So the image of \( \tilde{\rho}_0 \) can be conjugated into \( PSL_2(\mathbb{R}) \). But the image of \( \tilde{\rho}_0 \) is contained in \( PSU(2) \). [CD] Lemma 2.10] says that if an \( PSU(2) \)-presentation can be conjugated into an \( PSL_2(\mathbb{R}) \)-representation, then it is a reducible representation. So our \( \tilde{\rho}_0 \) is a reducible representation. We arrive at a contradiction, which completes the proof of Theorem \( \P \).


References


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