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Remarks on $SU(2)$-simple knots and $SU(2)$-cyclic 3-manifolds

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Dedicated to Steve Boyer on the occasion of his 65th birthday

Abstract. We give some remarks on two closely related issues as stated in the title. In particular we show that a Montesinos knot is $SU(2)$-simple if and only if it is a 2-bridge knot, extending a result of Zentner for 3-tangle summand pretzel knots. We conjecture with some evidence that an $SU(2)$-cyclic rational homology 3-sphere is an L-space.

For a knot K in S^3, $M_{\rm K}$ will be its exterior and μ a meridian slope of K. Up to a choice of an orientation for μ and a choice of the base point for $\pi_1(M_{\rm K})$, we may also consider μ as an element of $\pi_1(M_{\rm K})$. A representation $\rho : \pi_1(M_{\rm K})\to SU(2)$ is called trace free if the trace of $\rho(\mu)$ is zero (which is obviously well defined). An $SU(2)$-representation of $\pi_1(M_{\rm K})$ is called binary dihedral if its image is isomorphic to a binary dihedral group. Note that every binary dihedral representation of $\pi_1(M_{\rm K})$ is trace free [**K**, Proof of Theorem 10]. A knot K is called $SU(2)$-*simple* if every irreducible trace free $SU(2)$-representation of $\pi_1(M_{\rm K})$ is binary dihedral.

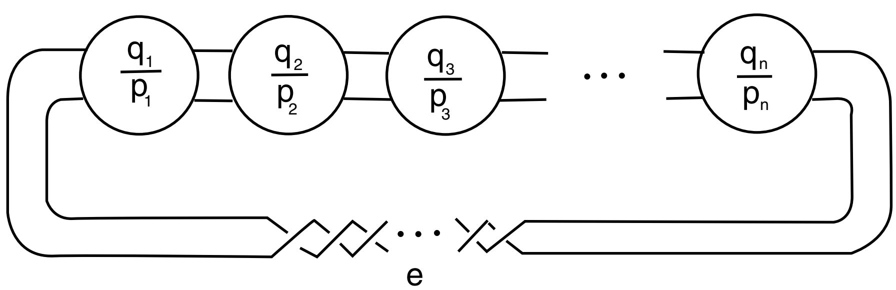

Figure 1. A Montesinos link $K(e; q_1/p_1, q_2/p_2, ..., q_n/p_n)$.

A Montesinos link is usually denoted by $K(e; q_1/p_1, q_2/p_2, ..., q_n/p_n)$ where q_i/p_i represents a rational tangle, $|p_i| > 1$ and $(q_i, p_i) = 1$ for all i (see Figure 1). By combining the e twists in the figure with one of the tangles, we may assume that $e = 0$, and we will simply write a Montesinos link as $K(q_1/p_1, \cdots, q_n/p_n)$ and sometimes we refer it as a cyclic tangle sum of n rational tangles. When $q_i = 1, i = 1, ..., n$, we get a pretzel link. In [**Z1**] it was shown that every pretzel

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knot $K(1/p, 1/q, 1/r)$, p, q, r pairwise coprime, is not $SU(2)$-simple. In this paper we extend this result to all Montesinos knots of at least three rational tangle summands.

Theorem 1. *Every Montesinos knot $K(q_1/p_1, \cdots, q_n/p_n)$, where $n \geq 3$ and $|p_i| > 1$ for all $i = 1, ..., n$, is not $SU(2)$-simple.*

For any $SU(2)$-representation $\rho : \pi_1(M_{\rm K}) \to SU(2)$, let $\bar\rho : \pi_1(M_{\rm K}) \to PSU(2)$ be the induced $PSU(2)$-representation. If $\rho : \pi_1(M_{\rm K}) \to SU(2)$ is a trace free representation, then $\rho(\mu)$ is an order 4 matrix and $\rho(\mu^2) = -I$, where I is the identity matrix of $SU(2)$. So $\bar\rho : \pi_1(M_{\rm K}) \to PSU(2)$ factors through the quotient group $\pi_1(M_{\rm K})/\langle \mu^2 \rangle$, where $\langle \mu^2 \rangle$ denotes the normal subgroup of $\pi_1(M_{\rm K})$ generated by μ^2. Let $\Sigma_2(K)$ denote the double branched cover of (S^3, K), then $\pi_1(\Sigma_2(K))$ is an index two subgroup of $\pi_1(M_{\rm K})/\langle \mu^2 \rangle$. It is known that a trace free irreducible representation $\rho : \pi_1(M_{\rm K}) \to SU(2)$ is binary dihedral if and only if the restriction of $\bar\rho$ on $\pi_1(\Sigma_2(K))$ has nontrivial cyclic image [**K**, Section I.E]. For any 2-bridge knot K, $\Sigma_2(K)$ is a lens space and so every irreducible trace free SU(2)-representation of $\pi_1(M_{\rm K})$ is binary dihedral, that is, every 2-bridge knot is $SU(2)$-simple. As a Montesinos knot is a 2-bridge knot if and only if it has lass than three rational tangle summands, we have

Corollary 2. *A Montesinos knot is $SU(2)$-simple if and only if it is a 2-bridge knot.*

By [**KM**, Corollary 7.17], every nontrivial knot in S^3 has an irreducible trace free $SU(2)$-representation. It follows that if the double branched cover of a nontrivial knot K is a homology 3-sphere, i.e. if the knot determinant $|\Delta_{\rm K}(-1)| = 1$ where $\Delta_{\rm K}(t)$ is the Alexander polynomial of K, then K is not $SU(2)$-simple.

A 3-manifold Y is called $SU(2)$-cyclic (resp. $PSU(2)$-cyclic) if every $SU(2)$-representation (resp. $PSU(2)$-representation) of $\pi_1(Y)$ has cyclic image. In general $PSU(2)$-cyclic is a stronger condition than $SU(2)$-cyclic, that is, $PSU(2)$-cyclic implies $SU(2)$-cyclic but not the other way around. Since $\Sigma_2(K)$ is an $\mathbb{Z}_2$-homology 3-sphere [**R**, Cor 3 of 8D], every $PSU(2)$-representation of $\pi_1(\Sigma_2(K))$ lifts to an $SU(2)$-representation [**BZ**, Page 752] and thus $\Sigma_2(K)$ is $SU(2)$-cyclic if and only if it is $PSU(2)$-cyclic. So if $\Sigma_2(K)$ is $SU(2)$-cyclic, then K is an $SU(2)$-simple knot. The following question concerns the converse.

Question 3. Is there an $SU(2)$-simple knot K in S^3 whose double branched cover $\Sigma_2(K)$ is not $SU(2)$-cyclic (that is, the double branched cover $\Sigma_2(K)$ has irreducible $PSU(2)$-representations but none of them extend to $M_{\rm K}$)?

One may consider an $SU(2)$-cyclic 3-manifold as an $SU(2)$-representation L-space. The following conjecture suggests that for a rational homology 3-sphere being $SU(2)$-cyclic is more restrictive than being a usual L-space in the Heegaard Floer homology sense.

Conjecture 4. *If a rational homology 3-sphere is $SU(2)$-cyclic, then it is an L-space.*

Certainly the converse of Conjecture 4 does not hold; there are many L-spaces which are not $SU(2)$-cyclic. For instance, the double branched covers of all alternating Montesinos knots of at least three tangle summands are not $SU(2)$-cyclic but are L-spaces.

Here are some evidences for the conjecture. Let $K_1 = T(p_1, q_1)$ and $K_2 = T(p_2, q_2)$ be two torus knots in S^3, and let M_1 and M_2 be their exteriors. Let $Y(T(p_1, q_1), T(p_2, q_2))$ be the graph manifold obtained by gluing M_1 and M_2 along their boundary tori by an orientation reversing homeomorphism $h : \partial M_1 \to \partial M_2$ which identifies the meridian slope in ∂M_1 to the Seifert fiber slope in ∂M_2 and identifies the Seifert fiber slope in ∂M_1 with the meridian slope in ∂M_2. By [**Mot**, Proposition 5] $Y(T(p_1, q_2), T(P_2, q_2))$ has only cyclic $PSL_2(\mathbb{C})$-representations. (Although it was assumed in [**Mot**] that all p_1, q_1, p_2, q_2 are positive, the same argument with obvious modification works without this assumption). Therefore $Y(T(p_1, q_1), T(p_2, q_2))$ is $SU(2)$-cyclic.

PROPOSITION 5. *$Y(T(p_1, q_1), T(p_2, q_2))$ is an L-space.*

PROOF. We prove this assertion by applying [**HW**, Theorem 1.6]. By that theorem, we just need to verify that $h(\mathcal{L}^\circ_{M_1}) \cup \mathcal{L}^\circ_{M_2} \cong \mathbb{Q} \cup \{1/0\}$, where $\mathcal{L}^0_{M_i}$ is the interior of the set of L-space filling slopes of M_i, $i = 1, 2$. Note that a general torus knot can be expressed as $T(p, q)$ with $(p, q) = 1$ and $|p|, q \geq 2$. By [**OS**, Corollary 1.4]

$$(1) \qquad \mathcal{L}^0(M_i) = \{ \begin{array}{ll} \text{slopes in the open interval } (p_iq_i - p_i - q_i, \infty), & \text{if } p_i > 0, \\ \text{slopes in the open interval } (-\infty, p_iq_i - p_i + q_i), & \text{if } p_i < 0. \end{array}$$

Let μ_i, λ_i be the meridian and longitude of K_i. Note that p_iq_i is the Seifert fiber slope in ∂M_i. We have $h(\mu_1) = \mu_2^{p_2q_2}\lambda_2$ and $h(\mu_1^{p_1q_1}\lambda_1) = \mu_2$. Hence for a general slope m/n in ∂M_1, where m, n are relative prime,

$$\begin{aligned} h(\mu_1^m\lambda_1^n) = h(\mu_1^{m-p_1q_1n}(\mu_1^{p_1q_2}\lambda_1)^n) &= (\mu_2^{p_2q_2}\lambda_2)^{m-p_1q_1n}\mu_2^n \\ &= \mu_2^{p_2q_2(m-p_1q_1n)+n}\lambda_2^{m-p_1q_1n}. \end{aligned}$$

Now suppose a/b is a slope in ∂M_2, where a, b are relatively prime. Choose $n = a - p_2q_2b$ and $m = p_1q_1(a - p_2q_2b) + b$, then m, n are relatively prime, $h(m/n) = a/b$, and

$$(2) \qquad \frac{m}{n} = p_1q_1 + \frac{b}{a - p_2q_2b} = p_1q_1 + \frac{1}{\frac{a}{b} - p_2q_2}.$$

Case 1. $p_1 > 0$ and $p_2 > 0$.

For any $a/b \notin \mathcal{L}^0(M_2)$, i.e. either $a/b = 1/0$ or a/b is finite and $a/b \leq p_2q_2 - p_2 - q_2$ by (1), choose correspondingly in ∂M_1 the slope $m/n = p_1q_1$ or as in (2) which yields $m/n \geq p_1q_1 + \frac{1}{-p_2-q_2} > p_1q_1 - 1$. So in either case $m/n \in \mathcal{L}^0(M_1)$ by (1) and $h(m/n) = a/b$, which means $h(\mathcal{L}^\circ_{M_1}) \cup \mathcal{L}^\circ_{M_2} \cong \mathbb{Q} \cup \{1/0\}$ in this case.

Case 2. $p_1 > 0$ and $p_2 < 0$.

For any $a/b \notin \mathcal{L}^0(M_2)$, we may assume that a/b is finite and so $a/b \geq p_2q_2 - p_2 + q_2$ by (1). So $\frac{a}{b} - p_2q_2$ is positive. Choose the slope m/n in ∂M_1 as in (2) which yields $m/n > p_1q_1$ in this case. So $m/n \in \mathcal{L}^0(M_1)$ by (1) and $h(m/n) = a/b$. Thus $h(\mathcal{L}^\circ_{M_1}) \cup \mathcal{L}^\circ_{M_2} \cong \mathbb{Q} \cup \{1/0\}$ holds in this case.

Case 3. $p_1 < 0$ and $p_2 > 0$.

This case is really Case 2 if we switch K_1 and K_2.

Case 4. $p_1 < 0$ and $p_2 < 0$.

For any $a/b \notin \mathcal{L}^0(M_2)$, again we may assume a/b is finite and so $a/b \geq p_2q_2 - p_2 + q_2$ by (1). Choose the slope m/n in ∂M_1 as in (2) which yields $m/n \leq p_1q_1 + \frac{1}{-p_2+q_2} < p_1q_1 + 1$. So $m/n \in \mathcal{L}^0(M_1)$, $h(m/n) = a/b$ and we have $h(\mathcal{L}^\circ_{M_1}) \cup \mathcal{L}^\circ_{M_2} \cong \mathbb{Q} \cup \{1/0\}$.

The proof of Proposition 5 is now completed. □

It was shown in [**Z2**] that if $p_1q_1p_2q_2 - 1$ is odd, then $Y(T(p_1,q_1), T(p_2,q_2))$ is the double branched cover of an alternating knot in S^3, so $Y(T(p_1,q_1), T(p_2,q_2))$ is an L-space and the knot in S^3 is an $SU(2)$-simple knot (and is an arborescent knot) but is not a 2-bridge knot.

Remark 6. It is pointed out by Steven Sivek and the referee that Proposition 5 actually follows from [**Z2**]; the proof of the result of [**Z2**] cited above generalizes immediately to give the conclusion that $Y(T(p_1,q_1), T(p_2,q_2))$ is the double branched cover of an alternating link in S^3 and thus is an L-space. Proving Proposition 5 this way gives a little more information for free, because branched double covers of alternating links are known to be L-spaces in pretty much every version of Floer homology including monopole Floer homology and (framed) instanton homology. By contrast, the proof using [**HW**] does not apply in instanton homology.

There are also examples of hyperbolic rational homology 3-spheres which are $SU(2)$-cyclic [**C**]. These examples are also double branched covers of alternating knots in S^3 and thus are L-spaces. These alternating knots are thus $SU(2)$-simple but are not arborescent.

Conjecture 4 can be equivalently stated as: if an rational homology 3-sphere is not an L-space, then it has an irreducible $SU(2)$-representation. There are evidences supporting the conjecture from this point of view. For instances Dehn surgery on any nontrivial knot in S^3 with any slope in the interval $(-1,1)$ yields a manifold which is not an L-space [**OS**] and is not $SU(2)$-cyclic either [**KM**].

Steven Sivek and the referee provide the following remark with further evidence for Conjecture 4.

Remark 7. A rational homology 3-sphere Y which is $SU(2)$-cyclic is conjecturally an instanton homology L-space (meaning $I^\#(Y)$ has rank $|H_1(Y)|$). On the other hand $I^\#(Y)$ is conjecturally isomorphic to $\widehat{HF}(Y)$. It was shown in [**BS**, Theorem 4.6] that if Y is a $SU(2)$-cyclic rational homology 3-sphere whose fundamental group is cyclically finite, then Y is an instanton homology L-space. Here the notion of cyclically finite was first defined in [**BN**] meaning that as ρ ranges over reducible representations of $\pi_1(Y)\to SU(2)$, all of the finite cyclic covers of Y corresponding to subgroups $ker(ad(\rho)) \lhd \pi_1(Y)$ are rational homology 3-spheres.

Proof of Theorem 1. Let $K = K(q_1/p_1, \cdots, q_n/p_n)$ be a Montesinos knot with $n \geq 3$. We need to show that $\pi_1(M_K)$ has an irreducible trace free $SU(2)$-representation which is not binary dihedral. Here is an outline of how the proof goes. We show that the double branched cover $\Sigma_2(K)$ has an irreducible $PSU(2)$-representation $\bar\rho_0$ which can be extended to an $PSU(2)$-representation $\bar\rho$ of $\pi_1(M_K)$ up to conjugation. This $PSU(2)$-representation $\bar\rho$ lifts to an $SU(2)$-representation ρ of $\pi_1(M_K)$ which is automatically trace free. Since $\bar\rho_0$ is an irreducible representation, ρ is not binary dihedral. The existence of $\bar\rho_0$ is provided by [**B**]. We first apply some ideas from [**Mat**] to show that $\bar\rho_0$ extends to a unique $PSL_2(\mathbb{C})$-representation $\bar\rho$ of $\pi_1(M_K)$. Then we further show that this $\bar\rho$ is conjugate to an $PSU(2)$-representation by applying some results from [**HP**][**CD**].

Now we give the details of the proof. For a finitely generated group Γ, $\bar R(\Gamma) = Hom(\Gamma, PSL_2(\mathbb{C}))$ denotes the $PSL_2(\mathbb{C})$ representation variety of Γ and $\bar X(\Gamma)$ the

$PSL_2(\mathbb{C})$ character variety of Γ. Let $t:\bar{R}(\Gamma)\to\bar{X}(\Gamma)$ be the map which sends a representation $\bar\rho$ to its character $\chi_{\bar\rho}$. We shall write an element in $PSL_2(\mathbb{C})$ as $\bar{A}$ which is the image of an element A in $SL_2(\mathbb{C})$ under the quotient map $SL_2(\mathbb{C})\to PSL_2(\mathbb{C})$ and for convenience we sometimes call elements in $PSL_2(\mathbb{C})$ as matrices. For any $\bar{A}\in PSL_2(\mathbb{C})$ define $tr^2(\bar{A})=(trace(A))^2$ which is obviously well defined. Recall that the character $\chi_{\bar\rho}$ of an $PSL_2(\mathbb{C})$-representation $\bar\rho$ is the function $\chi_{\bar\rho}:\Gamma\to\mathbb{C}$ defined by $\chi_{\bar\rho}(\gamma)=tr^2(\bar\rho(\gamma))$.

A character $\chi_{\bar\rho}$ is real if $\chi_{\bar\rho}(\gamma)\in\mathbb{R}$ for all $\gamma\in\Gamma$. If we consider $\bar{X}(\Gamma)$ as an algebraic subset in $\mathbb{C}^n$ (for some n), then real characters of $\bar{X}(\Gamma)$ correspond to real points of $\bar{X}(\Gamma)$, i.e. points of $\bar{X}(\Gamma)\cap\mathbb{R}^n$. If $\sigma:\mathbb{C}^n\to\mathbb{C}^n$ (for each $n\geq 1$) denotes the operation of coordinatewise taking complex conjugation, then any complex affine algebraic set Y in $\mathbb{C}^n$ defined over $\mathbb{Q}$ is invariant under σ and the set of real points of Y is precisely the fixed point set of σ in Y. Note that $\bar{R}(\Gamma)$ and $\bar{X}(\Gamma)$ are both algebraic sets defined over $\mathbb{Q}$ and that the map $t:\bar{R}(\Gamma)\to\bar{X}(\Gamma)$ is an algebraic map defined over $\mathbb{Q}$, we thus have the following commutative diagram of maps:

$$\begin{array}{ccc}\bar{R}(\Gamma) & \stackrel{\sigma}{\longrightarrow} & \bar{R}(\Gamma)\\ \downarrow t & & \downarrow t\\ \bar{X}(\Gamma) & \stackrel{\sigma}{\longrightarrow} & \bar{X}(\Gamma).\end{array}$$

It follows that $\sigma(\chi_{\bar\rho})=\chi_{\sigma(\bar\rho)}$.

Recall that a representation $\bar\rho\in\bar{R}(\Gamma)$ is called irreducible if the image of $\bar\rho$ cannot be conjugated into the set $\{\bar{A};\ A \text{ upper triangular}\}$ ([**BZ**, Definition on page 752]). Two irreducible representations in $\bar{R}(\Gamma)$ are conjugate if and only if they have the same character (This property is proved in [**BZ**, the second paragraph on page 753]).

If W is a compact manifold, $\bar{R}(W)$ and $\bar{X}(W)$ denote $\bar{R}(\pi_1 W)$ and $\bar{X}(\pi_1 W)$ respectively.

Let $K=K(q_1/p_1,\cdots,q_n/p_n)$ and $M=M_{\rm K}$. We may assume that all p_i are positive (by changing the sign of q_i if necessary). Let $p:\tilde{M}\to M$ be the 2-fold cyclic covering and let $\tilde\mu=p^{-1}(\mu)$ which is a connected simple closed essential curve in $\partial\tilde{M}$ which double covers μ. Then $p_*:\pi_1(\tilde{M})\to\pi_1(M)$ is an injection and we may consider $\pi_1(\tilde{M})$ as an index two normal subgroup of $\pi_1(M)$, in which $\tilde\mu=\mu^2$. Dehn filling $\tilde{M}(\tilde\mu)$ of $\tilde{M}$ with the slope $\tilde\mu$ is the double branched cover $\Sigma_2(K)$ of (S^3,K). The covering involution τ on $\tilde{M}$ extends to one on $\tilde{M}(\tilde\mu)$ which we still denote by τ. Montesinos proved in [**Mont1**][**Mont2**] that $\tilde{M}(\tilde\mu)$ admits a Seifert fibering invariant under the covering involution τ, the base orbifold of the Seifert fibred space is $S^2(p_1,...,p_n)$ which is the 2-sphere with n cone points of orders $p_1,...,p_n$, and τ descends down to an involution $\bar\tau$ on $S^2(p_1,...,p_n)$ which is a reflection in a circle passing through all the cone points (see Figure 2).

We denote the orbifold fundamental group of $S^2(p_1,...,p_n)$ by $\Delta(p_1,...,p_n)$ which has the following presentation:

$$\Delta(p_1,...,p_n)=\langle a_1,...,a_n; a_i^{p_i}=1, i=1,...,n, a_1a_2\cdots a_n=1\rangle.$$

Geometrically the element a_i is represented by the loop $b_{i-1}^{-1}b_i$ shown in Figure 2 (b_0 is the trivial loop). Note that there is a quotient homomorphism from $\pi_1(\tilde{M}(\tilde\mu))$ onto $\Delta(p_1,...,p_n)$.

It was shown in [**Mat**, Section 3.3] that when $n=3$ any irreducible $PSL_2(\mathbb{C})$-representation of $\pi_1(\tilde{M})$ which factors through $\pi_1(\tilde{M}(\tilde\mu))$ has a unique extension to

$\pi_1(M)$. Note that this extended representation can be lifted to a trace free $SL_2(\mathbb{C})$-representation of $\pi_1(M)$. We shall slightly extend this result to the following

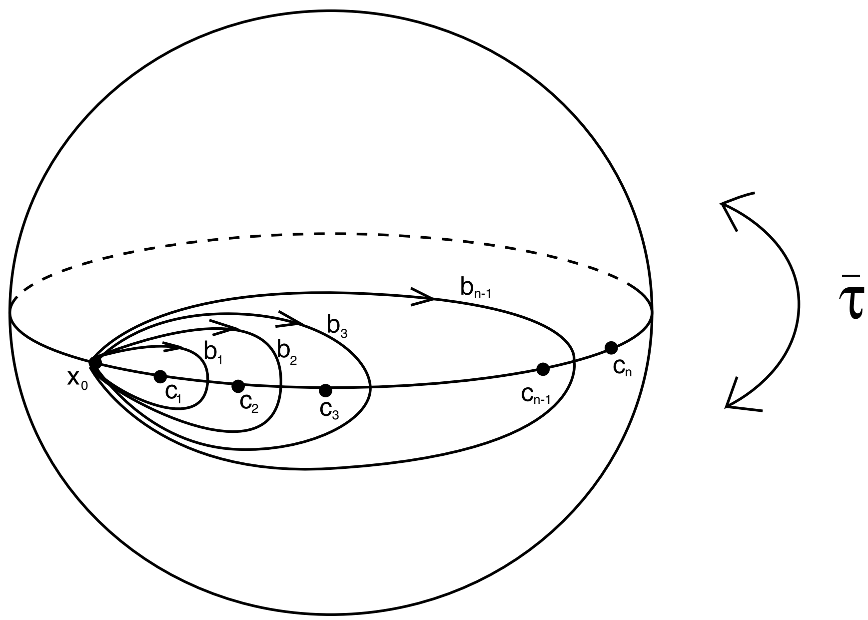

FIGURE 2. The orbifold $S^2(p_1, \cdots, p_n)$, its involution $\bar{\tau}$ and the generating set $b_1, ..., b_{n-1}$ for $\Delta(p_1, ..., p_n)$, where x_0 is the base point and $c_1, ..., c_n$ are cone points of orders $p_1, ..., p_n$ respectively.

PROPOSITION 8. *Let δ be the composition of the three quotient homomorphisms*

$$\pi_1(\widetilde{M}) \to \pi_1(\widetilde{M}(\tilde{\mu})) \to \Delta(p_1, p_2, ..., p_n) \to \Delta(p_1, p_2, p_3).$$

Let $\phi : \Delta(p_1, p_2, p_3) \to PSL_2(\mathbb{C})$ be any irreducible representation. Then $\bar{\rho}_0 = \phi \circ \delta$ has a unique extension to $\pi_1(M_K)$.

PROOF. The proof for uniqueness is verbatim as that given in [**Mat**] on page 38-39. We need to note that as $K(q_1/p_1, \cdots, q_n/p_n)$ is a knot at most one of p_i's is even. So [**Mat**, Lemma 2.4.9] still applies to our current case, i.e. the center of the image group of $\bar{\rho}_0$ is the trivial group.

CLAIM 9. There will be an extension $\bar{\rho}$ if and only if there is $\bar{A} \in PSL_2(\mathbb{C})$ such that $\bar{A}^2 = \bar{I}$ (where I is the identity matrix of $SL_2(\mathbb{C})$) and $\bar{A}\bar{\rho}_0(\beta)\bar{A}^{-1} = \bar{\rho}_0(\mu\beta\mu^{-1})$ for all $\beta \in \pi_1(\tilde{M})$.

Again this claim can be proved verbatim as that of [**Mat**, Claim 3.3.2].

So to finish the proof of Proposition 8, we just need to find an $\bar{A} \in PSL_2(\mathbb{C})$ with the properties stated in Claim 9, which is what we are going to do in the rest of the proof of Proposition 8. Recall that $\tilde{M}(\tilde{\mu})$ is the Dehn filling of $\tilde{M}$ with a solid torus N whose meridian slope is identified with the slope $\tilde{\mu}$. The core circle of N is the fixed point set of τ in $\tilde{M}(\tilde{\mu})$. Let D be a meridian disk of N such that the fixed point of τ in D (the center point of D) is disjoint from the singular fibers of the Seifert fibred space $\tilde{M}(\tilde{\mu})$. Choose a point $\tilde{x}$ in ∂D and let $\tilde{x}_0$ be the center point of D. Then arguing as on [**Mat**, Page 40] we have the following commutative diagram:

$$\begin{array}{ccccccc} \pi_1(\tilde{M},\tilde{x}) & \longrightarrow & \pi_1(\tilde{M}(\tilde{\mu}),\tilde{x}) & \longrightarrow & \pi_1(\tilde{M}(\tilde{\mu}),\tilde{x}_0) & \longrightarrow & \Delta(p_1,....,p_n) \\ \downarrow(\cdot)^{\mu} & & & & \downarrow\bar{\tau}_* & & \downarrow\bar{\tau}_* \\ \pi_1(\tilde{M},\tilde{x}) & \longrightarrow & \pi_1(\tilde{M}(\tilde{\mu}),\tilde{x}) & \longrightarrow & \pi_1(\tilde{M}(\tilde{\mu}),\tilde{x}_0) & \longrightarrow & \Delta(p_1,....,p_n) \end{array}$$

where $(\cdot)^{\mu} : \pi_1(\tilde{M},\tilde{x}) \to \pi_1(\tilde{M},\tilde{x})$ corresponds to the conjugation action by μ, i.e. $(\beta)^{\mu} = \mu\beta\mu^{-1}$ and $\Delta(p_1,...,p_n)$ is the orbifold fundamental group of $S^2(p_1,...,p_n)$ whose base point is the image x_0 of the point $\tilde{x}_0$ under the quotient map

$$\tilde{M}(\tilde{\mu}) \to S^2(p_1,...,p_n).$$

Figure 2 shows the generating set $b_1,...,b_{n-1}$ of the orbifold fundamental group $\Delta(p_1,...,p_n)$ of $S^2(p_1,...,p_n)$. In fact we have

$$a_1 = b_1, a_2 = b_1^{-1}b_2, a_3 = b_2^{-1}b_3, \cdots, a_{n-1} = b_{n-2}^{-1}b_{n-1}, a_n = b_{n-1}^{-1}$$

and conversely

$$b_1 = a_1, b_2 = a_1a_2, b_3 = a_1a_2a_3, \cdots, b_{n-1} = a_1a_2\cdots a_{n-1}, b_{n-1} = a_n^{-1}.$$

Obviously from Figure 2 the induced isomorphism $\bar{\tau}_* : \Delta(p_1,...,p_n) \to \Delta(p_1,...,p_n)$ sends b_i to b_i^{-1}, $i = 1,...,n-1$. So we have

$\bar{\tau}_*(a_1) = a_1^{-1}$, $\bar{\tau}_*(a_2) = b_1b_2^{-1} = a_1a_2^{-1}a_1^{-1}$, $\bar{\tau}_*(a_3) = b_2b_3^{-1} = a_1a_2a_3^{-1}a_2^{-1}a_1^{-1}$, $\cdots$, $\bar{\tau}_*(a_{n-1}) = b_{n-2}b_{n-1}^{-1} = a_1a_2\cdots a_{n-2}a_{n-1}^{-1}a_{n-2}^{-1}\cdots a_2^{-1}a_1^{-1}$, $\bar{\tau}_*(a_n) = \bar{\tau}_*(b_{n-1}^{-1}) = b_{n-1} = a_n^{-1}$.

Since the quotient homomorphism

$$\begin{aligned} &\Delta(p_1,\cdots,p_n) = \langle a_1,...,a_n; a_i^{p_i} = 1, i = 1,...,n, a_1a_2\cdots a_n = 1\rangle \\ &\longrightarrow \Delta(p_1,p_2,p_3) = \langle \bar{a}_1,\bar{a}_2,\bar{a}_3; \bar{a}_i^{p_i} = 1, i = 1,2,3, \bar{a}_1\bar{a}_2\bar{a}_3 = 1\rangle \end{aligned}$$

sends a_i to $\bar{a}_i$, $i = 1,2,3$, and send a_i to 1, $i = 4,...,n$. we see that $\bar{\tau}_*$ descents to an isomorphism $\bar{\tau}_{\#} : \Delta(p_1,p_2,p_3) \to \Delta(p_1,p_2,p_3)$ such that $\bar{\tau}_{\#}(\bar{a}_1) = \bar{a}_1^{-1}$, $\bar{\tau}_{\#}(\bar{a}_2) = \bar{a}_1\bar{a}_2^{-1}\bar{a}_1^{-1}$, $\bar{\tau}_{\#}(\bar{a}_3) = \bar{a}_1\bar{a}_2\bar{a}_3^{-1}\bar{a}_2^{-1}\bar{a}_1^{-1}$ and we have the following commutative diagram:

$$\begin{array}{ccccccccc} \pi_1(\tilde{M}) & \longrightarrow & \pi_1(\tilde{M}(\tilde{\mu})) & \longrightarrow & \Delta(p_1,....,p_n) & \longrightarrow & \Delta(p_1,p_2,p_3) & \xrightarrow{\phi} & PSL_2(\mathbb{C}) \\ \downarrow(\cdot)^{\mu} & & \downarrow\tau_* & & \downarrow\bar{\tau}_* & & \downarrow\bar{\tau}_{\#} & & \\ \pi_1(\tilde{M}) & \longrightarrow & \pi_1(\tilde{M}(\tilde{\mu})) & \longrightarrow & \Delta(p_1,....,p_n) & \longrightarrow & \Delta(p_1,p_2,p_3) & \xrightarrow{\phi} & PSL_2(\mathbb{C}) \end{array}$$

So $\bar{\tau}_{\#}(\bar{a}_1\bar{a}_2) = \bar{a}_2^{-1}\bar{a}_1^{-1} = (\bar{a}_1\bar{a}_2)^{-1}$. Since $\Delta(p_1,p_2,p_3)$ is generated by $\bar{a}_1, \bar{a}_2$, we see by applying [**BZ**, Lemma 3.1] that ϕ and $\phi \circ \bar{\tau}_{\#}$ have the same $PSL_2(\mathbb{C})$ character. (In fact if $\phi(\bar{a}_1) = \bar{A}_1$ and $\phi(\bar{a}_2) = \bar{A}_2$, then $\phi(\bar{a}_1\bar{a}_2) = \bar{A}_1\bar{A}_2 = \overline{A_1A_2}$, $(\phi \circ \bar{\tau}_{\#})(\bar{a}_1) = (\bar{A}_1)^{-1} = \overline{A_1^{-1}}$, $(\phi \circ \bar{\tau}_{\#})(\bar{a}_2) = \bar{A}_1(\bar{A}_2)^{-1}(\bar{A}_1)^{-1} = \overline{A_1A_2^{-1}A_1^{-1}}$ and $(\phi \circ \bar{\tau}_{\#})(\bar{a}_1\bar{a}_2) = (\bar{A}_2)^{-1}(\bar{A}_1)^{-1} = \overline{(A_1A_2)^{-1}}$. Now let F_2 be the free group on two generators ξ_1 and ξ_2. Let ρ_1 and ρ_2 be the $SL_2(\mathbb{C})$ representations of F_2 defined by $\rho_1(\xi_i) = A_i$, $i = 1,2$, and $\rho_2(\xi_1) = A_1^{-1}$, $\rho_2(\xi_2) = A_1A_2^{-1}A_1^{-1}$. Then one can easily verify that $tr(\rho_1(\xi_1)) = tr(\rho_2(\xi_1))$, $tr(\rho_1(\xi_2)) = tr(\rho_2(\xi_2))$ and $tr(\rho_1(\xi_1\xi_2)) = tr(\rho_2(\xi_1\xi_2))$. So [**BZ**, Lemma 3.1] applies.) So ϕ and $\phi \circ \bar{\tau}_{\#}$ are conjugate $PSL_2(\mathbb{C})$ representations, that is, there is $\bar{A} \in PSL_2(\mathbb{C})$ with $\bar{A}\phi\bar{A}^{-1} =$

$\phi \circ \bar{\tau}_{\#}$. Combining this with the definition of $\bar{\rho}_0$ and the last commutative diagram, we see that $\bar{A}\bar{\rho}_0(\beta)\bar{A}^{-1} = \bar{\rho}_0(\mu\beta\mu^{-1})$ for each $\beta \in \pi_1(\tilde{M})$. The proof of Proposition 8 is now finished. □

Now by [**B**], every triangle group $\Delta(p_1, p_2, p_3)$ has an irreducible $SO(3) \cong PSU(2)$-representation. Therefore there is an irreducible representation $\bar{\rho}_0$ as given in Proposition 8 with its image contained in $PSU(2)$. So the character $\chi_{\bar{\rho}_0}$ of $\bar{\rho}_0$ is real valued. Let $\bar{\rho}$ be the unique extension of $\bar{\rho}_0$ to $\pi_1(M)$ as guaranteed by Proposition 8. The rest of proof is to show that $\bar{\rho}$ is also a $PSU(2)$-representation.

Claim 10. The character $\chi_{\bar{\rho}}$ of $\bar{\rho}$ is real valued.

Suppose otherwise. Recall that $\sigma : \bar{X}(M) \to \bar{X}(M)$ is the operation of taking complex conjugation and a character is real valued if and only if it is a fixed point of σ. So $\chi_{\bar{\rho}} \neq \sigma(\chi_{\bar{\rho}}) = \chi_{\sigma(\bar{\rho})}$ are two different characters of irreducible representations and thus $\bar{\rho}$ and $\sigma(\bar{\rho})$ are non-conjugate representations. But $\chi_{\bar{\rho}_0} = \sigma(\chi_{\bar{\rho}_0}) = \chi_{\sigma(\bar{\rho}_0)}$ and $\bar{\rho}_0$ is irreducible. Hence $\bar{\rho}_0$ and $\sigma(\bar{\rho}_0)$ are conjugate representations, that is, there is $\bar{B} \in PSL_2(\mathbb{C})$ such that $\bar{\rho}_0 = \bar{B}\sigma(\bar{\rho}_0)\bar{B}^{-1}$. Hence $\bar{\rho}$ and $\bar{B}\sigma(\bar{\rho})\bar{B}^{-1}$ are non-conjugate representations which have the same restriction on $\pi_1(\tilde{M})$. We get a contradiction with Proposition 8.

By [**HP**, Lemma 10.1] an $PSL_2(\mathbb{C})$-character $\chi_{\bar{\rho}}$ of a finitely generated group is real valued if and only if the image of $\bar{\rho}$ can be conjugated into $PSU(2)$ or $PGL_2(\mathbb{R})$. So our current representation $\bar{\rho}$ can be conjugated into $PSU(2)$ or $PGL_2(\mathbb{R})$. If it can be conjugated into $PSU(2)$, then we are done because this conjugated representation lifts to a trace free $SU(2)$-representation which is not binary dihedral. So suppose that $\bar{\rho}$ is conjugate to an $PGL_2(\mathbb{R})$-representation $\bar{\rho}'$. As noted in [**HP**] right after Lemma 10.1, $PGL_2(\mathbb{R}) \subset PGL_2(\mathbb{C}) \cong PSL_2(\mathbb{C})$ has two components, the identity component is $PSL_2(\mathbb{R})$ and the other component consists of matrices of determinant -1 (which in $PSL_2(\mathbb{C})$ are represented by matrices with entries in $\mathbb{C}$ with zero real part). Considering the action of $PSL_2(\mathbb{C})$ on hyperbolic space $\mathbb{H}^3$ by orientation preserving isometries, the group $PGL_2(\mathbb{R})$ is the stabilizer of a total geodesic plane in $\mathbb{H}^3$, and an element of $PGL_2(\mathbb{R})$ is contained in $PSL_2(\mathbb{R})$ if and only if it preserves the orientation of the plane (cf [**JS**, Section 2.8]). Since $\pi_1(\tilde{M})$ is the unique index two normal subgroup of $\pi_1(M)$, $\pi_1(\tilde{M})$ is generated by elements γ^2, $\gamma \in \pi_1(M)$. (To see this assertion holds, it is easy to verify that the subgroup G of $\pi_1(M)$ generated by squares is a normal subgroup and the quotient group $\pi_1(M)/G$ is an abelian group in which every element has order two. Since $H_1(M) = \mathbb{Z}$, $\pi_1(M)/G$ has to be the cyclic group of order two.) Therefore the image of the restriction $\bar{\rho}'_0$ of $\bar{\rho}'$ on $\pi_1(\tilde{M})$ consists of elements preserving the orientation of the total geodesic plane mentioned above, i.e. the image of $\bar{\rho}'_0$ is contained in $PSL_2(\mathbb{R})$. So the image of $\bar{\rho}_0$ can be conjugated into $PSL_2(\mathbb{R})$. But the image of $\bar{\rho}_0$ is contained in $PSU(2)$. [**CD**, Lemma 2.10] says that if an $PSU(2)$-presentation can be conjugated into an $PSL_2(\mathbb{R})$-representation, then it is a reducible representation. So our $\bar{\rho}_0$ is a reducible representation. We arrive at a contradiction, which completes the proof of Theorem 1.

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